SIMPLE RANDOM WALK EXCURSION MEASURE IN THE PLANE

by

Michael John Kozdron

Department of Mathematics
Duke University

Date: ______________________________

Approved:

____
Gregory F. Lawler, Supervisor

____
J. Thomas Beale

____
David P. Kraines

____
Robert L. Wolpert

Dissertation submitted in partial fulfillment of
the requirements for the degree of Doctor
of Philosophy in the Department of
Mathematics in the Graduate School
of Duke University

2004

ABSTRACT

(Mathematics)

SIMPLE RANDOM WALK EXCURSION MEASURE IN THE PLANE

by

Michael John Kozdron

Department of Mathematics
Duke University

Date: ______________________________

Approved:

____
Gregory F. Lawler, Supervisor

____
J. Thomas Beale

____
David P. Kraines

____
Robert L. Wolpert

An abstract of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University

2004

Abstract

A number of two-dimensional physical systems studied in statistical mechanics are well-described by a lattice model. These models are frequently conjectured to possess conformally invariant scaling limits which may be used to make exact predictions for certain critical exponents describing the qualitative behaviour of the given system.

In this dissertation, we prove simple random walk excursion measure converges in the scaling limit to Brownian excursion measure on all bounded, simply connected domains in the complex plane with Jordan boundary. This result is exclusively two-dimensional as the Riemann mapping theorem and the conformal invariance of Brownian motion play a vital rôle.

We carefully construct Brownian excursion measure and prove it is, in fact, a conformal invariant. We also prove several Green's function estimates and establish a relationship between the continuous and discrete excursion Poisson kernels.

Using these estimates, we prove the convergence of the corresponding excursion measures in the Prohorov metric. This result enables us to prove a conjecture made by S. Fomin in 2001 concerning a relationship between the scaling limit of the hitting matrix determinant for simple random walk and a certain functional of loop-erased random walk.

This dissertation is dedicated to my wonderful wife

Jessica ♡

Contents

Acknowledgements

There are many people who have helped me throughout my career, thus contributing to this dissertation; this is my opportunity to publicly thank them.

First, and foremost, I must express my deepest appreciation to my supervisor, Gregory F. Lawler. I most certainly would have never completed my Ph.D. if it were not for his guidance, encouragement, and support, not to mention his infinite patience when it was not so easily seen. Indeed he has taught me almost all the mathematics (over 3000 pages of notes!) that I know. My graduate career can really be divided into two regimes. The first four years were spent at Duke University, and the final two visiting Cornell University, with a semester's visit to Institut Mittag-Leffler in Djursholm, Sweden, in between. I began working on excursion measure at Mittag-Leffler, and I am grateful to Greg for bringing me with him, first to Sweden, and then to Ithaca. I'd also like to thank him for his generous funding which has enabled me to travel to conferences in Europe and across North America. I will be forever indebted to him.

Spending these last two years at Cornell has been a wonderful experience, and I deeply appreciate the welcoming efforts made by Donna Smith and Cathy Stevens. I would especially like to thank Maria Terrell for letting me teach my own courses here, notably introductory statistics. One of the nicest surprises I had at Cornell was meeting Philip Protter. What began as a LaTeX job [43] for extra money has evolved into a rewarding friendship. I would also like to thank the members of my examination committee from Duke: Tom Beale, David Kraines, and Robert Wolpert, as well as Lewis Blake who always supported my classroom teaching.

I've also been fortunate that there have been several other students with whom I could work, talk, and drink. Certainly, Christian Beneš, Nathanaël Berestycki, José Antonio Trujillo Ferreras, and Brigitta Vermesi have all provided a continuing source of inspiration, and I am lucky to have made such good friends. I'd specifically like to thank José for his help with the Prohorov metric in Section 4.2. Christian helped me learn a number of probabilistic techniques including Skorohod embedding, but I'm most appreciative to Chris for introducing me to Chimay!

In high school, I had several influential math teachers, and I am especially grateful to Reg Cichos, my father Casey, and the late Don Modjeski: "Joke, joke, joke." Of course, the support and encouragement of my parents, Casey and Audrey, and my sister Allison, has been constant, and appreciated by me, though I have rarely said so. Allow me now to genuinely thank them for their unquestioning love. Most importantly, however, I must save my final words for my wonderful wife, Jessica. We have laughed together, and cried together, and she has never once failed to console me anytime I faltered on this journey. Simply put Jessica, I love you.

Chapter 1

Introduction

A very general goal in statistical physics is to understand the behaviour of a system at *criticality*; that is, at or near the temperature at which there is a phase transition. In a number of instances the physical system is well-described by a discrete, or lattice, model, and many two-dimensional lattice models are conjectured to have a conformally invariant scaling limit. By assuming conformal invariance, it is often possible to give exact predictions for certain exponents which describe the qualitative behaviour of the systems. The most basic example is that of simple random walk on $\mathbb{Z}^2$ which, appropriately normalized, converges to complex Brownian motion, a conformally invariant scaling limit. Other examples include loop-erased random walk, domino tilings, uniform spanning trees, percolation, and the Ising and Potts models. The self-avoiding walk, a model of polymer chains introduced by the chemist P. Flory [16] in the 1940's, is a model where minimal rigorous progress has been made, and is one of the motivations for much of this entire program of study; see also [59]. For an account of recent mathematical progress on these models see [55].

The purpose of this dissertation is to prove that simple random walk excursion measure converges in the scaling limit to Brownian excursion measure, a conformal invariant, on all bounded, simply connected domains $D \subset \mathbb{C}$ with Jordan boundary.

P. Lévy first proved that Brownian motion until its exit from a domain is conformally invariant. If $f : D \to D'$ is a conformal transformation, B_t is a Brownian motion started at $x \in D$, stopped at τ_D, its exit time from D, then $f(B_t)$ is a (time-changed) Brownian motion started at $f(x) \in D'$, stopped at $\tau_{D'}$; see [5]. The convergence of simple random walk to Brownian motion is made precise by Donsker's invariance principle, and is stated in terms of weak convergence of probability measures. Let S_j be a simple random walk and define the continuous time process Y_t by linear interpolation. If $X_t^{(n)} := n^{-1/2}\,Y_{nt}$ for each n and $\mathbb{P}_n$ is the measure induced by $X^{(n)}$ on $(C[0,\infty), \mathcal{B}(C[0,\infty))$, then $\mathbb{P}_n$ converges weakly to Wiener measure $\mathbb{W}$, the measure under which $B_t(\omega) := \omega(t)$ on $C[0,\infty)$ is a Brownian motion; see [23].

Our approach is similar; we prove simple random walk excursion measure converges weakly to Brownian excursion measure, although some care needs to be shown as these measures are, in general, not probabilities. Let $D \subset \mathbb{C}$ be a simply connected, bounded domain with Jordan boundary. If z, $w \in \partial D$, then a curve $\gamma : [0,T] \to \mathbb{C}$, $0 < T < \infty$, is called an *excursion from z to w in D* if $\gamma(0) = z$, $\gamma(T) = w$, and $\gamma(0,T) \subset D$. We call the measure $\mu_{\partial D,z,w}$ on such curves *excursion measure*. Let $D_N = \frac{1}{N}\mathbb{Z}^2 \cap D$ and suppose that x, $y \in \partial D_N$. If S_j is a simple random walk on $\frac{1}{N}\mathbb{Z}^2$ with $S_0 = x$, $S_n = y$, and $[S_1, \ldots, S_{n-1}] \subset D_N$, then S_j is called a *simple random walk excursion from x to y in D_N*. Write Ω_N for the space of all simple random walk excursions from x to y in D_N. Let $\nu_{\partial D_N,x,y}$ be *simple random walk excursion measure* which assigns weight $4^{-|\omega|}$ to each $\omega \in \Omega_N$. Suppose that Γ, $\Upsilon \subset \partial D$ are open boundary arcs with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$, and Γ_N, $\Upsilon_N \subset \partial D_N$ are the corresponding discrete boundary arcs. In Theorem 5.5.1, we prove that

$$\sum_{x\in\Gamma_N}\sum_{y\in\Upsilon_N} \nu_{\partial D_N,x,y} \;\Rightarrow\; \int_\Gamma\int_\Upsilon \mu_{\partial D_N,x,y}\,|\mathrm{d}y|\,|\mathrm{d}x| \quad \text{weakly.}$$

Brief history of Brownian excursions

One-dimensional Brownian motion excursions were formally introduced by K. Itô in 1971, though they had been understood as early as 1948 by Lévy. Suppose that B_t is Brownian motion in $\mathbb{R}$, and partition the path into (random) components consisting of the zero set $Z(\omega) = \{t : B_t(\omega) = 0\}$ and the "excursions away from 0." Itô noted that if Z is parameterized by the local time of B, then these excursions away from 0 can be viewed as points in a Poisson process. Furthermore, the characteristic measure of this Poisson process describes many interesting features of the original Brownian motion. As indicated in [45], this Poisson-point-process view of one-dimensional Brownian excursions has successfully been exploited by a number of researchers. In general, suppose that X_t is a Markov process in $\mathbb{R}$ and b is in the state space of X so that we can now consider $\{t : X_t(\omega) = b\}$ and the excursions away from b. A reasonably complete treatment in this case may be found in [10].

In contrast, the theory of multi-dimensional Brownian excursions is much more incomplete, and they were basically neglected until 1984. K. Burdzy [12] introduces excursion laws in a domain from a distinguished boundary point. His motivation was to study the properties of the initial part of the excursion, and he restricted attention primarily to Lipschitz domains since he could give local properties of excursions in explicit form only for these particular regions.

Excursions in a domain between fixed boundary points

The point of view of two-dimensional Brownian excursions in an arbitrary simply connected domain in $\mathbb{C}$ taken in this dissertation was introduced in 2000 by G. Lawler and W. Werner [39] while studying Brownian intersection exponents. For their purposes, an excursion in a simply connected domain $D \subset \mathbb{C}$ was a complex Brownian motion B_t, $0 \leq t \leq T$, with $B_0 \in \partial D$, $B_T \in \partial D$, and $B(0,T) \subset D$. However, they only considered domains D with smooth boundary such as the unit disk $\mathbb{D}$ or the upper half plane $\mathbb{H}$ (allowing $T = \infty$). Excursion measure in these cases was shown to be conformally invariant using the topology generated by (4.1). In Theorem 4.5.16, this fact is proved in a more general topology. Also in [39, §5.1], two-dimensional discrete excursion measure was defined as a measure on random walk paths. When the boundary of the domain $D \subset \mathbb{C}$ is smooth, it was conjectured that this discrete excursion measure converged to the Brownian excursion measure. They also discovered that certain critical exponents could be computed exactly from Brownian excursions in a rectangle. It had been previously proved [28, Chapter 5] that the intersection exponents for simple random walk could be given in terms of the intersection exponents for Brownian motion. This yielded, roughly, that critical exponents are conformally invariant. Using the Schramm-Loewner evolution (SLE_κ) introduced by O. Schramm, these exponents can now be calculated exactly. (See [35, 36, 37] for the original proofs and [34] for a survey of those results. A well-written popular science overview is found in [42].) Most recently, excursion measure has been used by B. Virág in his work on Brownian beads [54] to show the pieces of the Brownian path between cut points form a Poisson process, and by Lawler and Werner in connection with the Brownian loop soup [40].

Fomin's identity

In 2001, S. Fomin [18] proved an identity relating loop-erased random walk probabilities to determinants of matrices of hitting measures. He showed under certain conditions that the probability a first random walk starting at x^1 exits a domain at y^1, and a second random walk starting at x^2 exits the domain at y^2 and avoids the loop-erasure of the first path, is given by the determinant of the hitting matrix. Actually, he proved this identity in general for the loop-erasure of discrete stationary

Markov chains, and conjectured that it held for continuous processes as well, provided that the model was discretized and appropriate limits taken.

The correct limiting process of simple random walk between distinct boundary points to consider is the conformally invariant Brownian excursion. Thus, we are able to establish this conjecture using our results that the mass of discrete excursion measure, the hitting measure of the boundary, converges to the mass of the Brownian excursion measure, the excursion Poisson kernel. Theorem 6.4.2 is the culmination of these efforts.

A word on notation

The results in this dissertation are exclusively two-dimensional, and unless otherwise explicitly noted, assume that we are working in $\mathbb{C}$. As mentioned in the introduction to Chapter 2, we will denote points in the complex plane by any of w, x, y, z. Following [44], write $:=$ to mean "is defined to equal." Typographical necessity dictates that we write $\mathrm{cl}(D)$ for the closure of D; that is, $\mathrm{cl}(D) = \overline{D} := D \cup \partial D$.

As noted, we answer questions of convergence of processes in the scaling limit. The careful formulation of this, as in Donsker's invariance principle, is in terms of convergence of measures on paths. The natural metric for doing such is the Prohorov metric $\wp$. Hence, whenever we say that measures converge, it will be with respect to $\wp$. Recall that the Prohorov topology is also called the topology of weak convergence, and under certain conditions, convergence in $\wp$ is equivalent to weak convergence; that is, $\wp(\mu_n, \mu) \to 0$ if and only if $\mu_n \Rightarrow \mu$ weakly. This is explained in greatly expanded detail in Section 4.2.1. Especially note the important remark on page 67.

Furthermore, we will be considering a number of "boundary-to-boundary" quantities that are analogues of well-known "interior-to-boundary" ones. We will write subscripts D for interior-to-boundary and ∂D for boundary-to-boundary. For example, if $x \in D$, $y \in \partial D$, then the Poisson kernel $H_D(x, y)$ and Wiener measure $\mu_D(x, y)$ are interior-to-boundary quantities. If x, $y \in \partial D$, the boundary-to-boundary analogues are the excursion Poisson kernel $H_{\partial D}(x, y)$ and excursion measure $\mu_{\partial D}(x, y)$.

Summary of contents

A brief outline of the content of each chapter is as follows. Note that each chapter itself begins with a summary of that chapter's material. In Chapter 2, the necessary background material is reviewed including the required complex analysis. We also discuss some of the potential theory pertinent for the analysis of Brownian motion and random walk. Our main references for this material are [5] and [14]. In Chapter 3, we establish a number of Green's function estimates that will be essential for proving the convergence of discrete excursion measure. Chapter 4 is devoted to a careful study of Brownian excursion measure. We rigorously construct excursion measure, which requires the development of the topology associated with the metric space of curves on which this measure is concentrated. In Chapter 5, we prove the main theorem that discrete excursion measure in the plane converges in the scaling limit to Brownian excursion measure. Finally, in Chapter 6, we review Fomin's identity, and demonstrate its relationship with excursion measure. We discuss briefly loop-erased random walk, and recall some of the history of this process. Theorem 6.4.2 then resolves the conjecture made in [18].

Chapter 2

Background and Preliminary Results

The purpose of this chapter is to review some basic material that will be needed in subsequent chapters and to standardize our notation. Almost all of the complex analysis is well-known, and may be found in a variety of sources; references are supplied throughout. We prove several elementary results, but often refer the reader to the literature for details. The material on the *excursion Poisson kernel* is not difficult, and these results may be known, but they are not widespread.

The only Euclidean dimension that will concern us is $d = 2$; consequently we will associate $\mathbb{C} \cong \mathbb{R}^2$ in the natural way. Points in the complex plane will be denoted by any of w, x, y, or z. Although this is a departure from the traditional $z = x + iy$, it should cause no difficulty. If $x \in \mathbb{C}$, denote by $\mathcal{R}e(x)$ and $\mathcal{I}m(x)$ the real and imaginary parts of x, respectively, so that $x = (\mathcal{R}e(x), \mathcal{I}m(x)) = \mathcal{R}e(x) + i\,\mathcal{I}m(x)$. If a real parameter is needed, it will usually be denoted by either t or n, lending to an interpretation as time. A *domain* $D \subset \mathbb{C}$ is an open and connected set. It is implicit that D is larger than a single point; therefore, if D is a domain and $0 \in D$, then there exists some $r > 0$ such that $\{|z| \leq r\} \subset D$.

Throughout, B_t, $t \geq 0$, will denote a standard complex Brownian motion, and S_n, $n = 0, 1, \ldots$, will denote two-dimensional simple random walk, both started at the origin. We write $B[0,t] := \{z \in \mathbb{C} : B_s = z \text{ for some } 0 \leq s \leq t\}$, and $S[0,n] := [S_0, S_1, \ldots, S_n]$ for the set of lattice points visited by the random walk. We will generally use T for stopping times for Brownian motion, and τ for stopping times for random walk.

2.1 Simply connected subsets of $\mathbb{C}$ and $\mathbb{Z}^2$

We write $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ to denote the open unit disk. Recall that a complex-valued function f of a complex variable is *analytic* at $z_0 \in \mathbb{C}$ if it is complex-differentiable at every point in some neighbourhood of z_0. We remark that there is no agreement in the literature over the following definitions; our presentation follows [14]. A single-valued function f is *univalent* in a domain $D \subset \mathbb{C}$ if it is one-to-one in D; that is, if $f(z_1) \neq f(z_2)$ whenever $z_1, z_2 \in D$ with $z_1 \neq z_2$. (For an analytic function f, $f'(z_0) \neq 0$ if and only if f is locally univalent at z_0. However, we will not be concerned with local univalence.) An analytic, univalent function is called a *conformal mapping*. We write $\mathcal{S}$ to denote the set of functions f which are analytic and univalent in $\mathbb{D}$ satisfying the normalizing conditions $f(0) = 0$ and $f'(0) = 1$. In particular, we say that $f : D \to D'$ is a *conformal transformation* if f is a conformal mapping that is onto D'. It follows that $f'(z) \neq 0$ for $z \in D$, and $f^{-1} : D' \to D$ is also a conformal transformation.

The most important result from complex analysis that we will use is the Riemann mapping theorem. For details of the following two theorems, consult [14, §1.5].

Theorem 2.1.1 (Riemann Mapping Theorem). *Suppose D is a simply connected proper subset of $\mathbb{C}$ and $z_0 \in D$. Then there exists a unique conformal transformation f_D of D onto $\mathbb{D}$ satisfying $f_D(z_0) = 0$, $f'_D(z_0) > 0$.*

It will be important that we can extend the Riemann mapping theorem to the boundary; the following result due to Carathéodory will suffice.

Theorem 2.1.2 (Carathéodory Extension Theorem). *If D is a domain bounded by a Jordan curve ∂D, and $f_D : D \to \mathbb{D}$ is a conformal transformation, then f_D can be extended to a homeomorphism of $\overline{D} = D \cup \partial D$ onto the closed disk $\overline{\mathbb{D}}$.*

We also recall some of the most important elementary results from the study of univalent functions including the Koebe one-quarter theorem, and the Koebe growth and distortion theorem. Proofs may be found in [14, Theorems 2.3, 2.4, 2.5, 2.6]

Theorem 2.1.3 (Koebe One-Quarter Theorem). *If f is a conformal mapping of the unit disk with $f(0) = 0$, then the image of f contains the open disk of radius $|f'(0)|/4$ about the origin.*

Theorem 2.1.4 (Koebe Growth and Distortion Theorem). *If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \le \frac{4|z|}{1-|z|^2}, \quad \frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2},$$

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}.$$

If $D \subset \mathbb{C}$ with $0 \in D$, then we define the *radius* (with respect to the origin) of D to be $\text{rad}(D) := \sup\{|z| : z \in \partial D\}$, and the *inradius* (with respect to the origin) of D to be $\text{inrad}(D) := \text{dist}(0, \partial D) := \inf\{|z| : z \in \partial D\}$. The *diameter* of D is given by $\text{diam}(D) := \sup\{|x - y| : x, y \in D\}$. If $D \subset \mathbb{C}$, then we say that ∂D is *nice* if ∂D is a Jordan curve and is piecewise analytic. That is, the Jordan curve ∂D can be expressed as a finite union of analytic curves. For each $r > 0$, let $\mathcal{D}^r$ be the set of simply connected, bounded domains in $\mathbb{C}$ containing the origin of inradius r with nice boundary. We then write $\mathcal{D} := \bigcup_{r>0} \mathcal{D}^r$ for the set of bounded, simply connected domains in $\mathbb{C}$ with piecewise analytic boundary containing the origin. We also define

$$\mathcal{D}^* := \{\text{domains } D \subset \mathbb{C} : 0 \in D;\ D \text{ simply connected and bounded};\ \partial D \text{ Jordan}\},$$

and note that $\mathbb{D} \in \mathcal{D} \subset \mathcal{D}^*$. If D, $D' \in \mathcal{D}^*$, let $\mathcal{T}(D, D')$ be the set of all $f : D \to D'$ that are conformal transformations of D onto D'. If $D \in \mathcal{D}^r$ and $D' \in \mathcal{D}^{r'}$, then by the Riemann mapping theorem (Theorem 2.1.1), there exists a unique conformal transformation $f : D \to D'$ with $f(0) = 0$, $f'(0) > 0$. Thus, $\mathcal{T}(D, D') \neq \emptyset$. By the Carathéodory extension theorem (Theorem 2.1.2), since ∂D, $\partial D'$ are Jordan, for any $f \in \mathcal{T}(D, D')$ there exists an extension of f to a homeomorphism of $\overline{D}$ onto $\overline{D'}$. We will use this fact repeatedly throughout, without explicit mention of it.

A subset $A \subset \mathbb{Z}^2$ is *connected* if every two points in A can be connected by a nearest neighbour path staying in A. We say that a finite subset A is *simply connected* if both A and $\mathbb{Z}^2 \setminus A$ are connected. There are three reasonable ways to define the "boundary" of A.

- *(outer) boundary:* $\partial A := \{y \in \mathbb{Z}^2 \setminus A : |y - x| = 1 \text{ for some } x \in A\}$
- *inner boundary:* $\partial_i A := \partial(\mathbb{Z}^2 \setminus A) = \{x \in A : |y - x| = 1 \text{ for some } y \in \mathbb{Z}^2 \setminus A\}$
- *edge boundary:* $\partial_e A := \{(x, y) : x \in A,\ y \in \mathbb{Z}^2 \setminus A,\ |x - y| = 1\}$

To each finite $A \subset \mathbb{Z}^2$ we associate a domain $\tilde{A} \subset \mathbb{C}$ in the following way. For each edge $(x, y) \in \partial_e A$, considered as a line segment of length one, let $\ell_{x,y}$ be the perpendicular line segment of length one intersecting (x, y) in the midpoint. Let $\partial\tilde{A}$ denote the union of the line segments $\ell_{x,y}$, and let $\tilde{A}$ denote the bounded open subset of $\mathbb{C}$ bounded by $\partial\tilde{A}$ containing A. Observe that

$$\tilde{A} \cup \partial\tilde{A} = \bigcup_{x \in A} \mathcal{S}_x \quad \text{where} \quad \mathcal{S}_x := x + (\,[-1/2, 1/2] \times [-1/2, 1/2]\,)\,. \tag{2.1}$$

That is, $\mathcal{S}_x$ is the closed square of side length one centred at x whose sides are parallel to the coordinate axes. Also, note that $\tilde{A}$ is a simply connected domain if and only if A is a simply connected subset of $\mathbb{Z}^2$. We will frequently refer $\tilde{A}$ as the *"union of squares"* domain associated to A.

Let $\mathcal{A}$ denote the set of all finite simply connected subsets of $\mathbb{Z}^2$ containing the origin. If $A \in \mathcal{A}$, let $\mathrm{inrad}(A) := \min\{|z| : z \in \mathbb{Z}^2 \setminus A\}$ and $\mathrm{rad}(A) := \max\{|z| : z \in A\}$ denote the *inradius* and *radius* (with respect to the origin), respectively, of A, and define $\mathcal{A}^n$ to be the set of $A \in \mathcal{A}$ with $n \le \mathrm{inrad}(A) \le 2n$; thus $\mathcal{A} := \bigcup_{n \ge 0} \mathcal{A}^n$. Note that if $A \in \mathcal{A}$ and $0 \ne x \in \partial_i A$, then the connected component of $A \setminus \{x\}$ containing the origin is simply connected. (This is not true if we do not assume $x \in \partial_i A$.) Similarly, by induction, if $A \in \mathcal{A}$, $0 \ne x_1 \in \partial_i A$, and $[x_1, x_2, \ldots, x_j]$ is a nearest neighbour path in $A \setminus \{0\}$, then the connected component of $A \setminus \{x_1, \ldots, x_j\}$ containing the origin is simply connected.

Finally, if $A \in \mathcal{A}$ with associated domain $\tilde{A} \subset \mathbb{C}$, then we write $f_A := f_{\tilde{A}}$ for the conformal transformation of $\tilde{A}$ onto the unit disk $\mathbb{D}$ with $f_A(0) = 0$, $f_A'(0) > 0$.

2.2 Green's functions on $\mathbb{C}$

For x, $y \in \mathbb{D}$, let $g_{\mathbb{D}}(x, y)$ denote the standard Green's function in the unit disk (for Brownian motion) given by

$$g_{\mathbb{D}}(x, y) := \log|\overline{y}x - 1| - \log|y - x|\,. \tag{2.2}$$

This is not the usual form of the Green's function as found in, for example, [17]. However, they are trivially shown to be equal. Note that $g_{\mathbb{D}}(0,x) = g_{\mathbb{D}}(x) = -\log|x|$, and $g_{\mathbb{D}}(x,y) = g_{\mathbb{D}}(y,x)$. An equivalent formulation of the Green's function for D can be given in terms of Brownian motion. Suppose B_t is a standard Brownian motion in $\mathbb{C}$. If $x \in D$, we can define $g_D(x,\cdot)$ as the unique harmonic function on $D \setminus \{x\}$, vanishing on ∂D (in the sense that $g_D(x,y) \to 0$ as $y \to \partial D$), with

$$g_D(x,y) = -\log|x-y| + O(1) \quad \text{as} \quad |x-y| \to 0. \tag{2.3}$$

From this description we have that $g_D(x,y) = \mathbb{E}^x[\log|B_{T_D} - y|] - \log|x-y|$ for distinct points x, $y \in D$ where $T_D := \inf\{t : B_t \notin D\}$. In particular, if $0 \in D$, then

$$g_D(x) = \mathbb{E}^x[\log|B_{T_D}|] - \log|x| \quad \text{for} \quad x \in D. \tag{2.4}$$

Additional details may be found in [32, Chapter 2].

Suppose that D is a simply connected proper subset of $\mathbb{C}$ with $z_0 \in D$. If $f_D \in \mathcal{T}(D,\mathbb{D})$ with $f_D(z_0) = 0$, $f_D'(z_0) > 0$, then the Green's function for D is given by

$$g_D(z,w) = g_{\mathbb{D}}(f_D(z), f_D(w)) \quad \text{for} \quad z, w \in D. \tag{2.5}$$

Proposition 2.2.1 (Conformal Invariance of the Green's Function). *Suppose that D, $D' \in \mathcal{D}^*$, and let $f \in \mathcal{T}(D,D')$. If x, $y \in D$, then $g_D(x,y) = g_{D'}(f(x), f(y))$.*

On the other hand, suppose $D \subset \mathbb{C}$ is a simply connected domain and $g_D(z,w)$ is known. If $-\theta_D$ denotes the harmonic conjugate of g_D and $z_0 \in D$, then $f_D(z) = \exp\{-g_D(z,z_0) + i\theta_D(z,z_0)\}$ is a conformal mapping of D onto $\mathbb{D}$ with $f_D(z_0) = 0$. In fact, it can be shown that θ_D has period 2π as z winds around z_0 so that f_D is single-valued. In particular, if $0 \in D$ and we write $g_D(z) := g_D(z,0) = g_D(0,z)$, then

$$f_D(z) = \exp\{-g_D(z) + i\theta_D(z)\} \tag{2.6}$$

is the unique conformal transformation of D onto $\mathbb{D}$ with $f_D(0) = 0$, $f'_D(0) > 0$. In other words, determining the Green's function for a simply connected, proper subset of $\mathbb{C}$ is equivalent to finding the Riemann mapping function of that domain onto the unit disk. For more details, see [14, §1.8] and the references therein.

If $A \subset \mathbb{Z}^2$ with $A \in \mathcal{A}$, and if we let $g_A(x,y) := g_{\tilde{A}}(x,y)$ be the Green's function (for Brownian motion) in $\tilde{A}$, then by (2.5) we have that

$$g_A(x,y) = g_{\mathbb{D}}(f_A(x), f_A(y)) = \log \left| \frac{\overline{f_A(y)} f_A(x) - 1}{f_A(y) - f_A(x)} \right|. \tag{2.7}$$

Furthermore, let $g_A(x) := g_A(0,x)$ and $\theta_A := \theta_{\tilde{A}}$, so that by (2.6),

$$f_A(x) = \exp\{-g_A(x) + i\theta_A(x)\}. \tag{2.8}$$

We conclude by defining what it means for two boundary arcs to be separated. Note that separation is always defined in terms of distance in the unit circle.

Definition 2.2.2. Suppose that $A \in \mathcal{A}$ and $D \in \mathcal{D}^*$. Let Γ_1, $\Upsilon_1 \subset \partial_i A$ with $\overline{\Gamma_1} \cap \overline{\Upsilon_1} = \emptyset$, and let Γ_2, $\Upsilon_2 \subset \partial D$ with $\overline{\Gamma_1} \cap \overline{\Upsilon_2} = \emptyset$. The *separation of* Γ_i *and* Υ_i, $i = 1, 2$, written $\operatorname{sep}(\Gamma_i, \Upsilon_i)$, is defined to be

$$\operatorname{sep}(\Gamma_i, \Upsilon_i) := \inf\{|\theta_i(x) - \theta_i(y)| : x \in \Gamma_i, y \in \Upsilon_i\}, \tag{2.9}$$

where $\theta_1 = \theta_A$ and $\theta_2 = \theta_D$. The *spread of* Γ_i *and* Υ_i, $i = 1, 2$, written $\operatorname{spr}(\Gamma_i, \Upsilon_i)$, is defined to be

$$\text{spr}(\Gamma_i, \Upsilon_i) := \sup\{|\theta_i(x) - \theta_i(y)| : x \in \Gamma_i, y \in \Upsilon_i\}. \tag{2.10}$$

Remark. If Γ_1, $\Upsilon_1 \subset \partial A$ instead, then (2.9) and (2.10) hold with θ_A as in (2.19).

2.3 Green's functions on $\mathbb{Z}^2$

Suppose that S_n is a simple random walk on $\mathbb{Z}^2$ and $A \subset \mathbb{Z}^2$. If $\tau_A := \min\{j \geq 0 : S_j \notin A\}$, then following [28, page 34] we let

$$G_A(x,y) := \mathbb{E}^x[\sum_{j=0}^{\tau_A - 1} \mathbb{1}_{\{S_j = y\}}] = \sum_{j=0}^{\infty} \mathbb{P}^x\{S_j = y, \tau_A > j\} \tag{2.11}$$

denote the Green's function for random walk on A. Set $G_A(x) := G_A(x,0) = G_A(0,x)$. In analogy with the Brownian motion case, it follows from [28, Proposition 1.6.3] that

$$G_A(x) = \mathbb{E}^x[a(S_{\tau_A})] - a(x) \text{ for } x \in A \tag{2.12}$$

where a is the potential kernel for simple random walk defined by

$$a(x) := \lim_{m \to \infty} \sum_{j=0}^{m} \left(\mathbb{P}^0\{S_j = 0\} - \mathbb{P}^x\{S_j = 0\} \right).$$

It is also known [28, Theorem 1.6.2] that as $|x| \to \infty$,

$$a(x) = \frac{2}{\pi} \log |x| + k_0 + o(|x|^{-3/2}) \tag{2.13}$$

where $k_0 := (2\varsigma + 3 \ln 2)/\pi$ and ς is Euler's constant.

The error above will suffice for our purposes, even though stronger results are known. The asymptotic expansion of $a(x)$ given in [21] shows that the best-possible error is $O(|x|^{-2})$. See also [50, 51, 52].

2.4 Consequences of the Koebe theorems

From the Koebe one-quarter theorem (Theorem 2.1.3), and the Koebe growth and distortion theorem (Theorem 2.1.4), a number of consequences may be deduced.

Corollary 2.4.1. *For each $0 < r < 1$ there is a constant c_r such that if $f \in \mathcal{S}$ and $|z| \le r$, then $|f(z) - z| \le c_r |z|^2$.*

Proof. If we combine the first estimate in Theorem 2.1.4 with the estimate of $|f'(z)|$ in the third statement of that theorem, then we can obtain a uniform bound on $|f''(z)|$ over all $f \in \mathcal{S}$ and $|z| \le r$. □

Recall that $f_A \in \mathcal{T}(\tilde{A}, \mathbb{D})$ is the unique map with $f_A(0) = 0$, $f_A'(0) > 0$.

Corollary 2.4.2. *If $A \in \mathcal{A}^n$, then $-\log f_A'(0) = \log n + O(1)$.*

Proof. By definition, since $A \in \mathcal{A}^n$, we have that $n \le \operatorname{inrad}(\tilde{A}) \le 2n$. If we let $F_A := f_A^{-1}$, then $F_A : \mathbb{D} \to \tilde{A}$ and $F_A(0) = 0$. Thus, we can apply the Koebe one-quarter theorem (Theorem 2.1.3) to conclude that $n \le |F_A'(0)|/4 \le 2n$, or equivalently, $4n \le |[f_A^{-1}]'(0)| \le 8n$. As $|[f_A^{-1}]'(0)| = 1/|f_A'(0)|$ taking logs yields $\log 4 \le -\log |f_A'(0)| - \log n \le \log 8$ so that $-\log |f_A'(0)| = \log n + O(1)$. To conclude, note $f_A'(0) > 0$. □

Along with Corollary 2.4.1, the growth and distortion theorem yields the following.

Corollary 2.4.3. *If $A \in \mathcal{A}^n$ and $|x| \le n/16$, then $f_A(x) = x f_A'(0) + |x|^2 O(n^{-2})$.*

Proof. For $z \in \mathbb{D}$, let $F_A(z) := f_A(nz)/(nf'_A(0))$. Then $F_A \in \mathcal{S}$, so Corollary 2.4.1 with $r = 1/16$ gives $|F_A(z) - z| \leq C|z|^2$. Thus, if $z = x/n$, $|f_A(x) - xf'_A(0)| \leq Cf'_A(0)|x|^2 n^{-1}$. By the previous corollary, $f'_A(0) = O(n^{-1})$, so the result follows. □

In particular, we have an estimate for $g_A(x)$.

Corollary 2.4.4. *If $A \in \mathcal{A}^n$ and $|x| \leq n/16$, then*

$$g_A(x) + \log|x| = -\log f'_A(0) + |x|O(n^{-1}). \tag{2.14}$$

Proof. From the proof of Corollary 2.4.3 we have $|f_A(x)| = |x|f'_A(0)[1 + |x|O(n^{-1})]$. Noting that $|f_A(x)| = \exp\{-g_A(x)\}$, taking logarithms, and a simple estimate yield the result. □

We remark that this corollary implies $\lim_{|x|\to 0}(g_A(x)+\log|x|) = -\log f'_A(0)$ which gives an explicit expression for the error term in (2.3).

2.5 Beurling estimates and related results

Throughout this section, suppose that $A \in \mathcal{A}^n$ with associated "union of squares" domain $\tilde{A} \subset \mathbb{C}$. Suppose that B_t is a Brownian motion in $\mathbb{C}$, and $T_A = T_{\tilde{A}} := \inf\{t : B_t \notin \tilde{A}\}$. From the Beurling projection theorem [5, Theorem (V.4.1)] the following theorem may be derived, whose proof can be found in [32].

Theorem 2.5.1. *There is a constant $c < \infty$ such that if $\gamma : [0,1] \to \mathbb{C}$ is a curve with $\gamma(0) = 0$, $|\gamma(1)| = 1$, $\gamma(0,1) \subset \mathbb{D}$, and $x \in \mathbb{D}$, then*

$$\mathbb{P}^x\{B[0,T_{\mathbb{D}}] \cap \gamma[0,1] = \emptyset\} \leq c\,|x|^{1/2}. \tag{2.15}$$

Several useful corollaries may then be concluded. While both the above theorem and the following corollaries may be termed "Beurling estimates," it is Corollary 2.5.3 that will be the most important for our purposes.

Corollary 2.5.2. *There is a constant $c < \infty$ such that if $\gamma : [a, b] \to \mathbb{C}$ is a curve with $|\gamma(a)| = r$, $|\gamma(b)| = R$, $0 < r < R < \infty$, $\gamma(a, b) \subset \mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$, and $|x| \leq r$, then*

$$\mathbb{P}^x\{B[0, T_{\mathbb{D}_R}] \cap \gamma[a, b] = \emptyset\} \leq c\,(r/R)^{1/2}. \tag{2.16}$$

Proof. By Brownian scaling we can assume $R = 1$. Then (2.16) follows almost immediately from (2.15). □

Corollary 2.5.3 (Beurling Estimate). *There is a constant $c < \infty$ such that if $x \in \tilde{A}$, then for all $r > 0$,*

$$\mathbb{P}^x\{|B_{T_A} - x| > r \ \mathrm{dist}(x, \partial\tilde{A})\} \leq c\, r^{-1/2}. \tag{2.17}$$

Proof. Without loss of generality, we may assume by Brownian scaling that $x = 0$ and $\mathrm{inrad}(\tilde{A}) =: d \in [1/2, 1]$. If $\mathrm{rad}(\tilde{A}) \leq r$, then this estimate is trivial. If not, then there is a curve in $\partial\tilde{A}$ from the circle of radius d to the circle of radius r, and (2.17) follows from the Beurling estimate (2.16). □

In particular, if $|x| > n/2$, the probability starting at x of reaching $(n/2)\mathbb{D}$ before leaving $\tilde{A}$ is bounded above by $cn^{-1/2}\,\mathrm{dist}(x, \partial\tilde{A})^{1/2}$. From the Koebe one-quarter theorem (Theorem 2.1.3) it easily follows that $g_A(x) \leq c$ for $|x| \geq n/4$; hence we get

$$g_A(x) \leq c\, n^{-1/2}\ \mathrm{dist}(x, \partial\tilde{A})^{1/2}, \quad A \in \mathcal{A}^n, \quad |x| \geq n/4. \tag{2.18}$$

Recall from (2.8) that $f_A(x) = \exp\{-g_A(x)+i\theta_A(x)\}$ for $x \in \tilde{A}$. Hence, if $x \in \partial_i A$, then $g_A(x) \le cn^{-1/2}$, so that $f_A(x) = \exp\{i\theta_A(x)\} + O(n^{-1/2})$. If $z \in \partial A$, then since $f_A(z)$ is not defined, we let $\theta_A(z)$ be the average of $\theta_A(x)$ over all $x \in A$ (for which $f_A(x)$ *is* defined) with $|x - z| = 1$. The Beurling estimate and a simple Harnack principle show that

$$\theta_A(z) = \theta_A(x) + O(n^{-1/2}), \quad (x, z) \in \partial_e A. \tag{2.19}$$

There are analogous Beurling-type results in the discrete case. Suppose that S_n, $n = 0, 1, \ldots$, is a simple random walk on $\mathbb{Z}^2$, and $\tau_A := \min\{j \ge 0 : S_j \notin A\}$. The discrete Beurling projection theorem below is proved in [28, Theorem 2.5.2].

Theorem 2.5.4 (Discrete Beurling Projection Theorem). *If $A \in \mathcal{A}$ has radius R, then $\lim_{|x|\to\infty} \mathbb{P}^x\{S_{\tau_{A^c}} = 0\} \le CR^{-1/2}$.*

One consequence of this result is the discrete Beurling estimate; see also [33].

Corollary 2.5.5 (Discrete Beurling Estimate). *There is a constant $c < \infty$ such that if $r > 0$, then $\mathbb{P}^x\{|S_{\tau_A} - x| > r\ \mathrm{dist}(x, \partial A)\} \le c\, r^{-1/2}$.*

In particular, if $|x| > n/2$, the probability starting at x of reaching $(n/2)\mathbb{D}$ before leaving A is bounded above by $cn^{-1/2}\,\mathrm{dist}(x, \partial A)^{1/2}$. It is easy to show that $G_A(x) \le c$ for $|x| \ge n/4$; hence in this case we get

$$G_A(x) \le c\, n^{-1/2}\ \mathrm{dist}(x, \partial A)^{1/2}, \quad A \in \mathcal{A}^n, \quad |x| \ge n/4. \tag{2.20}$$

Specifically, if $x \in \partial_i A$, then $G_A(x) \le cn^{-1/2}$.

If $A \in \mathcal{A}$ and $0 \ne x \in \partial_i A$, then since $G_A(0) = G_{A\setminus\{x\}}(0) + \mathbb{P}\{\tau_A > \tau_{A\setminus\{x\}}\}G_A(x)$

it follows that

$$G_A(0) = G_{A\setminus\{x\}}(0) + \frac{G_A(x)^2}{G_A(x,x)}.$$

We can replace $A\setminus\{x\}$ in the above formula with the connected component of $A\setminus\{x\}$ containing the origin. In particular, if $A \in \mathcal{A}^n$ and $x \in \partial_i A$, then we conclude that $G_A(0) - G_{A\setminus\{x\}}(0) \le G_A(x)^2 \le c\, n^{-1}$.

2.6 The Poisson kernel

The results of this section are all standard, and may be found in a variety of sources, including [5] and [32]. They are stated for reference.

2.6.1 Definition and basic properties

Suppose that B_t is a standard two-dimensional Brownian motion, $D \subset \mathbb{C}$ is a bounded domain, and $T_D := \inf\{t > 0 : B_t \notin D\}$. Suppose further that for all $y \in \partial D$, $\mathbb{P}^y\{T_D = 0\} = 1$; in other words, assume that all points of ∂D are *regular for* D^c. Let Δ denote the usual Laplacian in $\mathbb{C}$. If $F : \partial D \to \mathbb{R}$ is a function which is continuous on ∂D, then it is very well-known that there exists a unique solution to the Dirichlet problem for D: find $u : \overline{D} \to \mathbb{R}$ such that $\Delta u(x) = 0$ for $x \in D$, $u(y) = F(y)$ for $y \in \partial D$, and u is continuous on $\overline{D}$. In fact, this solution is given by

$$u(x) = \mathbb{E}^x[F(B_{T_D})] = \int_{\partial D} F(y)\, \mathbb{P}^x\{B_{T_D} \in \mathrm{d}y\}.$$

We call $\mathbb{P}^x\{B_{T_D} \in \mathrm{d}y\}$ *harmonic measure in* D *from* x, and we denote its density with respect to arc length by $H_D(x, y)$, the *Poisson kernel*. (For details on harmonic measure, consult [1].)

Remark. In the language of measure theory, harmonic measure is absolutely continuous with respect to arc length so by the Radon-Nikodým theorem [58], the Poisson kernel is simply the appropriate density. That is, if we write $|\mathrm{d}y|$ for arc length measure on ∂D, then

$$\mathbb{P}^x\{B_{T_D} \in \mathrm{d}y\} = H_D(x,y)\,|\mathrm{d}y|. \tag{2.21}$$

In the differential equations literature (say [17], for example) the Poisson kernel is the normal derivative of the Green's function. That is, if $D \in \mathcal{D}^*$, $x \in D$, $y \in \partial D$ with ∂D locally analytic at y, then

$$H_D(x,y) = \frac{\partial}{\partial \mathbf{n}_y} g_D(x,y) \tag{2.22}$$

where $\mathbf{n}_y$ is the (inward pointing) unit normal vector to D at y.

However, we will primarily be concerned with $D \in \mathcal{D}^*$, and for such a domain every point of ∂D is regular for D^c. Hence, if $x \in D$ and $y \in \partial D$, then both harmonic measure $\mathbb{P}^x\{B_{T_D} \in \mathrm{d}y\}$, and the Poisson kernel $H_D(x,y)$ are well-defined.

Example (Poisson kernel for $\mathbb{D}_r$). Let $\mathbb{D}_r := \{x \in \mathbb{C} : |x| < r\}$ denote the open disk of radius $r > 0$ in $\mathbb{C}$. For $x \in \mathbb{D}_r$ and $y \in \partial\mathbb{D}_r$ we have

$$H_{\mathbb{D}_r}(x,y) = \frac{1}{2\pi r}\,\frac{r^2 - |x|^2}{|y-x|^2}. \tag{2.23}$$

Proposition 2.6.1. *Consider $H_{\mathbb{D}}$, the Poisson kernel for the unit disk.*

- *For fixed $y \in \partial\mathbb{D}$, $H_{\mathbb{D}}(\cdot, y)$ is harmonic in $\mathbb{D}$.*
- *If $x_n \in \mathbb{D}$ and $x_n \to y \in \partial\mathbb{D}$, then $H_{\mathbb{D}}(x_n, y) \to \infty$. However, if $x_n \to y' \in \partial\mathbb{D}$ with $y' \neq y$, then $H_{\mathbb{D}}(x_n, y) \to 0$.*

- $H_{\mathbb{D}}(0, y) = \frac{1}{2\pi}$, *and for all* $x \in \mathbb{D}$, $\int_0^{2\pi} H_{\mathbb{D}}(x, e^{i\theta})\, \mathrm{d}\theta = 1$.

- *Harnack's inequality: If* $|x| \le r < 1$, $y \in \partial\mathbb{D}$, *then*

$$\frac{1-r}{1+r} = \frac{1-r^2}{(1+r)^2} \le 2\pi\, H_{\mathbb{D}}(x, y) \le \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r}.$$

- *If* $x = Re^{i\theta}$ *and* $y = e^{i\theta'}$ *with* $0 \le R < 1$, *then*

$$H_{\mathbb{D}}(x, y) = \frac{1}{2\pi} \frac{1-R^2}{1-2R\cos(\theta-\theta')+R^2}.$$

Note that if Γ is an arc in $\partial\mathbb{D}$, and we consider the Dirichlet problem with boundary function $F(y) = \mathbb{1}_{\{y\in\Gamma\}}$, then

$$H_{\mathbb{D}}(x, \Gamma) := \mathbb{P}^x\{B_{T_{\mathbb{D}}} \in \Gamma\} = \frac{1}{2\pi}\int_\Gamma \frac{1-|x|^2}{|y-x|^2}\,|\mathrm{d}y| \tag{2.24}$$

where $|\mathrm{d}y|$ is arc length measure on $\partial\mathbb{D}$; i.e., if $y = e^{i\theta}$, then $|\mathrm{d}y| = \mathrm{d}\theta$.

We conclude with the following well-known, and easily proved, fact that the Poisson kernel $H_D(z, \cdot)$ is continuous in z. (See [26] for the proof written down.)

Proposition 2.6.2. *If* $D \in \mathcal{D}^*$, $z \in D$, *and* $y \in \partial D$ *with* ∂D *locally analytic at* y, *and if* $z_n \to z$ *with* $z_n \in D$ *for each* n, *then* $H_D(z_n, y) \to H_D(z, y)$ *as* $n \to \infty$.

2.6.2 Behaviour under conformal transformation

The Riemann mapping theorem allows us to describe the behaviour of the Poisson kernel under a conformal transformation. For details, see [5, Chapter V] or [32].

Proposition 2.6.3. *If* D, $D' \in \mathcal{D}^*$; $x \in D$; $y \in \partial D$; ∂D *is locally analytic at* y; $f : D \to D'$ *is a conformal transformation of* D *onto* D'; *and* $\partial D'$ *is locally analytic* *at* $f(y)$, *then* $\mathbb{P}^x\{B_{T_D} \in \mathrm{d}y\} = \mathbb{P}^{f(x)}\{{B'}_{T_{D'}} \in f(\mathrm{d}y)\}$ *where* B' *is a (time-change of) Brownian motion. Equivalently,*

$$H_D(x,y) = |f'(y)|\, H_{D'}(f(x), f(y)). \tag{2.25}$$

In fact, this time-change can be given explicitly. If

$$A_s = A_{s,f,\omega} := \int_0^s |f'(B_r)|^2 \,\mathrm{d}r \quad \text{and} \quad \sigma_t = \sigma_{t,f,\omega} := \inf\{s : A_s \geq t\},$$

then $B'_t := f(B_{\sigma_t})$ is a Brownian motion.

2.7 The excursion Poisson kernel

2.7.1 Definition and basic properties

In the present section, we introduce the *excursion Poisson kernel* which behaves in a manner similar to the Poisson kernel, and will turn out to give the mass of the *boundary-to-boundary excursion measure*. We restrict to $D \in \mathcal{D}$ because we need local analyticity to take normal derivatives and to define $f'(x)$, $x \in \partial D$; in Section 4.7 we discuss extensions to $D \in \mathcal{D}^*$. For $D \in \mathcal{D}$, however, the excursion Poisson kernel is defined in terms of the usual Poisson kernel, with the appropriate scaling, viz.

Definition 2.7.1. Suppose that $D \in \mathcal{D}$ and x, $y \in \partial D$, $y \neq x$, with ∂D locally analytic at both x and y. Let $\mathbf{n}_x$ be the (inward pointing) unit normal vector to D at x, and define the *excursion Poisson kernel* $H_{\partial D}(x,y)$ to be

$$H_{\partial D}(x,y) := \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} H_D(x + \varepsilon \mathbf{n}_x, y).$$

It is a simple consequence of harmonicity that the excursion Poisson kernel exists for $D \in \mathcal{D}$. Indeed, the usual Poisson kernel is harmonic in D so that it is the real part of an analytic function. Since D is assumed to have piecewise analytic boundary, by analytic continuation, this function has an analytic extension to the boundary (see [2, Theorem 4, page 235], for example). Hence, the excursion Poisson kernel is exactly the normal derivative of the (extended) Poisson kernel; see also Theorem 2.10.3. That is, if $D \in \mathcal{D}$, and x, $y \in \partial D$ with ∂D locally analytic at both x and y, then

$$H_{\partial D}(x,y) = \frac{\partial}{\partial \mathbf{n}_x} H_D(x,y). \tag{2.26}$$

In the case of $\mathbb{D}$, explicit calculations are possible, as the following propositions show.

Proposition 2.7.2. *If x, $y \in \partial\mathbb{D}$, $y \neq x$, then*

$$H_{\partial\mathbb{D}}(x,y) = \frac{1}{\pi}\frac{1}{|y-x|^2}. \tag{2.27}$$

Proof. By definition, $H_{\partial\mathbb{D}}(x,y) := \lim_{\varepsilon\to 0+} \varepsilon^{-1} H_{\mathbb{D}}(x+\varepsilon\mathbf{n}_x, y)$. For any $x \in \partial\mathbb{D}$, we have $x+\varepsilon\mathbf{n}_x = (1-\varepsilon)x$. Since $|x| = 1$, we conclude that

$$H_{\partial\mathbb{D}}(x,y) = \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon} H_{\mathbb{D}}((1-\varepsilon)x,y) = \lim_{\varepsilon\to 0+}\frac{1}{2\pi\varepsilon}\frac{1-(1-\varepsilon)^2|x|^2}{|y-(1-\varepsilon)x|^2} = \frac{1}{\pi}\frac{1}{|y-x|^2}. \qquad \square$$

Corollary 2.7.3. *If we write $x = e^{i\theta}$ and $y = e^{i\theta'}$ with $\theta \neq \theta'$, then*

$$H_{\partial\mathbb{D}}(e^{i\theta}, e^{i\theta'}) = H_{\partial\mathbb{D}}(e^{i\theta'}, e^{i\theta}) = H_{\partial\mathbb{D}}(1, e^{i(\theta'-\theta)}) = \frac{1}{2\pi}\frac{1}{1-\cos(\theta'-\theta)}.$$

Proof. Since $|y-x|^2 = |e^{i\theta'} - e^{i\theta}|^2 = 2 - 2\cos(\theta'-\theta)$ by the law of cosines and $\cos(\theta'-\theta) = \cos(\theta-\theta')$, we have $H_{\partial\mathbb{D}}(e^{i\theta}, e^{i\theta'}) = H_{\partial\mathbb{D}}(e^{i\theta'}, e^{i\theta}) = H_{\partial\mathbb{D}}(1, e^{i(\theta'-\theta)})$. $\square$

Proposition 2.7.4. *Suppose that* $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_1 + 2\pi$. *Then,*

$$\int_{\theta_1}^{\theta_2} \frac{1}{1-\cos(\theta-\theta')}\,\mathrm{d}\theta = \frac{\cos(\theta_1-\theta')+1}{\sin(\theta_1-\theta')} - \frac{\cos(\theta_2-\theta')+1}{\sin(\theta_2-\theta')}$$

and

$$\int_{\theta_3}^{\theta_4}\int_{\theta_1}^{\theta_2} \frac{1}{1-\cos(\theta-\theta')}\,\mathrm{d}\theta\,\mathrm{d}\theta' = \log\left(\frac{[1-\cos(\theta_4-\theta_2)][1-\cos(\theta_3-\theta_1)]}{[1-\cos(\theta_3-\theta_2)][1-\cos(\theta_4-\theta_1)]}\right).$$

In particular, if $0 < |\theta| < \pi$, *then* $(1-\cos(\theta))^{-1} = 2\theta^{-2}[1+O(\theta^2)]$, *so that*

$$\begin{aligned}&\int_{\theta_3}^{\theta_4}\int_{\theta_1}^{\theta_2} H_{\partial\mathbb{D}}(e^{i\theta}, e^{i\theta'})\,\mathrm{d}\theta\,\mathrm{d}\theta' \\ &= \frac{(\theta_4-\theta_3)(\theta_2-\theta_1)}{\pi(\theta_3-\theta_2)(\theta_4-\theta_1)} + O\left(\frac{(\theta_4-\theta_3)^2(\theta_2-\theta_1)^2}{(\theta_3-\theta_2)^2(\theta_4-\theta_1)^2}\right) + O((\theta_4-\theta_3)(\theta_2-\theta_1)).\end{aligned}$$

Proposition 2.7.5. *If* Γ *is an arc on the unit circle, and* $x \in \partial\mathbb{D}$, $x \notin \Gamma$, *then*

$$\lim_{\varepsilon\to0+} \frac{1}{\varepsilon}\mathbb{P}^{x+\varepsilon\mathbf{n}_x}\{B_{T_\mathbb{D}} \in \Gamma\} = \frac{1}{\pi}\int_\Gamma \frac{1}{|x-y|^2}\,|\mathrm{d}y|.$$

Proof. Define $p(u) := \mathbb{P}^u\{B_{T_\mathbb{D}} \in \Gamma\}$ so that,

$$\frac{1}{\varepsilon}p(x+\varepsilon\mathbf{n}_x) = \frac{1}{2\pi}\int_\Gamma \frac{2-\varepsilon}{|(1-\varepsilon)x-y|^2}\,|\mathrm{d}y|.$$

If we let $h_\varepsilon(x,y) := (2-\varepsilon)|(1-\varepsilon)x-y|^{-2}$, then $\lim_{\varepsilon\to0+} h_\varepsilon(x,y) = 2|x-y|^{-2}$. Now $\int_\Gamma |x-y|^{-2}\,|\mathrm{d}y| < \infty$ since $|x-y| \neq 0$, so we may apply the dominated convergence theorem and conclude that

$$\lim_{\varepsilon\to0+}\int_\Gamma h_\varepsilon(x,y)\,|\mathrm{d}y| = \int_\Gamma \lim_{\varepsilon\to0+} h_\varepsilon(x,y)\,|\mathrm{d}y|. \qquad \square$$

Remark. Compare this proposition with (2.24). This result holds more generally, and shows that the excursion Poisson kernel is aptly named. Suppose that $D \in \mathcal{D}$, and $\Gamma \subset \partial D$ is an arc. If $x \in D$, then

$$H_D(x, \Gamma) := \mathbb{P}^x\{B_{T_D} \in \Gamma\} = \int_\Gamma H_D(x, y)\,|\mathrm{d}y|. \tag{2.28}$$

However, if $x \in \partial D$, $x \notin \Gamma$, then

$$H_{\partial D}(x, \Gamma) := \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \mathbb{P}^{x+\varepsilon \mathbf{n}_x}\{B_{T_D} \in \Gamma\} = \int_\Gamma H_{\partial D}(x, y)\,|\mathrm{d}y|.$$

We conclude with the easily proved boundary analogue of Proposition 2.6.2. (Consult [26] to see the details of the proof written down.)

Proposition 2.7.6. *Let $D \in \mathcal{D}$, and let x, $y \in \partial D$ with ∂D locally analytic at both x and y. Write $\partial D = \Gamma \cup \Gamma'$ where Γ is an analytic open boundary arc with $x \in \Gamma$ and $y \in \Gamma'$. If $x_n \to x$ with $x_n \in \Gamma$ for each n, then $H_{\partial D}(x_n, y) \to H_{\partial D}(x, y)$ as $n \to \infty$.*

2.7.2 Behaviour under conformal transformation

The most important property of the excursion Poisson kernel, and another reason it is so-named, is the conformal covariance property. We prove the following as a consequence of Proposition 2.6.3. It is implicit throughout that for x, $y \in \partial D$, we have $x \neq y$.

Proposition 2.7.7. *If D, $D' \in \mathcal{D}$; x, $y \in \partial D$; ∂D is locally analytic at x and y; $f : D \to D'$ is a conformal transformation of D onto D'; and $\partial D'$ is locally analytic at $f(x)$ and $f(y)$, then $H_{\partial D}(x, y) = |f'(x)|\,|f'(y)|\,H_{\partial D'}(f(x), f(y))$.*

Proof. By definition, $H_{\partial D}(x,y) := \lim_{\varepsilon\to 0+} \varepsilon^{-1} H_D(x+\varepsilon \mathbf{n}_x, y)$. Therefore,

$$\begin{aligned} H_{\partial D}(x,y) &= \lim_{\varepsilon\to 0+} \varepsilon^{-1} |f'(y)| \, H_{D'}(f(x+\varepsilon \mathbf{n}_x), f(y)) \quad \text{(Proposition 2.6.3)} \\ &= |f'(y)| \lim_{\varepsilon\to 0+} \varepsilon^{-1} H_{D'}(f(x)+\varepsilon f'(x)\mathbf{n}_x + o(\varepsilon), f(y)) \quad \text{(Taylor's theorem)} \\ &= |f'(x)|\,|f'(y)| \lim_{\varepsilon\to 0+} (\varepsilon|f'(x)|)^{-1} H_{D'}(f(x)+\varepsilon |f'(x)| \mathbf{n}_{f(x)} + o(\varepsilon), f(y)) \\ &= |f'(x)|\,|f'(y)| \lim_{\varepsilon_1\to 0+} \varepsilon_1^{-1} H_{D'}(f(x)+\varepsilon_1 \mathbf{n}_{f(x)} + o(\varepsilon_1), f(y)) \\ &= |f'(x)|\,|f'(y)| \, H_{\partial D'}(f(x), f(y)) \end{aligned}$$

where we have written $\varepsilon_1 := \varepsilon|f'(x)|$, and have noted that $\mathbf{n}_{f(x)} = f'(x)\mathbf{n}_x/|f'(x)|$ and $|\mathbf{n}_{f(x)}| = |\mathbf{n}_x| = 1$. □

Recall that if $f_D \in \mathcal{T}(D,\mathbb{D})$ with $f_D(0) = 0$, $f_D'(0) > 0$, then we can write $f_D(x) = \exp\{-g_D(x) + i\theta_D(x)\}$ as in (2.6).

Proposition 2.7.8. *If $D \in \mathcal{D}$ and x, $y \in \partial D$ with ∂D locally analytic at x and y, then*

$$H_{\partial D}(x,y) = \frac{2\pi\, H_D(0,x)\, H_D(0,y)}{1-\cos(\theta_D(x)-\theta_D(y))}. \tag{2.29}$$

Proof. Note that $4\pi^2 H_{\mathbb{D}}(0,e^{i\theta_D(x)})H_{\mathbb{D}}(0,e^{i\theta_D(y)}) = 1$, so that by Corollary 2.7.3,

$$2\pi H_{\partial\mathbb{D}}(e^{i\theta_D(x)}, e^{i\theta_D(y)}) = \frac{1}{1-\cos(\theta_D(x)-\theta_D(y))} = \frac{4\pi^2 H_{\mathbb{D}}(0,e^{i\theta_D(x)})H_{\mathbb{D}}(0,e^{i\theta_D(y)})}{1-\cos(\theta_D(x)-\theta_D(y))}.$$

But from the conformal covariance of the excursion Poisson kernel (Proposition 2.7.7), it follows that $H_{\partial D}(x,y) = |f_D'(x)|\,|f_D'(y)| H_{\partial\mathbb{D}}(e^{i\theta_D(x)}, e^{i\theta_D(y)})$, and by the conformal invariance of the usual Poisson kernel (Proposition 2.6.3),

$$H_D(0,x)H_D(0,y) = |f_D'(x)|\,|f_D'(y)| H_{\mathbb{D}}(0,e^{i\theta_D(x)})H_{\mathbb{D}}(0,e^{i\theta_D(y)}).$$

□

Proposition 2.7.9 (Conformal Invariance of Excursion Poisson Kernel). *Suppose that $D \in \mathcal{D}$, and let Γ, $\Upsilon \subset \partial D$ be analytic open boundary arcs with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. Let $D' \in \mathcal{D}$, and suppose that $f \in \mathcal{T}(D, D')$ with $f(0) = 0$, $f'(0) > 0$. Write Γ', Υ' for the images under f of Γ, Υ, respectively. If*

$$H_{\partial D}(\Gamma, \Upsilon) := \int_{\Upsilon} \int_{\Gamma} H_{\partial D}(x, y) \, |\mathrm{d}x| \, |\mathrm{d}y|, \tag{2.30}$$

then $H_{\partial D}(\Gamma, \Upsilon) = H_{\partial D'}(\Gamma', \Upsilon')$.

Proof. From the definition of $H_{\partial D}(\Gamma, \Upsilon)$ and Proposition 2.7.7, we conclude by changing variables that

$$\begin{aligned} H_{\partial D}(\Gamma, \Upsilon) = \int_{\Upsilon} \int_{\Gamma} H_{\partial D}(x, y) \, |\mathrm{d}x| \, |\mathrm{d}y| &= \int_{\Upsilon} \int_{\Gamma} |f'(x)||f'(y)| H_{\partial D'}(f(x), f(y)) \, |\mathrm{d}x| \, |\mathrm{d}y| \\ &= \int_{\Upsilon'} \int_{\Gamma'} H_{\partial D'}(x', y') \, |\mathrm{d}x'| \, |\mathrm{d}y'| \\ &= H_{\partial D'}(\Gamma', \Upsilon'). \end{aligned}$$

□

Remark. In Section 4.7, we extend the definition of $H_{\partial D}(\Gamma, \Upsilon)$ to $D \in \mathcal{D}^*$. To prove Fomin's conjecture in Chapter 6, we define the hitting matrix of excursion Poisson kernels, and establish as a straightforward extension of Proposition 2.7.7 that the determinant of the hitting matrix is conformally covariant. We then define, similar to (2.30), an integrated determinant which is conformally invariant.

2.8 Almost uniform random variables

Let $R > 2$, and let B_t be a Brownian motion started at $B_0 = x$, $|x| = R$. If $T := \inf\{t > 0 : |B_t| = 1\}$, then for R large, the density of B_T is "almost uniform."

Proposition 2.8.1. *The probability density function $p = p_{B_T,R}$ of the random variable B_T is*

$$p(y) = \frac{1}{2\pi} + O(\frac{1}{R}), \qquad y \in \partial\mathbb{D}.$$

Proof. Consider the domain $D = \{z \in \mathbb{C} : |z| > 1\}$. The map $z \mapsto z^{-1}$ is a conformal transformation from D onto $\mathbb{D}$ sending the unit circle to the unit circle, and the circle of radius R to the circle of radius $1/R$. Since the Poisson kernel is conformally covariant (Proposition 2.6.3), we have for $|y| = 1$, $|x| = R$,

$$p(y) = |y|^{-2} H_{\mathbb{D}}(x^{-1}, y^{-1}) = \frac{1}{2\pi} \frac{R^2 - 1}{|x - y|^2}.$$

However, by Harnack's inequality (Proposition 2.6.1), we have

$$\frac{R-1}{R+1} \leq 2\pi H_{\mathbb{D}}(x^{-1}, y^{-1}) \leq \frac{R+1}{R-1}$$

from which it is easily deduced that $2\pi p(y) - 1 = O(R^{-1})$. □

2.9 Conformal mappings of $\mathbb{D}$ onto $\mathbb{D}$

It is well-known that every conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$ is of the form

$$f_\alpha(z) = e^{i\theta} \frac{z - \alpha}{\overline{\alpha} z - 1} \tag{2.31}$$

for some $\theta \in [0, 2\pi)$ and $|\alpha| < 1$. By the Carathéodory extension theorem (Theorem 2.1.2), f_α can be extended to a homeomorphism of the closed unit disk onto the closed unit disk. For the remainder of this section suppose that $f := f_\alpha$ is the map (2.31) and that x, $y \in \partial\mathbb{D}$.

Proposition 2.9.1. *If f is as in (2.31) and $y \in \partial\mathbb{D}$, then $|f'(y)| = 2\pi\, H_{\mathbb{D}}(\alpha, y)$.*

Proof. If $f(z) = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1}$, then $f'(z) = e^{i\theta}\frac{|\alpha|^2-1}{(\overline{\alpha}z-1)^2}$, so that

$$|f'(y)| = \frac{1-|\alpha|^2}{|\overline{\alpha}y-1|^2} = \frac{1-|\overline{\alpha}|^2}{|\overline{\alpha}-1/y|^2} = 2\pi\, H_{\mathbb{D}}(\overline{\alpha}, 1/y),$$

since $|y| = 1$. Note that if $y \in \partial\mathbb{D}$, then $1/y = \overline{y}$. It is easily seen that $H_{\mathbb{D}}(\overline{\alpha}, \overline{y}) = H_{\mathbb{D}}(\alpha, y)$, and the result follows immediately. □

This proposition can be proved directly from Proposition 2.6.3 by choosing $z = \alpha$ and noting that $2\pi\, H_{\mathbb{D}}(0, f(y)) = 1$. In particular, $0 < |f'(y)| < \infty$ for $y \in \partial\mathbb{D}$.

Proposition 2.9.2. *If x, $y \in \partial\mathbb{D}$, $x \neq y$, and $f \in \mathcal{T}(\mathbb{D}, \mathbb{D})$ with $f(x) = x$ and $f(y) = y$, then $|f'(x)|\,|f'(y)| = 1$.*

Proof. If $f(x) = x$ and $f(y) = y$, then we immediately obtain from Proposition 2.7.7 that $H_{\partial\mathbb{D}}(x, y) = |f'(x)|\,|f'(y)|\, H_{\partial\mathbb{D}}(f(x), f(y)) = |f'(x)|\,|f'(y)|\, H_{\partial\mathbb{D}}(x, y)$. Hence, we conclude that $|f'(x)|\,|f'(y)| = 1$. □

Example. Suppose that $f(z) = \frac{z-\alpha}{1-\alpha z}$ for $\alpha \in (-1, 1)$. Then $f \in \mathcal{T}(\mathbb{D}, \mathbb{D})$ and both 1 and -1 are fixed points of f. Since $f'(z) = \frac{1-\alpha^2}{(1-\alpha z)^2}$ we see that

$$f'(1)\, f'(-1) = \frac{1-\alpha^2}{(1-\alpha)^2} \cdot \frac{1-\alpha^2}{(1+\alpha)^2} = \frac{1+\alpha}{1-\alpha} \cdot \frac{1-\alpha}{1+\alpha} = 1.$$

In fact, for x, $y \in \partial\mathbb{D}$, $x \neq y$, suppose that $h \in \mathcal{T}(\mathbb{D}, \mathbb{D})$ with $h(x) = -1$, $h(y) = 1$. Setting $F = h^{-1} \circ f \circ h$ so that $F \in \mathcal{T}(\mathbb{D}, \mathbb{D})$ with $F(x) = x$ and $F(y) = y$ gives

$$F'(x) = (h^{-1} \circ f \circ h)'(x) = \frac{1}{h'\left((h^{-1} \circ f \circ h)(x)\right)}\, f'(h(x))\, h'(x) = f'(-1)$$

and similarly $F'(y) = f'(1)$. This yields a stronger version of Proposition 2.9.2.

Proposition 2.9.3. *If $x,\ y \in \partial\mathbb{D}$, $x \neq y$, and $f \in \mathcal{T}(\mathbb{D},\mathbb{D})$ with $f(x) = x$ and $f(y) = y$, then $f'(x)\, f'(y) = 1$.*

2.10 Relationship between $H_{\partial D}$ and g_D for $D \in \mathcal{D}$

In this section we derive a formula relating the excursion Poisson kernel and the Green's function for Brownian motion. To do so, it is necessary to first observe a relationship between the usual Poisson kernel and the Green's function, proving it initially in the unit disk $\mathbb{D}$ and then extending it to general $D \in \mathcal{D}$.

Lemma 2.10.1. *If $z \in \mathbb{D}$ and $y \in \partial\mathbb{D}$, then*

$$\lim_{\varepsilon \to 0+} \frac{g_{\mathbb{D}}(z,(1-\varepsilon)y)}{2\pi\,\varepsilon\, H_{\mathbb{D}}(z,y)} = 1. \tag{2.32}$$

Proof. We treat the cases $z = 0$ and $z \neq 0$ separately. If $z = 0$, then $2\pi H_{\mathbb{D}}(0,y) = 1$ from (2.23), and $g_{\mathbb{D}}(0,(1-\varepsilon)y) = -\log|(1-\varepsilon)y| = -\log(1-\varepsilon) - \log|y| = -\log(1-\varepsilon)$ from (2.2). Since

$$\lim_{\varepsilon \to 0+} \frac{-\log(1-\varepsilon)}{\varepsilon} = 1,$$

(2.32) holds for $z = 0$. Suppose instead that $z \neq 0$. Without loss of generality we may assume that $y = 1$. It follows from (2.23) that $1 - |z|^2 = 2\pi|z-1|^2 H_{\mathbb{D}}(z,1)$ and from (2.2) that

$$g_{\mathbb{D}}(z, 1-\varepsilon) = \log\left|\frac{z(1-\varepsilon)-1}{z-(1-\varepsilon)}\right|.$$

However, (2.32) also holds when $z \neq 0$ since

$$\frac{z+1}{z-(1-\varepsilon)} + \frac{\overline{z}+1}{\overline{z}-(1-\varepsilon)} = \frac{2|z|^2 - 2(1-\varepsilon) + z\varepsilon + \overline{z}\varepsilon}{|z-(1-\varepsilon)|^2} \sim \frac{2|z|^2-2}{|z-1|^2},$$

so that as $\varepsilon \to 0+$,

$$\log\left|\frac{z(1-\varepsilon)-1}{z-(1-\varepsilon)}\right| \sim -\frac{\varepsilon}{2}\left(\frac{z+1}{z-(1-\varepsilon)}+\frac{\overline{z}+1}{\overline{z}-(1-\varepsilon)}\right)+\frac{\varepsilon^2|z+1|^2}{2|z-(1-\varepsilon)|^2}$$
$$\sim -\frac{\varepsilon}{2}\cdot\frac{2|z|^2-2}{|z-1|^2} = 2\pi\,\varepsilon\, H_{\mathbb{D}}(z,1). \qquad \square$$

The next lemma easily follows from Lemma 2.10.1 using the conformal invariance of the Green's function and the conformal covariance of the Poisson kernel.

Lemma 2.10.2. *If $D \in \mathcal{D}$, $z \in D$, and $y \in \partial D$ with ∂D locally analytic at y so that $\mathbf{n}_y$, the inward pointing normal to D at y, exists, then*

$$H_D(z,y) = \lim_{\varepsilon\to 0+}\frac{g_D(z, y+\varepsilon\mathbf{n}_y)}{2\pi\,\varepsilon}.$$

Proof. Recall from (2.5) that if $D \in \mathcal{D}$ and $f \in \mathcal{T}(D,\mathbb{D})$ with $f(0)=0$, $f'(0)>0$, then $g_D(z,w) = g_{\mathbb{D}}(f(z),f(w))$ for z, $w \in D$. Also recall from Proposition 2.6.3 that $H_D(z,y) = |f'(y)|\, H_{\mathbb{D}}(f(z),f(y))$ for $z\in D$, $y \in \partial D$. Furthermore, by Taylor's theorem, $f(y+\varepsilon\mathbf{n}_y) = f(y) + \varepsilon|f'(y)|\mathbf{n}_{f(y)} + o(\varepsilon) = (1-\varepsilon|f'(y)|)f(y) + o(\varepsilon)$ since $\mathbf{n}_{f(y)} = -f(y)$ for $f(y)\in\partial\mathbb{D}$. Consequently, we have

$$\begin{aligned}\lim_{\varepsilon\to 0+}\frac{g_D(z,y+\varepsilon\mathbf{n}_y)}{2\pi\,\varepsilon\, H_D(z,y)} &= \lim_{\varepsilon\to 0+}\frac{g_{\mathbb{D}}(f(z), f(y+\varepsilon\mathbf{n}_y))}{2\pi\,\varepsilon\,|f'(y)|\, H_{\mathbb{D}}(f(z),f(y))}\\ &= \lim_{\varepsilon\to 0+}\frac{g_{\mathbb{D}}(f(z),(1-\varepsilon|f'(y)|)f(y)+o(\varepsilon))}{2\pi\,\varepsilon\,|f'(y)|\, H_{\mathbb{D}}(f(z),f(y))}\\ &= \lim_{\varepsilon_1\to 0+}\frac{g_{\mathbb{D}}(f(z),(1-\varepsilon_1)f(y))}{2\pi\,\varepsilon_1\, H_{\mathbb{D}}(f(z),f(y))} \quad \text{where } \varepsilon_1 := \varepsilon|f'(y)|\\ &= 1,\end{aligned}$$

by Lemma 2.10.1 since $f(z)\in\mathbb{D}$ and $f(y)\in\partial\mathbb{D}$. $\square$

The theorem is now an immediate application of Lemma 2.10.2 to the definition of the excursion Poisson kernel (Definition 2.7.1).

Theorem 2.10.3. *If $D \in \mathcal{D}$, and x, $y \in \partial D$ with ∂D locally analytic at both x and y, then*

$$H_{\partial D}(x,y) = \lim_{\varepsilon \to 0+} \frac{g_D(x+\varepsilon \mathbf{n}_x, y+\varepsilon \mathbf{n}_y)}{2\pi\, \varepsilon^2}.$$

Remark. In other words, this result is the boundary-to-boundary analogue of (2.22), namely if $D \in \mathcal{D}$ and x, $y \in \partial D$ with ∂D locally analytic at both x and y, then

$$H_{\partial D}(x,y) = \frac{\partial}{\partial \mathbf{n}_x} \frac{\partial}{\partial \mathbf{n}_y} g_D(x,y).$$

Compare this with (2.26).

2.11 The discrete excursion Poisson kernel

While the Poisson kernel $H_D(x,y)$ can be thought of as the "probability that a Brownian motion starting at x hits ∂D at y," this is not entirely precise since complex Brownian motion does not hit points. Indeed, as in Section 2.6, the Poisson kernel $H_D(x,y)$ is the density with respect to arc length of $\mathbb{P}^x\{B_{T_D} \in \mathrm{d}y\}$, harmonic measure in D from x. However, the discrete analogue has an exact interpretation as the probability that a simple random walk S_n on $\mathbb{Z}^2$ starting at $x \in A$ will hit ∂A at y since there is a positive probability that $S_{\tau_A} = y$.

Definition 2.11.1. Suppose that $A \in \mathcal{A}^n$, and let $\tau_A := \min\{j > 0 : S_j \in \partial A\}$. For $x \in A$ and $y \in \partial A$, we define the *discrete Poisson kernel*, written $h_A(x,y)$, as

$$h_A(x,y) := \mathbb{P}^x\{S_{\tau_A} = y\}.$$

Remark. The object defined above has many names in the literature; among them are *hitting probability measure* [49], *harmonic measure in A from x*, *hitting probability* [28], or *hitting distribution* [20].

We define the discrete analogue of the excursion Poisson kernel to be the probability that a random walk starting at $x \in \partial A$ takes its first step into A, and then exits A at y.

Definition 2.11.2. Suppose that $A \in \mathcal{A}^n$, and let $\tau_A := \min\{j > 0 : S_j \in \partial A\}$. For $x, y \in \partial A$, define the *discrete excursion Poisson kernel*, written $h_{\partial A}(x, y)$, as

$$h_{\partial A}(x, y) := \mathbb{P}^x\{S_{\tau_A} = y, S_1 \in A\}.$$

We now state two simple *last-exit decompositions*—one for h_A and one for $h_{\partial A}$.

Proposition 2.11.3. *If $A \in \mathcal{A}^n$, $x \in A$, $y \in \partial A$, then*

$$h_A(x, y) = \frac{1}{4} \sum_{(z,y) \in \partial_e A} G_A(x, z). \tag{2.33}$$

where G_A is the Green's function for simple random walk on A as in (2.11).

Proof. From the definitions of the discrete Poisson kernel and the Green's function,

$$\begin{aligned}
\mathbb{P}^x\{S_{\tau_A} = y\} = \sum_{k=1}^{\infty} \mathbb{P}^x\{S_k = y, \tau_A = k\} &= \sum_{z \in V} \sum_{k=1}^{\infty} \mathbb{P}^x\{S_{\tau_A} = y, S_{k-1} = z, \tau_A = k\} \\
&= \sum_{z \in V} \sum_{k=1}^{\infty} \frac{1}{4}\, \mathbb{P}^x\{S_{k-1} = z, \tau_A \geq k\} \\
&= \frac{1}{4} \sum_{z \in V} G_A(x, z),
\end{aligned}$$

where we have written $V := \{z \in A : (z, y) \in \partial_e A\}$. □

Note that Proposition 2.11.3 holds in slightly more generality, and may be found in [49, P10.1(c)]. It is also noted in [49] that (2.33) is the correct discrete analogue of (2.22). The following result is the "boundary-to-boundary" version, and is therefore the correct discrete analogue of (2.26).

Proposition 2.11.4. *If* $A \in \mathcal{A}^n$, $x \in \partial A$, $y \in \partial A$, *then*

$$h_{\partial A}(x,y) = \frac{1}{4} \sum_{(z,x)\in\partial_e A} h_A(z,y).$$

Proof. Using the definition of the discrete excursion Poisson kernel and the strong Markov property, we conclude that

$$\begin{aligned} h_{\partial A}(x,y) = \sum_{z\in V} \mathbb{P}^x\{S_{\tau_A} = y, S_1 = z\} &= \sum_{z\in V} \mathbb{P}^x\{S_{\tau_A} = y | S_1 = z\}\, \mathbb{P}^x\{S_1 = z\} \\ &= \frac{1}{4} \sum_{z\in V} \mathbb{P}^z\{S_{\tau_A} = y\} \\ &= \frac{1}{4} \sum_{z\in V} h_A(z,y), \end{aligned}$$

where again we have written $V := \{z \in A : (z,y) \in \partial_e A\}$. □

Chapter 3

Green's Function Estimates

In this chapter, we lay the foundation for our proof that discrete excursion measure converges to Brownian excursion measure by deriving several useful Green's function estimates. We begin, in the first section, by reviewing some facts about coupling two stochastic processes on the same probability space; see [22] or [41]. We then use the important theorem of Komlós, Major, and Tusnády [24, 25] to construct a Brownian motion and a simple random walk on the same probability space and to derive a strong approximation result. In Section 3.2, we establish some estimates relating the Green's function for Brownian motion to the Green's function for simple random walk in certain domains; Proposition 3.2.3 is the culmination of these efforts. While the Green's function estimates of Section 3.2 hold in general, they will be most useful for points away from the boundary. In Sections 3.3 and 3.4, we establish better estimates for the case of points near the boundary, provided they are not too close to each other. These results are then summarized in Theorem 3.5.1 and Theorem 3.5.4 of the final section. Throughout this chapter, suppose that $A \in \mathcal{A}^n$ with associated "union of squares" domain $\tilde{A} \in \mathcal{D}$. As in Section 2.1, let $f_A \in \mathcal{T}(\tilde{A}, \mathbb{D})$ with $f_A(0) = 0$, $f'_A(0) > 0$, and recall from (2.8) that $f_A(x) = \exp\{-g_A(x) + i\theta_A(x)\}$, where g_A is the Green's function for Brownian motion in $\tilde{A}$.

3.1 Strong approximation

Definition 3.1.1. Suppose that $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ are two probability spaces. A *coupling* of the probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ is a probability measure $\mathbb{P}$ defined on the product measurable space $(\Omega, \mathcal{F}) := (\Omega_1 \times \Omega_2, \sigma(\mathcal{F}_1 \times \mathcal{F}_2))$ whose marginal probabilities are $\mathbb{P}_1$ and $\mathbb{P}_2$.

Example (Independence coupling). For $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$, let $\mathbb{P}$ be given by $\mathbb{P}(A_1 \times A_2) := \mathbb{P}_1(A_1) \cdot \mathbb{P}_2(A_2)$, and set $\mathbb{P}(A) = 0$ for $A \in \sigma(\mathcal{F}_1 \times \mathcal{F}_2) \setminus \mathcal{F}_1 \times \mathcal{F}_2$ so that $\mathbb{P}$ is obviously a coupling of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. If $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ and $\pi_i(\omega_1, \omega_2) = \omega_i$, then the marginal probabilities of $\mathbb{P}$ are $\mathbb{P} \circ \pi_1^{-1} = \mathbb{P}_1$ and $\mathbb{P} \circ \pi_2^{-1} = \mathbb{P}_2$.

A *coupling* of two stochastic processes X and Y is a random process (X', Y') on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the components of (X', Y') have the laws of X and Y, respectively. The usefulness of coupling two stochastic processes is that it may be possible to produce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that yields more information about the two stochastic processes than just that provided by the marginals of $\mathbb{P}$. The landmark theorem of Komlós, Major, and Tusnády [24, 25] is an example of a successful, and useful, coupling of Brownian motion and simple random walk. While they initially proved this theorem only for one-dimensional processes, it has been extended in a variety of different directions by a number of researchers. (See [53] and [60], for example.) Since we are concerned exclusively with complex Brownian motion and simple random walk on $\mathbb{Z}^2$, the results noted in [4] suffice.

Theorem 3.1.2 (Komlós-Major-Tusnády). *There exist constants λ, C, K, and there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which can be defined a two-dimensional Brownian motion B and a two-dimensional simple random walk S with $B_0 = S_0$,* *such that for all $x > 0$ and each $n \in \mathbb{N}$,*

$$\mathbb{P}\{\max_{1\le k\le n}\left|\frac{1}{\sqrt{2}}B_k - S_k\right| > C\log n + x\} < Ke^{-\lambda x}.$$

The one-dimensional proof may be found in [25], and the immediate extension to two dimensions is written down in [4, Lemma 3]. As noted in [25], λ can be chosen as large as desired by choosing C large enough; consequently $|B_n - S_{2n}| = O(\log n)$ a.s. The following easy corollary to the previous theorem is found in [4, Lemma 4].

Corollary 3.1.3. *There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which can be defined a two-dimensional Brownian motion B and a two-dimensional simple random walk S with $B_0 = S_0$ such that for each $n \in \mathbb{N}$, and for any $\lambda > 0$,*

$$\mathbb{P}\{\max_{0\le k\le n}\sup_{|t-k|\le\frac{1}{2}}|B_t - S_k| > C\log n\} = O(n^{-\lambda}),$$

provided $C = C(\lambda)$ is large enough.

For our purposes, we will need to consider the maximum up to a *random* time, not just a fixed time. The following strong approximation will suffice. Our choice of n^8 is arbitrary, and will turn out to be good enough.

Corollary 3.1.4 (Strong Approximation). *There is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which can be defined a two-dimensional Brownian motion B and a two-dimensional simple random walk S with $B_0 = S_0$, and a constant C such that*

$$\mathbb{P}\{\max_{0\le k\le \sigma_n}\sup_{|t-k|\le\frac{1}{2}}|B_t - S_k| > C\log n\} = O(n^{-10}),$$

where $\sigma_n^1 := \inf\{t : |S_t - S_0| \ge n^8\}$, $\sigma_n^2 := \inf\{t : |B_t - B_0| \ge n^8\}$, and $\sigma_n := \sigma_n^1 \vee \sigma_n^2$.

Proof. Suppose that $I_n := \{\sigma_n \leq n^{36}\}$. Then by the reflection principle for random walk [28], the central limit theorem, and the reflection principle for Brownian motion, it follows that $\mathbb{P}(I_n^c) = O(n^{-10})$. Suppose that $\lambda = 10/36$, and let $C' = C'(\lambda)$ be the required constant from Corollary 3.1.3. Therefore,

$$\mathbb{P}\{\max_{0\leq k\leq \sigma_n} \sup_{|t-k|\leq \frac{1}{2}} |B_t - S_k| > C\log n\}$$

$$= \mathbb{P}\{\max_{0\leq k\leq \sigma_n} \sup_{|t-k|\leq \frac{1}{2}} |B_t - S_k| > C\log n; I_n\} + \mathbb{P}\{\max_{0\leq k\leq \sigma_n} \sup_{|t-k|\leq \frac{1}{2}} |B_t - S_k| > C\log n; I_n^c\}$$

$$\leq \mathbb{P}\{\max_{0\leq k\leq n^{36}} \sup_{|t-k|\leq \frac{1}{2}} |B_t - S_k| > C'\log n\} + \mathbb{P}(I_n^c)$$

$$= O(n^{-36\lambda}) + O(n^{-10}),$$

by Corollary 3.1.3, and the proof is complete. □

Proposition 3.1.5. *There exists a constant c such that for every n, a Brownian motion B and a simple random walk S can be defined on the same probability space so that if $A \in \mathcal{A}^n$, $1 < r \leq n^{20}$, and $x \in A$ with $|x| \leq n^3$, then*

$$\mathbb{P}^x\{|B_{T_A} - S_{\tau_A}| \geq cr\log n\} \leq cr^{-1/2}.$$

Proof. For any given n, let B and S be defined as in Corollary 3.1.4 above, and let C be the constant in that corollary. Define $T'_A := \inf\{t \geq 0 : \mathrm{dist}(B_t, \partial\tilde{A}) \leq 2C\log n\}$, $\tau'_A := \inf\{t \geq 0 : \mathrm{dist}(S_t, \partial A) \leq 2C\log n\}$, and consider the events

$$V_1 := \{\max_{0\leq k\leq \sigma_n} \sup_{|t-k|\leq \frac{1}{2}} |B_t - S_k| > C\log n\},\;\; V_2 := \{\sup_{T'_A\leq t\leq T_A} |B_t - B_{T'_A}| \geq r\log n\},$$

$$\text{and } V_3 := \{\sup_{\tau'_A\leq t\leq \tau_A} |S_t - S_{\tau'_A}| \geq r\log n\}.$$

By the Beurling estimates (Corollaries 2.5.3 and 2.5.5), and the strong Markov property, it follows that $\mathbb{P}(V_2 \cup V_3) = O(r^{-1/2})$. From Corollary 3.1.4, $\mathbb{P}(V_1) = O(n^{-10}) = O(r^{-1/2})$. Therefore, $\mathbb{P}(V_1 \cup V_2 \cup V_3) = O(r^{-1/2})$, so that the proof is complete. Observe that $|B_{T_A} - S_{\tau_A}| \le (r + 2C)\log n$ on the complement of $V_1 \cup V_2 \cup V_3$. □

If we combine the strong approximation with Theorem 3.1.2 then we can easily deduce the following estimate.

Proposition 3.1.6. *There exists a decreasing sequence* $\delta_n \downarrow 0$ *such that if* $A \in \mathcal{A}^n$ *with associated "union of squares" domain* $\tilde{A} \in \mathcal{D}$, *and* Γ, $\Upsilon \subset \partial A$ *with* $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$ *and associated boundary arcs* $\tilde{\Gamma}$, $\tilde{\Upsilon} \subset \partial \tilde{A}$, *then* $h_A(0, \Gamma) = H_{\tilde{A}}(0, \tilde{\Gamma}) + O(\delta_n)$ *and* $h_A(0, \Upsilon) = H_{\tilde{A}}(0, \tilde{\Upsilon}) + O(\delta_n)$. *Consequently,*

$$h_A(0, \Gamma)\, h_A(0, \Upsilon) = H_{\tilde{A}}(0, \tilde{\Gamma})\, H_{\tilde{A}}(0, \tilde{\Upsilon}) + O(\delta_n)$$

where the error term depends on both Γ, Υ.

Proof. If c is the constant in Proposition 3.1.5, and V is the set $V := \{x \in \partial A : \mathrm{dist}(x, \Gamma) \le c n^{1/8} \log n\}$, then $h_A(0, \Gamma) = H_{\tilde{A}}(0, \tilde{V}) + O(n^{-1/16})$. However, a simple gambler's ruin estimate for Brownian motion shows that $H_{\tilde{A}}(0, \tilde{V}) = H_{\tilde{A}}(0, \tilde{\Gamma}) + O(n^{-7/8} \log n)$, so the result follows with $\delta_n = n^{-7/8} \log n$. □

3.2 Relationship between G_A and g_A for $A \in \mathcal{A}^n$

We now use our strong approximation result of Proposition 3.1.5 to derive a uniform estimate comparing $g_A(x, y)$ to $G_A(x, y)$ for $A \in \mathcal{A}^n$ that will be most useful if x, y are not too close to ∂A.

Proposition 3.2.1. *There exists a c such that if $A \in \mathcal{A}^n$ and $|x| \le n^2$, then*

$$|\,\mathbb{E}^x[\log|B_{T_A}|] - \mathbb{E}^x[\log|S_{\tau_A}|]\,| \le c\, n^{-1/3}\log n.$$

Proof. For any n, let B and S be as in Proposition 3.1.5, and let $V = V(n)$ be the event that $|B_{T_A} - S_{\tau_A}| \le n^{2/3}\log n$. By that proposition, $\mathbb{P}(V) \ge 1 - cn^{-1/3}$. Since $\mathrm{inrad}(A) \ge n$, we know that on the event V,

$$|\;\log|B_{T_A}| - \log|S_{\tau_A}|\;| \le c'\, n^{-1/3}\;\log n.$$

Note that $\mathbb{E}^x[\;\log|B_{T_A}|\;\mathbb{1}_{V^c}\;\mathbb{1}_{\{|B_{T_A}|\le n^5\}}\;] \le c\;\log n\;\mathbb{P}(V^c) \le c\,n^{-1/3}\;\log n$. Using the Beurling estimates it is easy to see that

$$\mathbb{E}^x[\;\log|B_{T_A}|\;\mathbb{1}_{\{|B_{T_A}|\ge n^5\}}\;] = O(n^{-1/3}\log n),$$

and similarly for $\log|S_{\tau_A}|$ in the last two estimates. Hence,

$$\mathbb{E}^x[\;(\;\log|B_{T_A}| + \log|S_{\tau_A}|\;)\;\mathbb{1}_{V^c}\;] \le c\,n^{-1/3}\;\log n.$$ □

Proposition 3.2.2. *If $A \in \mathcal{A}^n$, then*

$$G_A(x) = \frac{2}{\pi}\,g_A(x) + k_x + O(n^{-1/3}\log n), \tag{3.1}$$

where k_0 is as defined in (2.13), and $k_x := k_0 + (2/\pi)\log|x| - a(x)$ for $x \ne 0$. Note that $|k_x| \le c|x|^{-3/2}$.

Proof. Recall from (2.4) that $g_A(x) = \mathbb{E}^x[\log|B_{T_A}|] - \log|x|$ and from (2.12) that

$G_A(x) = \mathbb{E}^x[a(S_{\tau_A})] - a(x)$ with $a(x)$ as in (2.13). If $|x| \leq n^2$, then (3.1) follows from Corollary 3.2.1, and if $|x| \geq n^2$, then (3.1) follows directly from the bounds on $g_A(x)$ and $G_A(x)$ in (2.18) and (2.20), respectively. □

For any $A \in \mathcal{A}^n$, let $A^{*,n}$ be the set

$$A^{*,n} := \{x \in A : g_A(x) \geq n^{-1/16}\}. \tag{3.2}$$

The choice of 1/16 for the exponent is somewhat arbitrary, and slightly better estimates might be obtained by choosing a different exponent. However, since we do not expect the error estimate derived here to be optimal, we will just make this definition.

Proposition 3.2.3. *If $A \in \mathcal{A}^n$, and $x \in A^{*,n}$, $y \in A$, then*

$$G_A(x,y) = \frac{2}{\pi}\, g_A(x,y) + k_{y-x} + O(n^{-7/24} \log n). \tag{3.3}$$

Proof. From the Beurling estimates, (2.18), and (2.20), it follows that if $A \in \mathcal{A}^n$ and $x \in A^{*,n}$, then $\operatorname{dist}(x, \partial\tilde{A}) \geq cn^{7/8}$ and $\operatorname{dist}(x, \partial A) \geq cn^{7/8}$. That is, if we translate A to make x the origin, then the inradius of the translated set is at least $cn^{7/8}$. Hence, we can use Proposition 3.2.2 to deduce (3.3). □

3.3 An estimate for hitting the boundary

Suppose that A is any finite, connected subset of $\mathbb{Z}^2$, not necessarily simply connected, and let $V \subset \partial A$ be non-empty. Recall that $\tilde{A} \subset \mathbb{C}$ is the "union of squares" domain associated to A as in Section 2.1. For every $y \in V$, consider the collection of edges containing y, namely $\mathcal{E}_y := \{(x,y) \in \partial_e A\}$, and set $\mathcal{E}_V := \bigcup_{y \in V} \mathcal{E}_y$. If $\ell_{x,y}$ is the perpendicular line segment of length one intersecting (x, y) in the midpoint as in Section 2.1, then define $\tilde{V} := \bigcup_{(x,y)\in\mathcal{E}_V} \ell_{x,y}$ to be the associated boundary arc in $\partial\tilde{A}$. Suppose that $\tau_A := \min\{j : S_j \in \partial A\}$, $T_A = T_{\tilde{A}} := \inf\{t : B_t \in \partial\tilde{A}\}$, and throughout this section, write

$$h(x) = h_A(x, V) := \mathbb{P}^x\{S_{\tau_A} \in V\} = \sum_{y\in V} h_A(x, y), \tag{3.4}$$

and

$$H(x) = H_{\tilde{A}}(x, \tilde{V}) := \mathbb{P}^x\{B_{T_A} \in \tilde{V}\} = \int_{\tilde{V}} H_{\tilde{A}}(x, y)\,|\mathrm{d}y|, \tag{3.5}$$

where h_A and $H_{\tilde{A}}$ are the discrete Poisson kernel and the Poisson kernel, respectively.

Definition 3.3.1. If $F : \mathbb{Z}^2 \to \mathbb{R}$, let L denote the *discrete Laplacian* defined by

$$LF(x) := \frac{1}{4}\sum_{|x-y|=1}(F(y) - F(x)) = \frac{1}{4}\sum_{|e|=1}(F(x+e) - F(x)).$$

Call a function F *discrete harmonic at x* if $LF(x) = 0$. If $LF(x) = 0$ for all $x \in A \subseteq \mathbb{Z}^2$, then F is called *discrete harmonic in A*.

Remark. As in [28], it is easy to check that for any function F on $\mathbb{Z}^2$, the discrete Laplacian is related to simple random walk by

$$LF(x) = \mathbb{E}^x[F(S_1) - F(S_0)]. \tag{3.6}$$

Let Δ denote the usual Laplacian in $\mathbb{C}$ as in Section 2.6, and recall that F is *harmonic at x* if $\Delta F(x) = 0$. Note that L is a natural discrete analogue of Δ. (A relatively complete investigation of this analogy, including the solution to the discrete Dirichlet problem via random walks, may be found in [28].) If $r > 1$, $F : \mathbb{C} \to \mathbb{R}$, and

$\Delta F(x) = 0$ for all $x \in \mathbb{C}$ with $|x| < r$, then from Taylor's series and uniform bounds on the derivatives of harmonic functions [5], we conclude that

$$|LF(0)| \leq \|F\|_\infty O(r^{-3}). \tag{3.7}$$

It follows immediately from the strong Markov property that h defined by (3.4) is discrete harmonic in A. It is well-known [17] that the Poisson kernel, and therefore H defined by (3.5), is harmonic in $\tilde{A}$. Our goal in the remainder of this section is to prove the following proposition.

Proposition 3.3.2. *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if A is a finite connected subset of $\mathbb{Z}^2$, $V \subset \partial A$, and $x \in A$ with $H(x) \geq \varepsilon$, then $h(x) \geq \delta$.*

We first note that for every $n < \infty$, there is a $\delta' = \delta'(n) > 0$ such that the proposition holds for all A of cardinality at most n and all $\varepsilon > 0$. This is because h and H are strictly positive (since $V \neq \emptyset$) and the collection of connected subsets of $\mathbb{Z}^2$ containing the origin of cardinality at most n is finite. Hence, we can choose

$$\delta'(n) = \min \mathbb{P}^x\{S_{\tau_A} = y\} \tag{3.8}$$

where the minimum is over all finite connected A of cardinality at most n, all $x \in A$, and all $y \in \partial A$. We now extend this to all A for x near the boundary.

Lemma 3.3.3. *For every $\varepsilon > 0$ and for every $M < \infty$, there exists a $\delta > 0$, such that if A is a finite connected subset of $\mathbb{Z}^2$, $V \subset \partial A$, and $x \in A$ with $H(x) \geq \varepsilon$ and $\mathrm{dist}(x, \partial\tilde{A}) \leq M$, then $h(x) \geq \delta$.*

Proof. By the recurrence of planar Brownian motion and the Beurling estimate, we can find an $N = N(M, \varepsilon)$ such that $\mathbb{P}^x\{\mathrm{diam}(B[0, T_A]) \leq N\} \geq 1 - (\varepsilon/2)$ whenever

$\text{dist}(x, \partial\tilde{A}) \leq M$. Hence, if $\mathbb{P}^x\{B_{T_A} \in \tilde{V}\} \geq \varepsilon$ and $\text{dist}(x, \partial\tilde{A}) \leq M$, then

$$\mathbb{P}^x\{B_{T_A} \in \tilde{V}; \text{diam}(B[0, T_A]) \leq N\} \geq \varepsilon/2,$$

and the lemma holds with $\delta = \delta'(3N)$, say, where δ' is defined as in (3.8) above. □

For every $M < \infty$ and finite $A \subset \mathbb{Z}^2$, let

$$\sigma_M = \sigma_{M,A} := \min\{j \geq 0 : \text{dist}(S_j, \partial A) \leq M\}.$$

Since A is finite and h is a discrete harmonic function on A, it is necessarily bounded so that $h(S_{n\wedge\sigma_M})$ is a bounded martingale. It then follows from the optional sampling theorem [58] that $h(x) = \mathbb{E}^x[h(S_{\sigma_M})]$ for all $x \in A$ since σ_M is an a.s. finite stopping time. The next lemma gives a bound on the error in this equation if we replace h with H.

Lemma 3.3.4. *For every $\varepsilon > 0$, there exists an $M < \infty$ such that if A is a finite connected subset of $\mathbb{Z}^2$, $V \subset \partial A$, and $x \in A$, then $|\, H(x) - \mathbb{E}^x[H(S_{\sigma_M})]\, | \leq \varepsilon$.*

Proof. For any function F on A and any $x \in A$,

$$F(x) = \mathbb{E}^x[F(S_{\sigma_M})] - \mathbb{E}^x[\sum_{j=0}^{\sigma_M - 1} LF(S_j)]. \tag{3.9}$$

(Note that F is bounded since A is finite.) Applying (3.9) to H gives

$$|\, H(x) - \mathbb{E}^x[H(S_{\sigma_M})]\, | \leq \sum_{\text{dist}(y,\partial A)\geq M} G_A(x,y)\, |LH(y)|.$$

Since H is harmonic and bounded by 1, (3.7) implies $|LH(y)| \leq c\ \text{dist}(y, \partial A)^{-3}$. A routine estimate shows that there is a constant c such that for all $A \in \mathcal{A}$, $x \in A$, and $r \geq 1$, if a simple random walk is within distance r of the boundary, then the probability that it will hit the boundary in the next r^2 steps is bounded below by $c/\log r$. Consequently, we have

$$\sum_{r \leq \text{dist}(y, \partial A) \leq 2r} G_A(x,y) \leq c\, r^2 \log r.$$

Combining these estimates gives

$$\sum_{r \leq \text{dist}(y, \partial A) \leq 2r} G_A(x,y)\, |LH(y)| \leq c r^{-1} \log r,$$

and hence

$$\sum_{\text{dist}(y, \partial A) \geq M} G_A(x,y)\, |LH(y)| \leq c\, M^{-1}\, \log M. \tag{3.10}$$

The proof is completed by choosing M sufficiently large which will guarantee that the right side of (3.10) is smaller than ε. □

Proof of Proposition 3.3.2. Fix $\varepsilon > 0$, and suppose $H(x) \geq \varepsilon$. By the Lemma 3.3.4, we can find an $M = M(\varepsilon)$ such that

$$\mathbb{E}^x[H(S_{\sigma_M})] = \sum_{\text{dist}(y, \partial A) \leq M} J(x,y)\, H(y) \geq \varepsilon/2,$$

where $J(x,y) = J_A(x,y;M) := \mathbb{P}^x\{S_{\sigma_M} = y\}$. Hence

$$\sum_{H(y) \geq \varepsilon/4,\, \text{dist}(y, \partial A) \leq M} J(x,y) \;\geq \sum_{H(y) \geq \varepsilon/4,\, \text{dist}(y, \partial A) \leq M} J(x,y)\, H(y) \geq \varepsilon/4.$$

By Lemma 3.3.3, there is a $c = c(\varepsilon, M)$ such that $h(y) \geq c$ if $H(y) \geq \varepsilon/4$ and

$\mathrm{dist}(y, \partial A) \leq M$. Hence,

$$h(x) = \sum_{\mathrm{dist}(y,\partial A)\leq M} J(x, y)\, h(y) \geq c\,\varepsilon/4. \qquad \square$$

3.4 The main boundary estimates

Proposition 3.2.3 is a good estimate if $f_A(x)$ and $f_A(y)$ are not close to $\partial\mathbb{D}$. While this proposition is true even for points near the boundary, it is not very useful because the error terms are much larger than the value of the Green's function. Indeed, if $A \in \mathcal{A}^n$ and $x \in \partial_i A$, then $g_A(x) = O(n^{-1/2})$ and $G_A(x) = O(n^{-1/2})$, but $O(n^{-1/2}) \ll O(n^{-7/24} \log n)$, the error term in Proposition 3.2.3.

In this section we establish Proposition 3.4.1 which gives estimates for x and y close to the boundary provided that they are not too close to each other. Recall that $A^{*,n} := \{x \in A : g_A(x) \geq n^{-1/16}\}$ as in (3.2).

The following estimates can be derived easily from (2.7); see also Proposition 2.7.4. If $z = f_A(x) = (1 - r)e^{i\theta}$, $z' = f_A(y) \in \mathbb{D}$ with $|z - z'| \geq r$, then

$$g_A(x, y) = g_{\mathbb{D}}(z, z') = \frac{g_{\mathbb{D}}(z)\ (1 - |z'|^2)}{|z' - e^{i\theta}|^2}\ [1 + O(\frac{r}{|z - z'|})]. \tag{3.11}$$

Similarly, if $z' = f_A(y) = (1 - r')e^{i\theta'}$ with $r \geq r'$ and $|z - z'| \geq r$,

$$g_A(x, y) = g_{\mathbb{D}}(z, z') = \frac{g_{\mathbb{D}}(z)\ g_{\mathbb{D}}(z')}{1 - \cos(\theta - \theta')}\ [1 + O(\frac{r}{|\theta - \theta'|})]. \tag{3.12}$$

Proposition 3.4.1. *Suppose $A \in \mathcal{A}^n$. If $x \in A \setminus A^{*,n}$ and*

$$J_{x,n} := \{z \in A : |f_A(z) - \exp\{i\theta_A(x)\}| \geq n^{-1/16} \log^2 n\}, \tag{3.13}$$

then for $y \in J_{x,n}$,

$$G_A(x,y) = G_A(x)\, \frac{1-|f_A(y)|^2}{|f_A(y)-e^{i\theta_A(x)}|^2}\, [\, 1 + O(\frac{n^{-1/16}\log n}{|f_A(y)-e^{i\theta_A(x)}|})\,], \qquad y \in A^{*,n}, \quad (3.14)$$

$$G_A(x,y) = \frac{(\pi/2)\, G_A(x)\, G_A(y)}{1-\cos(\theta_A(x)-\theta_A(y))}\, [\, 1+O(\frac{n^{-1/16}\log n}{|\theta_A(y)-\theta_A(x)|})\,], \qquad y \in A \setminus A^{*,n}. \quad (3.15)$$

Thus, in view of the estimates (3.11) and (3.12) above, there is nothing surprising about the leading terms in (3.14) and (3.15). Proposition 3.4.1 essentially says that these relations are valid, at least in the dominant term, if we replace g_A with $(\pi/2)G_A$. The error terms in (3.14) and (3.15) are probably not optimal; however, this is what we obtain in our proof and it seems worthwhile to state them explicitly. The hardest part of the proof is a lemma that states that if the random walk starts at a point x with $f_A(x)$ near $\partial\mathbb{D}$, then, given that the walk does not leave A, $f_A(S_j)$ moves a little towards the center of the disk before its argument changes too much.

Lemma 3.4.2. *For* $A \in \mathcal{A}^n$, *let* $\eta = \eta(A,n) := \min\{j \geq 0 : S_j \in A^{*,n} \cup A^c\}$. *There exist constants* c, α *such that if* $A \in \mathcal{A}^n$, $x \in A \setminus A^{*,n}$, *and* $r > 0$, *then*

$$\mathbb{P}^x\{\max_{0 \leq j \leq \eta-1} |f_A(S_j) - f_A(x)| \geq r\, n^{-1/16}\} \leq c\, e^{-\alpha r}\, \mathbb{P}^x\{S_\eta \in A^{*,n}\},$$

$$\mathbb{P}^x\{|f_A(S_\eta) - f_A(x)| \geq r\, n^{-1/16} \mid S_\eta \in A^{*,n}\} \leq c\, e^{-\alpha r}.$$

In particular, there is a c_0 *such that if*

$$\xi = \xi(A,n,c_0) := \min\{j \geq 0 : |f_A(S_j) - f_A(x)| \geq c_0\, n^{-1/16}\ \log n\},$$

then

$$\mathbb{P}^x\{\xi < \eta\} \leq c_0\, n^{-5}\, \mathbb{P}^x\{S_\eta \in A^{*,n}\}.$$

In order to prove this lemma, we will need to establish several ancillary results. Therefore, we devote Section 3.4.1 which follows to the complete proof of this lemma, and then prove Proposition 3.4.1 in the separate Section 3.4.2.

3.4.1 Proof of Lemma 3.4.2

If $A \in \mathcal{A}$ and $x \in A$, let $d_A(x)$ be the distance from $f_A(x)$ to the unit circle. Note that $d_A(x) = \mathrm{dist}(f_A(x), \partial\mathbb{D}) = 1 - |f_A(x)| = 1 - \exp\{-g_A(x)\}$ in view of (2.8). As a first step in proving Lemma 3.4.2, we need the following.

Lemma 3.4.3. *There exist constants c, c', c'', ε such that if $A \in \mathcal{A}$, $x \in A$ with $d_A(x) \le c$, and $\sigma = \sigma(x, A, c, c')$ is defined by*

$$\sigma := \min\{j \ge 0 : S_j \notin A,\ d_A(S_j) \ge (1+c)d_A(x),\ \text{or } |\theta_A(S_j) - \theta_A(x)| \ge c' d_A(x)\},$$

then $\mathbb{P}^x\{S_\sigma \notin A\} \ge \varepsilon$, $\mathbb{P}^x\{S_\sigma \in A;\, d_A(S_\sigma) \ge (1+c)d_A(x)\} \ge \varepsilon$, and

$$\mathbb{P}^x\{|\theta_A(S_\sigma) - \theta_A(x)| \le c'' d_A(x) \mid S_\sigma \in A\} = 1. \tag{3.16}$$

Remark. (3.16) is not completely obvious since the random walk takes discrete steps.

Proof. We start by stating three inequalities whose verification we leave to the reader. These are simple estimates for conformal maps on domains that are squares or the unions of two squares. Recall that $\mathcal{S}_x$ is the closed square of side one centred at x whose sides are parallel to the coordinate axes. There exists a constant $c_2 \in (1, \infty)$ such that if $A \in \mathcal{A}$; x, y, $w \in A$; $x \ne y$; $|w - x| = 1$, then $d_A(z) \le c_2 d_A(x)$ for $z \in \mathcal{S}_x$; $|f_A(z) - f_A(x)| \le c_2\, |f_A(z') - f_A(x)|$ for z, $z' \in \mathcal{S}_y$; and $|f_A(w) - f_A(x)| \le c_2 d_A(x)$. The first of these inequalities implies that if $z \in \mathcal{S}_y$ and $d_A(z) \ge 3c_2 d_A(x)$, then

$d_A(y) \geq 3d_A(x)$. Fix $A \in \mathcal{A}$, $x \in A$ with $d_A(x) \leq 1/(100c_2^2)$, and let

$$J = J(x, A) := \left\{ y \in A : \mathcal{S}_y \cap \{z \in \tilde{A} : |f_A(z) - f_A(x)| < 5c_2 d_A(x)\} \neq \emptyset \right\}.$$

That is, $y \in J$ if there is a $z \in \mathcal{S}_y$ with $|f_A(z) - f_A(x)| < 5c_2 d_A(x)$. Note that J is a connected subset of A (although it is not clear whether it is *simply* connected) and $\tilde{J} \subseteq \{z \in \tilde{A} : |f_A(z) - f_A(x)| < 5c_2^2 d_A(x)\}$. In particular, $d_A(y) \leq 6c_2^2 d_A(x)$ and $|\theta_A(y) - \theta_A(x)| < c' d_A(x)$ for all $y \in J$.

There is a positive probability ρ_1 that a Brownian motion in $\mathbb{D}$ starting at $f_A(x)$ leaves $\mathbb{D}$ before leaving the disk of radius $2d_A(x)$ about $f_A(x)$. By conformal invariance, this implies that with probability at least ρ_1, a Brownian motion starting at x leaves $\tilde{J}$ at $\partial\tilde{A}$. Hence by Proposition 3.3.2, there is an ε_1 such that $\mathbb{P}^x\{S_{\tau_J} \notin A\} \geq \varepsilon_1$.

Similarly, there is a positive probability ρ_2 that a Brownian motion in the disk starting at $f_A(x)$ reaches $[1 - 6c_2^2 d_A(x)]\,\mathbb{D}$ before leaving $\mathbb{D}$ and before leaving the set $\{z : |\theta_A(z) - \theta_A(x)| \leq d_A(x)\}$. Note that $d_A(z)^2 \geq |f_A(z) - f_A(x)|^2 - d_A(x)^2$ on this set. In particular, with probability at least ρ_2, a Brownian motion starting at x leaves $\tilde{J}$ at a point z with $d_A(z) \geq 4c_2 d_A(x)$. Such a point z must be contained in an $\mathcal{S}_y$ with $d_A(y) \geq 4d_A(x)$. Hence, again using Proposition 3.3.2, there is a positive probability ε_2 that a random walk starting at x reaches a point $y \in J$ with $d_A(y) \geq 4d_A(x)$ before leaving J. In the notation of Lemma 3.4.3 choose $c := 1/(100c_2^2)$, let c' be the c' mentioned above, and let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Then we have already shown that $\mathbb{P}^x\{S_\sigma \notin A\} \geq \varepsilon$ and $\mathbb{P}^x\{S_\sigma \in A;\, d_A(S_\sigma) \geq (1+c)d_A(x)\} \geq \varepsilon$. Also, if y, $w \in A$ with $|y - w| = 1$, $d_A(y) \leq d_A(x)$, and $|\theta_A(y) - \theta_A(x)| \leq c' d_A(x)$, then

$$|f_A(w) - f_A(x)| \leq |f_A(w) - f_A(y)| + |f_A(y) - f_A(x)| \leq c''' d_A(x),$$

which implies that $|\theta_A(w) - \theta_A(x)| \leq c'' d_A(x)$ for an appropriate c''. This gives the last assertion in Lemma 3.4.3 and completes the proof. □

Corollary 3.4.4. *There exist constants c, c', α such that if $a \in (0, 1/2)$, $A \in \mathcal{A}$, and $x \in A$ with $d_A(x) < a$, then the probability that a random walk starting at x reaches the set $\{y \in A : d_A(y) \geq a\}$ without leaving the set $\{y \in A : |\theta_A(y) - \theta_A(x)| \leq c'a\}$ is at least $c(d_A(x)/a)^\alpha$.*

Proof. Let $Q(r, a, b)$ be the infimum over all $A \in \mathcal{A}$ and $x \in A$ with $d_A(x) \geq r$ of the probability that a random walk starting at x reaches the set $\{y \in A : d_A(y) \geq a\}$ without leaving the set $\{y \in A : |\theta_A(y) - \theta_A(x)| \leq b\}$. It follows from Lemma 3.4.3 that there exist $q > 0$, $\rho < 1$, and a c'' such that $Q(\rho^k, \rho^j, b) \geq q\, Q(\rho^{k-1}, \rho^j, b - c''\rho^k)$. By iterating this we get $Q(\rho^k, \rho^j, 2\ c''\ (1-\rho)^{-1}\ \rho^j) \geq q^{k-j}$. This and obvious monotonicity properties give the result. □

Remark. A similar proof gives an upper bound of $c_1(d_A(x)/a)^{\alpha_1}$.

For any $a \in (0, 1)$ and any $\theta_1 < \theta_2$, let $\eta(a, \theta_1, \theta_2)$ be the first time $t \geq 0$ that a random walk leaves the set $\{y \in A : d_A(y) \leq a, \theta_1 \leq \theta_A(y) \leq \theta_2\}$. Let

$$q(x, a, \theta_1, \theta_2) := \mathbb{P}^x\{d_A(S_{\eta(a,\theta_1,\theta_2)}) > a \mid S_{\eta(a,\theta_1,\theta_2)} \in A\},$$

and note that if $\theta_1 \leq \theta_1' \leq \theta_2' \leq \theta_2$, then $q(x, a, \theta_1', \theta_2') \leq q(x, a, \theta_1, \theta_2)$.

Proposition 3.4.5. *There exist constants c, c_1 such that if $a \in (0, 1/2)$, $A \in \mathcal{A}$, and $x \in A$, then $q(x, a, \theta_A(x) - c_1 a, \theta_A(x) + c_1 a) \geq c$.*

Proof. For every $r > 0$ and $m \in \mathbb{N}$ let $h(m, r) := \inf q(x, a, \theta_A(x) - ra, \theta_A(x) + ra)$ where the infimum is taken over all $a \in (0, 1/2)$, $A \in \mathcal{A}$, and all $x \in A$ with

$d_A(x) \geq 2^{-m}a$. The proposition is equivalent to saying that there is a c_1 such that

$$\inf_m h(m, c_1) > 0.$$

It follows from Corollary 3.4.4 that there is a c' such that $h(m, c') > 0$ for each m; more specifically, there exist c_2, β such that $h(m, r) \geq c_2 e^{-\beta m}$ for $r \geq c'$.

Suppose $x \in A$ with $2^{-(m+1)}a \leq d_A(x) < 2^{-m}a$. Start a random walk at x and stopped at t^*, defined to be the first time t when one of the following is satisfied: $S_t \notin A$, $d_A(S_t) \geq 2^{-m}a$, or $|\theta_A(S_t) - \theta_A(x)| \geq m^2 2^{-m}a$. By iterating Lemma 3.4.3, we see that the probability that the last of these three possibilities occurs is bounded above by $c'' e^{-\beta' m^2}$. Choose M sufficiently large such that for $m \geq M$, the last term is less than $c_2 e^{-2\beta(m+1)}$, and such that

$$\mathbb{P}^x\{|\theta_A(S_{t^*}) - \theta_A(x)| \leq m^3 2^{-m}a \mid S_{t^*} \in A\} = 1. \tag{3.17}$$

Note that (3.16) shows (3.17) holds for all sufficiently large m; for such m, if $r \geq c'$, then $h(m+1, r) \geq [1 - e^{-\beta m}]\, h(m, r - m^3 2^{-m})$. In particular, if

$$r > c' + \sum_{m=1}^{\infty} m^3 2^{-m},$$

then

$$h(m, r) \geq [\prod_{j=1}^{\infty}(1 - e^{-\beta j})]\, h(M, c') > 0. \qquad \square$$

Remark. This proposition gives an estimate on unconditioned simple random walk. If we consider simple random walk, conditioned to reach the origin before leaving A, a similar result holds, at least if $a \geq O(n^{-1/2} \log^3 n)$. One can see this since conditioning to hit the origin weights paths by the probability of hitting the origin which is proportional to G_A. But our Green's function estimates show that $G_A(y) \geq c\, n^{-1/2} \log^3 n$ for $d_A(y) \geq n^{-1/2} \log^3 n$ and $G_A(y) \leq c'\, n^{-1/2} \log^3 n$ for $d_A(y) \leq n^{-1/2} \log^3 n$. Hence, conditioning can only affect things by a multiplicative constant. (In fact, conditioning the random walk to reach the origin makes things better and we can prove the corresponding result for all a, but we will not need this stronger result.)

Iterating Proposition 3.4.5 gives Corollary 3.4.6, from which Lemma 3.4.2 follows.

Corollary 3.4.6. *There exist c, β such that if $a \in (0, 1/2)$, $r > 0$, $A \in \mathcal{A}$, and $x \in A$, then $q(x, a, \theta_A(x) - ra, \theta_A(x) + ra) \geq 1 - ce^{-\beta r}$.*

3.4.2 Proof of Proposition 3.4.1

Proof of Proposition 3.4.1. Suppose $A \in \mathcal{A}^n$ and $(z, y) \in \partial_e A^{*,n}$. Since g_A is harmonic in the disk of radius $O(n^{7/8})$ about z, standard estimates for positive harmonic functions give

$$|g_A(z) - g_A(y)| \leq O(n^{-7/8})\, g_A(z) \leq o(n^{-7/8}).$$

Since $g_A(y) < n^{-1/16} \leq g_A(z)$, we conclude $g_A(z) = n^{-1/16} + o(n^{-7/8})$, and similarly for $g_A(y)$. Hence,

$$G_A(z) = (2/\pi)\, n^{-1/16} + O(n^{-1/3} \log n) = (2/\pi)\, n^{-1/16}\, [1 + O(n^{-13/48} \log n)]$$

by Corollary 3.2.2, and similarly for $G_A(y)$. Therefore, for any $x \in A \setminus A^{*,n}$,

$$\begin{aligned} G_A(x) &= \mathbb{P}^x\{S_\eta \in A^{*,n}\}\, \mathbb{E}^x[G_A(S_\eta) \mid S_\eta \in A^{*,n}] \\ &= (2/\pi)\, n^{-1/16}\, \mathbb{P}^x\{S_\eta \in A^{*,n}\}\, [1 + O(n^{-13/48} \log n)]. \end{aligned} \tag{3.18}$$

In a similar fashion, note that if $x,\ y \in A^{*,n}$, then $g_A(x,y) \geq cn^{-1/8}$, and hence by Proposition 3.2.3, if $|x-y| \geq n^{1/2}$, then $G_A(x,y) = (2/\pi)\ g_A(x,y)\ [1+O(n^{-1/6}\log n)]$. If $A \in \mathcal{A}^n$, $x \in A \setminus A^{*,n}$, and $y \in A^{*,n}$, then the strong Markov property gives

$$G_A(x,y) = \sum_{z \in A^{*,n}} \mathbb{P}^x\{S_\eta = z\} G_A(z,y).$$

If $x \in A \setminus A^{*,n}$, let $R(x) = R(x,n,A) := \{z \in A : |f_A(z) - f_A(x)| \leq c_0 n^{-1/16} \log n\}$, where c_0 is the constant in Lemma 3.4.2. From that lemma we see that

$$\sum_{z \in A^{*,n} \cap R(x)} \mathbb{P}^x\{S_\eta = z\} = [1 - O(n^{-5})] \sum_{z \in A^{*,n}} \mathbb{P}^x\{S_\eta = z\}.$$

But $c\, n^{-1/8} \leq G_A(z,y) \leq c' \log n$ for $z,\ y \in A^{*,n}$; hence,

$$G_A(x,y) = [1 + o(n^{-4})] \sum_{z \in A^{*,n} \cap R(x)} \mathbb{P}^x\{S_\eta = z\}\ G_A(z,y).$$

If $|f_A(y) - f_A(x)| \geq n^{-1/16}\ \log^2 n$, and $z \in A^{*,n} \cap R(x)$, then from (3.11),

$$g_A(z,y) = \frac{n^{-1/16}\ (1 - |f_A(y)|^2)}{|f_A(y) - e^{i\theta_A(x)}|^2}\ [\ 1 + O(\frac{n^{-1/16}\ \log n}{|f_A(y) - e^{i\theta_A(x)}|})\ ].$$

Hence, using Proposition 3.2.3,

$$G_A(x,y) = \mathbb{P}^x\{S_\eta \in A^{*,n}\}\ \frac{(2/\pi)\ n^{-1/16}\ (1 - |f_A(y)|^2)}{|f_A(y) - e^{i\theta_A(x)}|^2}\ [\ 1 + O(\frac{n^{-1/16}\ \log n}{|f_A(y) - e^{i\theta_A(x)}|})\ ].$$

Combining this with (3.18) gives (3.14). If $y \in \partial_i A^{*,n}$, then we can write

$$G_A(x,y) = G_A(x) \frac{n^{-1/16}}{1 - \cos(\theta_A(x) - \theta_A(y))}\ [\ 1 + O(\frac{n^{-1/16} \log n}{|y - e^{i\theta_A(x)}|})\ ]. \tag{3.19}$$

Now suppose $y \in J_{x,n} \setminus A^{*,n}$. Then we can write

$$G_A(x,y) = G_{A \setminus A^{*,n}}(x,y) + \sum_{z \in A^{*,n}} \mathbb{P}^x\{S_\eta = z\} G_A(z,y),$$

and using (3.14) on $G_A(z,y)$ gives $G_A(x,y) \geq cn^{-1/8}\mathbb{P}^x\{S_\eta \in A^{*,n}\}\mathbb{P}^y\{S_\eta \in A^{*,n}\}$. However, provided $R(x) \cap R(y) = \emptyset$, which is true for n sufficiently large, Lemma 3.4.2 shows that $G_{A \setminus A^{*,n}}(x,y) \leq cn^{-10}\mathbb{P}^x\{S_\eta \in A^{*,n}\}\mathbb{P}^y\{S_\eta \in A^{*,n}\}$. Therefore,

$$G_A(x,y) = [1 + o(n^{-9})] \sum_{z \in A^{*,n}} \mathbb{P}^x\{S_\eta = z\} G_A(z,y).$$

and we can use (3.19) to deduce (3.15). □

3.5 Summary of results

We conclude with a summary of our results. These should probably be titled lemmas since they will be applied in later chapters to extend Fomin's identity, and to prove that discrete excursion measure converges to Brownian excursion measure. Recall that $A^{*,n} := \{x \in A : g_A(x) \geq n^{-1/16}\}$ as in (3.2), and that $J_{x,n} := \{z \in A : |f_A(z) - \exp\{i\theta_A(x)\}| \geq n^{-1/16} \log^2 n\}$ as in (3.13).

Theorem 3.5.1. *There exists a decreasing sequence $\varepsilon_n \downarrow 0$ such that if $A \in \mathcal{A}^n$, then*

$$G_A(0) = -\frac{2}{\pi} \log f_A'(0) + k_0 + O(\varepsilon_n^3), \tag{3.20}$$

where k_0 is the constant in (2.13). Moreover, if $x, y \in \partial_i A$ with $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$,

$$G_A(x,y) = \frac{(\pi/2)\, G_A(x)\, G_A(y)}{1 - \cos(\theta_A(x) - \theta_A(y))} \, [1 + O(\frac{\varepsilon_n^3}{|\theta_A(x) - \theta_A(y)|})]. \tag{3.21}$$

Proof. Let $\varepsilon_n^3 := n^{-1/16} \log^2 n$. Clearly

$$G_A(0) = 1 + \frac{1}{4} \sum_{|e|=1} G_A(e).$$

By Proposition 3.2.2, for any $|e| = 1$, $G_A(e) = (2/\pi)\, g_A(e) + k_e + O(\varepsilon_n^3)$ where $k_e = k_0 - a(e)$. However, it is easily shown that $a(e) = 1$ so that

$$G_A(0) = \frac{1}{4} \sum_{|e|=1} \frac{2}{\pi} g_A(e) + k_0 + O(\varepsilon_n^3).$$

But by Corollary 2.4.4, $g_A(e) = -\log f_A'(0) + O(n^{-1})$; hence (3.20) follows. Observe that if $x,\, y \in \partial_i A$, then $x,\, y \in A \setminus A^{*,n}$ as a consequence of the Beurling estimates. Further, if $y \in \partial_i A$, and $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$, then it follows that $y \in J_{x,n}$. We can therefore apply the second part of Proposition 3.4.1 and conclude (3.21). □

Remark. Using Corollary 2.7.3, we can restate (3.21) as

$$G_A(x,y) = \pi^2\, G_A(x)\, G_A(y)\, H_{\partial \mathbb{D}}(e^{i\theta_A(x)}, e^{i\theta_A(y)})\, [1 + O(\frac{\varepsilon_n^3}{|\theta_A(x) - \theta_A(y)|})].$$

If we now combine Theorem 3.5.1 with the decomposition of $h_A(x,y)$ from Proposition 2.11.3, then we may deduce the following.

Corollary 3.5.2. *There exists a decreasing sequence $\varepsilon_n \downarrow 0$ such that if $A \in \mathcal{A}^n$, and $x \in \partial_i A$, $y \in \partial A$ with $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$, then*

$$h_A(x,y) = \frac{(\pi/2)\, G_A(x)\, h_A(0,y)}{1 - \cos(\theta_A(x) - \theta_A(y))}\, [1 + O(\frac{\varepsilon_n^3}{|\theta_A(x) - \theta_A(y)|})].$$

The following corollary is now the result of combining Corollary 3.5.2 with the decomposition of $h_{\partial A}(x,y)$ from Proposition 2.11.4. Compare this result to Proposition 2.7.8. In a manner similar to Lemma 3.3.4, it gives us the error in (2.29) if we replace H with h.

Corollary 3.5.3. *There exists a decreasing sequence $\varepsilon_n \downarrow 0$ such that if $A \in \mathcal{A}^n$, and x, $y \in \partial A$ with $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$, then*

$$h_{\partial A}(x,y) = \frac{(\pi/2)\; h_A(0,x)\; h_A(0,y)}{1-\cos(\theta_A(x)-\theta_A(y))}\;[1+O(\frac{\varepsilon_n^3}{|\theta_A(x)-\theta_A(y)|})].$$

Our next estimate says that simple random walk starting at $\partial_i A$ conditioned to reach the origin before leaving A has a good chance to move away from the boundary quickly.

Theorem 3.5.4. *There exists a decreasing sequence $\varepsilon_n \downarrow 0$ and a constant c such that for any $A \in \mathcal{A}^n$, $x \in \partial_i A$, $a \geq \varepsilon_n$, if $\xi = \xi(n,A,a)$ denotes the first time j at which $g_A(S_j) \geq a$, then conditioned on the event that the walk reaches the origin before leaving A, the probability that*

$$\sup_{0\leq j\leq \xi} |\theta_A(S_j) - \theta_A(x)| \leq a/c$$

is at least c.

Proof. If we let $\varepsilon_n := n^{-1/2}\log^3 n$, then the result follows from Proposition 3.4.5 and the remark thereafter. □

Remark. Note that Theorem 3.5.1 holds if $\varepsilon_n^3 \geq n^{-1/16}\log^2 n$, and Theorem 3.5.4 holds if $\varepsilon_n \geq n^{-1/2}\log^3 n$. Since $n^{-1/48}\log^{2/3} n \gg n^{-1/2}\log^3 n$, all these results are valid with ε_n replaced with $n^{-1/48}\log^{2/3} n$.

Chapter 4

Excursions and Excursion Measure

In this chapter we introduce excursion measure. Although our primary concern is with excursions in a domain between boundary points, we will need to discuss measures on curves from an interior point to a boundary point. In the first section, we define an appropriate metric space of curves $(\mathcal{K}, \mathbb{d})$, and develop the associated topology. We then define an excursion precisely, and discuss several operations on curves including concatenation, truncation, reversal, and shifting. The most important, however, is the definition of a curve under a conformal transformation. In Section 4.2, we discuss measures on arbitrary metric spaces, including transforming measures under conformal mappings, and how to integrate measures. We also introduce our main tool for comparing measures, namely the Prohorov metric. In the following section, these results are specialized to our metric space $(\mathcal{K}, \mathbb{d})$. In Section 4.4 we construct the interior-to-boundary excursion measure and use it in Section 4.5 to construct the boundary-to-boundary excursion measure for $D \in \mathcal{D}$. In Sections 4.6 and 4.7 we generalize the construction of $\mu_{\partial D}$, $H_{\partial D}$, respectively, to arbitrary $D \in \mathcal{D}^*$. Finally, we conclude by defining simple random walk excursions in Section 4.8 and discrete excursion measure in Section 4.9. Recall from Section 2.1 that if D, $D' \in \mathcal{D}^*$, then $\mathcal{T}(D, D')$ denotes the set of all conformal transformations of D onto D'.

4.1 Metric spaces of curves

We will be considering a variety of measures on spaces of curves; these measures will be defined on Borel σ-algebras of metric spaces of curves. Throughout, a *curve* $\gamma : I \to \mathbb{C}$ shall always mean a continuous mapping of an interval $I \subseteq [0, \infty)$ into $\mathbb{C}$.

4.1.1 The space $\mathcal{K}$

Let $\mathcal{K}$ denote the set of curves $\gamma : [0, t_\gamma] \to \mathbb{C}$ where $0 < t_\gamma < \infty$, and write $\gamma[0, t_\gamma] := \{z \in \mathbb{C} : \gamma(t) = z \text{ for some } 0 \le t \le t_\gamma\}$ and similarly for $\gamma(0, t_\gamma)$. There are three natural metrics that we will consider on $\mathcal{K}$. Define the metric

$$d^*_{\mathcal{K}}(\gamma, \gamma') := \inf_{\varphi} \left[\sup_{0 \le s \le t_\gamma} |\, \gamma(s) - \gamma'(\varphi(s)) \,| \right] \tag{4.1}$$

where the infimum is over all increasing homeomorphisms $\varphi : [0, t_\gamma] \to [0, t_{\gamma'}]$. Call $\tilde{\gamma}$ a *reparameterization* of $\gamma \in \mathcal{K}$ with parameterization φ if $\varphi : [0, t_\gamma] \to [0, t_{\tilde{\gamma}}]$ is an increasing homeomorphism such that $\gamma(t) = \tilde{\gamma}(\varphi(t))$ for each $0 \le t \le t_\gamma$. If $\tilde{\gamma}$ is a reparameterization of γ under φ, then γ is a reparameterization of $\tilde{\gamma}$ under φ^{-1}, and we write $\gamma \overset{\text{par}}{\sim} \tilde{\gamma}$. Finally, let $\mathcal{K}^*$ be the set of equivalence classes of curves $\gamma \in \mathcal{K}$ under the relation $\overset{\text{par}}{\sim}$, so that the metric $d^*_{\mathcal{K}}$ identifies curves which are equal modulo time reparameterization. In fact, $(\mathcal{K}^*, d^*_{\mathcal{K}})$ is a complete metric space.

Remark. The metric $d^*_{\mathcal{K}}$ is used by Lawler and Werner in [39], and the proof that $(\mathcal{K}^*, d^*_{\mathcal{K}})$ is a complete metric space may be found in [3, Lemma 2.1].

In order to account for the time parameterization, however, we let

$$d_{\mathcal{K}}(\gamma, \gamma') = \inf_{\varphi} \left[d_{\mathcal{K}}(\gamma, \gamma'; \varphi) \right] := \inf_{\varphi} \left[\sup_{0 \le s \le t_\gamma} \{\, |\, \gamma(s) - \gamma'(\varphi(s)) \,| + |s - \varphi(s)| \,\} \right] \tag{4.2}$$

where again the infimum is over all increasing homeomorphisms $\varphi : [0, t_\gamma] \to [0, t_{\gamma'}]$. The metric $d_\mathcal{K}$ does not identify curves which are equal modulo time reparameterization, and, unfortunately, the metric space $(\mathcal{K}, d_\mathcal{K})$ is not complete.

As we shall see, a convenient choice of parameterization is $\varphi(s) = t_{\gamma'} s / t_\gamma$. Define

$$\mathbb{d}(\gamma, \gamma') := \sup_{0 \le s \le 1} |\, \gamma(t_\gamma s) - \gamma'(t_{\gamma'} s)\, | + |t_\gamma - t_{\gamma'}|. \tag{4.3}$$

and note that it is straightforward to verify $\mathbb{d}$ is a metric on $\mathcal{K}$.

Proposition 4.1.1. $(\mathcal{K}, \mathbb{d})$ *is a metric space.*

In Section 4.1.2 below, we show that $(\mathcal{K}, \mathbb{d})$ is not complete. Although, we have $d_\mathcal{K}(\gamma, \gamma') \le \mathbb{d}(\gamma, \gamma')$ by definition, it is easily seen that these two metrics are not equivalent; i.e., there does not exist a constant C so that $\mathbb{d}(\gamma, \gamma') \le C d_\mathcal{K}(\gamma, \gamma')$.

Lemma 4.1.2. *If* γ_1, $\gamma_2 \in \mathcal{K}$, *then* $\mathbb{d}(\gamma_1, \gamma_2) \le d_\mathcal{K}(\gamma_1, \gamma_2) + \mathrm{osc}(\gamma_2, 2 d_\mathcal{K}(\gamma_1, \gamma_2))$, *where* $\mathrm{osc}(\gamma, \delta) := \sup\{|\gamma(t) - \gamma(s)| : |t - s| \le \delta\}$ *is the modulus of continuity of* γ.

Proof. If $0 < s < 1$, and $\varphi : [0, t_{\gamma_1}] \to [0, t_{\gamma_2}]$ is any increasing homeomorphism, then with $d_\mathcal{K}(\gamma_1, \gamma_2; \varphi)$ as in (4.2), and $s_1 := t_{\gamma_1} s$, $s_2 := t_{\gamma_2} s$, since $|\gamma_1(s_1) - \gamma_2(\varphi(s_1))| + |t_{\gamma_1} - t_{\gamma_2}| \le d_\mathcal{K}(\gamma_1, \gamma_2; \varphi)$ and $|\varphi(s_1) - s_2| \le |\varphi(s_1) - s_1| + |s_1 - s_2| \le 2 d_\mathcal{K}(\gamma_1, \gamma_2; \varphi)$, we conclude that

$$\begin{aligned} |\gamma_1(s_1) - \gamma_2(s_2)| &\le |\gamma_1(s_1) - \gamma_2(\varphi(s_1))| + |\gamma_2(\varphi(s_1)) - \gamma_2(s_2)| \\ &\le d_\mathcal{K}(\gamma_1, \gamma_2; \varphi) - |t_{\gamma_1} - t_{\gamma_2}| + \mathrm{osc}(\gamma_2, 2 d_\mathcal{K}(\gamma_1, \gamma_2; \varphi)) \end{aligned}$$

Thus, $\mathbb{d}(\gamma_1, \gamma_2) \le d_\mathcal{K}(\gamma_1, \gamma_2; \varphi) + \mathrm{osc}(\gamma_2, 2 d_\mathcal{K}(\gamma_1, \gamma_2; \varphi))$ and by taking the infimum over all φ, the result follows. □

4.1.2 The Banach space $\mathcal{X}$

Since the metric spaces of curves $(\mathcal{K}, d_{\mathcal{K}})$ and $(\mathcal{K}, \mathbb{d})$ defined in the previous subsection are not complete (as will be shown shortly), we will consider a larger space $\mathcal{X}$, and identify $\mathcal{K}$ in a natural way with a subspace of $\mathcal{X}$. Since $\mathcal{X}$ will be complete, in our cases of interest we will be able to identify subspaces of $(\mathcal{K}, \mathbb{d})$ with closed subspaces of $\mathcal{X}$, and therefore these subspaces of $(\mathcal{K}, \mathbb{d})$ will inherit completeness.

Let $C[0,1]$ denote the space of continuous complex-valued functions on $[0,1]$ with metric $d_\infty(\gamma_1^*, \gamma_2^*) := \sup_{0\le r\le 1} |\gamma_1^*(r) - \gamma_2^*(r)|$. Denote the usual metric on $\mathbb{R}$ by abs so that $\text{abs}(s,t) := |s-t|$. Consider the separable Banach space $\mathcal{X} := C[0,1] \times \mathbb{R}$ with metric $d_{\mathcal{X}} := d_\infty + \text{abs}$. Thus elements of $\mathcal{X}$ are pairs (γ^*, t) where $\gamma^* \in C[0,1]$, $t \in \mathbb{R}$, and the distance between elements in $\mathcal{X}$ is

$$d_{\mathcal{X}}((\gamma_1^*, s), (\gamma_2^*, t)) = \sup_{0\le r\le 1} |\gamma_1^*(r) - \gamma_2^*(r)| + |s-t|.$$

We can embed $\mathcal{K}$ into $\mathcal{X}$ via $\iota : \mathcal{K} \hookrightarrow \mathcal{X}$, $\gamma \mapsto (\gamma^*, t_\gamma)$, where $\gamma^*(r) := \gamma(t_\gamma r)$, $0 \le r \le 1$. However, $\iota(\mathcal{K}) = \{(\gamma^*, t) \in \mathcal{X} : t > 0\} =: \mathcal{X}^+$ is not a closed subspace of $\mathcal{X}$. The metric spaces $(\mathcal{X}^+, d_{\mathcal{X}})$ and $(\mathcal{K}, d_{\mathcal{X},\mathcal{K}})$ are isomorphic, where $d_{\mathcal{X},\mathcal{K}}$ is the induced metric in $\mathcal{K}$ associated to the metric $d_{\mathcal{X}}$ in $\mathcal{X}$. That is, if γ_1, $\gamma_2 \in \mathcal{K}$, then $\iota(\gamma_1) = (\gamma_1^*, t_{\gamma_1})$, $\iota(\gamma_2) = (\gamma_2^*, t_{\gamma_2})$, so that $d_{\mathcal{X},\mathcal{K}}(\gamma_1, \gamma_2) = d_{\mathcal{X}}((\gamma_1^*, t_{\gamma_1}), (\gamma_2^*, t_{\gamma_2}))$. Since

$$d_{\mathcal{X}}((\gamma_1^*, t_{\gamma_1}), (\gamma_2^*, t_{\gamma_2})) = \sup_{0\le r\le 1} |\gamma_1(t_{\gamma_1} r) - \gamma_2(t_{\gamma_2} r)| + |t_{\gamma_1} - t_{\gamma_2}| = \mathbb{d}(\gamma_1, \gamma_2) \tag{4.4}$$

it follows that $d_{\mathcal{X},\mathcal{K}} = \mathbb{d}$ and $(\mathcal{K}, \mathbb{d}) \cong (\mathcal{X}^+, d_{\mathcal{X}})$.

Example. Suppose $\gamma \in \mathcal{K}$ is given by $\gamma(r) = r + ir$, $0 \le r \le 1$, and for $n = 1, 2, \ldots$, let $\gamma_n(r) = nr + inr$, $0 \le r \le 1/n$. Notice that $\gamma_n^* = \gamma^* = \gamma$. Thus, $\iota(\gamma_n) =$

$(\gamma_n^*, t_{\gamma_n}) = (\gamma^*, 1/n)$ so clearly $\{(\gamma_n^*, t_{\gamma_n})\}$ is a Cauchy sequence in $\mathcal{X}$, and $\{\gamma_n\}$ is a Cauchy sequence in $(\mathcal{K}, \mathbb{d})$. Since $\mathcal{X}$ is complete, it has a limit, namely $(\gamma^*, 0) \in \mathcal{X}$. However, $(\gamma^*, 0) \notin \mathcal{X}^+ = \iota(\mathcal{K})$ so that $(\gamma^*, 0)$ does not have a counterpart in $\mathcal{K}$. This shows that $(\mathcal{K}, \mathbb{d})$ is not complete, and illustrates the reason for considering $\mathcal{X}$.

However, if the limit does have a counterpart in $\mathcal{K}$ (i.e., if $(\gamma^*, t) \in \mathcal{X}^+$ so that $\iota^{-1}(\gamma^*, t) \in \mathcal{K}$), then we have the following positive result.

Theorem 4.1.3. *Let $(\gamma_n^*, t_n) \in \mathcal{X}^+$ for $n = 1, 2, \ldots$, so that $\gamma_n := \iota^{-1}(\gamma_n^*, t_n) \in \mathcal{K}$. Suppose that for some $(\gamma^*, t) \in \mathcal{X}$, $d_{\mathcal{X}}((\gamma_n^*, t_n), (\gamma^*, t)) \to 0$. If $t > 0$ so that $(\gamma^*, t) \in \mathcal{X}^+$, then $\gamma := \iota^{-1}(\gamma^*, t) \in \mathcal{K}$, and $d_{\mathcal{X}}((\gamma_n^*, t_n), (\gamma^*, t)) \to 0$ if and only if $d_{\mathcal{K}}(\gamma_n, \gamma) \to 0$ as $n \to \infty$, or equivalently, $\mathbb{d}(\gamma_n, \gamma) \to 0$ if and only if $d_{\mathcal{K}}(\gamma_n, \gamma) \to 0$ as $n \to \infty$.*

Proof. We begin with proving the second statement. Note that by the definitions of $d_{\mathcal{K}}$ and $\mathbb{d}$ in (4.2) and (4.3), respectively, $d_{\mathcal{K}}(\gamma_n, \gamma) \leq \mathbb{d}(\gamma_n, \gamma)$. For the other inequality, we have from Lemma 4.1.2 that $\mathbb{d}(\gamma_n, \gamma) \leq d_{\mathcal{K}}(\gamma_n, \gamma) + \operatorname{osc}(\gamma, 2d_{\mathcal{K}}(\gamma_n, \gamma))$. To prove the first statement, observe that by definition $d_{\mathcal{X}}((\gamma_n^*, t_{\gamma_n}), (\gamma^*, t_\gamma)) = \mathbb{d}(\gamma_n, \gamma)$, as noted in (4.4). □

We see that $d_{\mathcal{K}}$ and $\mathbb{d}$ generate the same topology on $\mathcal{K}$. Thus when we need to discuss convergence or continuity in $\mathcal{K}$, it can be respect to either metric since the previous theorem shows that these two notions are equivalent. Such calculations, therefore, can be made in whichever metric is more convenient for the given problem.

If $a > 0$, let $\mathcal{K}_a = \{\gamma \in \mathcal{K} : t_\gamma \geq a\}$, and set $\iota(\mathcal{K}_a) = \{(\gamma^*, t) \in \mathcal{X} : t \geq a\} =: \mathcal{X}_a$. Note that $\mathcal{X}_a$ *is* a closed subspace of $\mathcal{X}$ so that $(\mathcal{X}_a, d_{\mathcal{X}}) \cong (\mathcal{K}_a, \mathbb{d})$ is complete. However, $\mathcal{K}_a$ is not complete under $d_{\mathcal{K}}$. As an example, consider $\gamma_n(r) = r^n$, $0 \leq r \leq 1$, which is a Cauchy sequence in $(\mathcal{K}_1, d_{\mathcal{K}})$ that has no limit. By Lemma 4.1.2, if $\{\gamma_n\}$ is a Cauchy sequence in $(\mathcal{K}_a, d_{\mathcal{K}})$ that is equicontinuous, then it is a Cauchy sequence in $(\mathcal{K}_a, \mathbb{d})$ and therefore has a limit. In what follows, we will refer to spaces of curves which are primarily subspaces of $\mathcal{K}$. As we have seen we can consider such spaces as (isomorphic to) subspaces of $\mathcal{X}$. Unless it is necessary to explicitly mention this isomorphism, we prefer to work with $(\mathcal{K}, \mathbb{d})$ rather than $(\mathcal{X}, d_{\mathcal{X}})$.

4.1.3 The spaces $\mathcal{K}(D)$ and $\mathcal{K}_\Gamma^\Upsilon(D)$

If D is a simply connected proper subset of $\mathbb{C}$, and $\gamma \in \mathcal{K}$, then we say that γ *is in* D if $\gamma(0, t_\gamma) \subset D$. This does not require that either $\gamma(0) \in D$ or $\gamma(t_\gamma) \in D$. We define the space $\mathcal{K}(D)$ as $\mathcal{K}(D) := \{\gamma \in \mathcal{K} : \gamma \text{ is in } D\}$. For z, $w \in \overline{D}$, let $\mathcal{K}_z(D)$ be the set of $\gamma \in \mathcal{K}(D)$ with $\gamma(0) = z$, let $\mathcal{K}^w(D)$ be the set of $\gamma \in \mathcal{K}(D)$ with $\gamma(t_\gamma) = w$, and define $\mathcal{K}_z^w(D) := \mathcal{K}_z(D) \cap \mathcal{K}^w(D)$. Finally, if Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$, write $\mathcal{K}_\Gamma^\Upsilon(D) := \bigcup_{z\in\Gamma, w\in\Upsilon} \mathcal{K}_z^w(D)$.

4.1.4 Definition of excursion

Definition 4.1.4. If $\gamma \in \mathcal{K}(D)$, then we say that γ *is an excursion in* D if $\gamma(0) \in \partial D$ and $\gamma(t_\gamma) \in \partial D$.

Definition 4.1.5. Suppose that $\gamma \in \mathcal{K}(D)$. We say γ *is an excursion from* z *to* w *in* D if $\gamma(0) = z \in \partial D$ and $\gamma(t_\gamma) = w \in \partial D$. Equivalently, if $\gamma \in \mathcal{K}_z^w(D)$ with z, $w \in \partial D$, then γ is an excursion from z to w in D.

Definition 4.1.6. Suppose that $\gamma \in \mathcal{K}(D)$, and Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. We say γ *is an excursion from* Γ *to* Υ *in* D, or a (Γ, Υ)*-excursion in* D, if $\gamma(0) \in \Gamma$ and $\gamma(t_\gamma) \in \Upsilon$. Equivalently, if $\gamma \in \mathcal{K}_\Gamma^\Upsilon(D)$, then γ is a (Γ, Υ)-excursion in D.

Remark. As we will see later, excursion measure $\mu_{\partial D}(\Gamma, \Upsilon)$ is concentrated on $\mathcal{K}_\Gamma^\Upsilon(D)$.

4.1.5 Conformal image of a curve

Suppose that D and D' are simply connected domains in $\mathbb{C}$, and $f : D \to D'$ is a conformal transformation. For $\gamma \in \mathcal{K}(D)$, let

$$A_s = A_{s,f,\gamma} := \int_0^s |f'(\gamma(r))|^2 \, \mathrm{d}r \quad \text{and} \quad \sigma_t = \sigma_{t,f,\gamma} := \inf\{s : A_s \geq t\}.$$

If $\gamma \in \mathcal{K}(D)$ with $A_{t_\gamma} < \infty$, and if f extends to the endpoints of γ, then we define the image of γ under f, denoted $f \circ \gamma \in \mathcal{K}(D')$, by setting $t_{f\circ\gamma} := A_{t_\gamma}$ and $f \circ \gamma(t) := f(\gamma(\sigma_t))$ for $0 \leq t \leq A_{t_\gamma}$ (or equivalently for $0 \leq \sigma_t \leq t_\gamma$). Since $A_{\cdot,f,\gamma}$ is non-negative, continuous, and strictly increasing, it follows that $\sigma_{\cdot,f,\gamma}$ is well-defined.

Example (Brownian scaling). The following is a special case of the above. Let D be a simply connected proper subsets of $\mathbb{C}$, and for $a \in \mathbb{C} \setminus \{0\}$, let $f_a(z) = az$. If $\gamma \in \mathcal{K}(D)$, then we define the *Brownian scaling map* $\mathfrak{T}_a : \mathcal{K}(D) \to \mathcal{K}(f_a(D))$ by setting $t_{\mathfrak{T}_a\gamma} := |a|^2 t_\gamma$ and $\mathfrak{T}_a\gamma(t) := a\,\gamma\,(|a|^{-2}t)$ for $0 \leq t \leq t_{\mathfrak{T}_a\gamma}$.

In particular, if D, $D' \in \mathcal{D}$, γ is an excursion in D, and $f \in \mathcal{T}(D, D')$ so that f *does* extend to the endpoints of γ, then $f \circ \gamma =: \gamma' \in \mathcal{K}(D')$ is an excursion in D'. Note that $t_{\gamma'} = A_{t_\gamma}$ (or $\sigma_{t_{\gamma'}} = t_\gamma$) and $\gamma'(t) = f(\gamma(\sigma_t))$ for $0 \leq t \leq t_{\gamma'}$.

Example. As an application of Brownian scaling, suppose that $f(z) = (1+\varepsilon)z$ for $z \in \mathbb{D}$, $0 < \varepsilon < 1$, and let γ be an excursion from x to y in $\mathbb{D}$. Then $\gamma' := f \circ \gamma$ is an excursion from $(1+\varepsilon)x$ to $(1+\varepsilon)y$ in $(1+\varepsilon)\mathbb{D}$ given explicitly by

$$\gamma'(t) = (1+\varepsilon)\,\gamma\left((1+\varepsilon)^{-2}t\right) \tag{4.5}$$

for $0 \leq t \leq t_{\gamma'} = (1+\varepsilon)^2 t_\gamma$.

Proposition 4.1.7. *For each curve $\gamma \in \mathcal{K}(\mathbb{D})$, there exists a constant $C = C(\gamma)$, such that if $f(z) = (1+\varepsilon)z$ for $z \in \mathbb{D}$, $0 < \varepsilon < 1$, then $\mathbb{d}(\gamma, f \circ \gamma) \leq C\varepsilon$.*

Proof. Let $\gamma' := f \circ \gamma$ as defined in the example above, and note that $t_{\gamma'} = (1+\varepsilon)^2 t_\gamma$. From (4.3) and (4.5), we compute that

$$\begin{aligned}
\mathbb{d}(\gamma, \gamma') &= \sup_{0 \leq s \leq 1} |\, \gamma(t_\gamma s) - \gamma'(t_{\gamma'} s) \,| + |t_\gamma - t_{\gamma'}| \\
&= \sup_{0 \leq s \leq 1} \left|\, \gamma(t_\gamma s) - (1+\varepsilon)\, \gamma\left((1+\varepsilon)^{-2} t_{\gamma'} s\right) \,\right| + |t_\gamma - (1+\varepsilon)^2 t_\gamma| \\
&= \varepsilon \sup_{0 \leq s \leq 1} |\, \gamma(t_\gamma s) \,| + (2\varepsilon + \varepsilon^2) t_\gamma.
\end{aligned}$$

Since $t_\gamma < \infty$, and $\gamma(s) \in \overline{\mathbb{D}}$, $0 \leq s \leq t_\gamma$, we can take $C = C(\gamma) = 1 + 3t_\gamma$. □

If E is a domain containing $\overline{D}$ and f is analytic and univalent on E (that is, f is a conformal mapping of E), then it follows from the Koebe growth and distortion theorem (Theorem 2.1.4) that $|f'|$, $|f''|$, and $1/|f'|$ are uniformly bounded on D, and the function $\gamma \mapsto f \circ \gamma$ from $\mathcal{K}(D)$ to $\mathcal{K}(f(D))$ is continuous. If $D \in \mathcal{D}$, then since ∂D is piecewise analytic, $\partial D = \bigcup_{i=1}^n \Gamma_i$ for some finite union of analytic curves Γ_i. Hence, any conformal mapping f of D can be analytically continued across each Γ_i so $\gamma \mapsto f \circ \gamma : \mathcal{K}(D) \to \mathcal{K}(f(D))$ is continuous; we denote this induced map by f.

4.1.6 Reversal of curves

If $\gamma \in \mathcal{K}$, then we define the *reversal of* γ, denoted γ^r, by setting $t_{\gamma^r} = t_\gamma$, and $\gamma^r(t) := \gamma(t_\gamma - t)$, for $0 \leq t \leq t_{\gamma^r}$. It is easily seen that $\mathbb{d}(\gamma, \gamma^r) = \sup_{0 \leq s \leq t_\gamma} |\gamma(s) - \gamma(t_\gamma - s)| \leq \operatorname{diam}(\gamma[0, t_\gamma])$. Note that for every $w \in \mathbb{C}$, $\gamma \mapsto \gamma^r$ is a continuous map from $\mathcal{K}_w$ to $\mathcal{K}^w$. Furthermore, if γ, $\eta \in \mathcal{K}$, then $d_{\mathcal{K}}(\gamma, \eta) = d_{\mathcal{K}}(\gamma^r, \eta^r)$ and $\mathbb{d}(\gamma, \eta) = \mathbb{d}(\gamma^r, \eta^r)$.

4.1.7 Concatenation of curves

If $\gamma_1,\ \gamma_2 \in \mathcal{K}$ with $\gamma_1(t_{\gamma_1}) = \gamma_2(0)$, then we define the *concatenation of* γ_1 *and* γ_2, denoted $\gamma_1 \oplus \gamma_2$, by setting $t_{\gamma_1 \oplus \gamma_2} = t_{\gamma_1} + t_{\gamma_2}$, and

$$\gamma_1 \oplus \gamma_2(t) = \begin{cases} \gamma_1(t), & \text{if } 0 \le t \le t_{\gamma_1}, \\ \gamma_2(t - t_{\gamma_1}), & \text{if } t_{\gamma_1} \le t \le t_{\gamma_1 \oplus \gamma_2}. \end{cases}$$

Note that $(\gamma_1, \gamma_2) \mapsto \gamma_1 \oplus \gamma_2$ is a continuous map from $\mathcal{K}^w \times \mathcal{K}_w$ to $\mathcal{K}$ for every $w \in \mathbb{C}$.

4.1.8 Truncation operators

If $0 \le r < s \le t_\gamma$, then we define the *truncation operator* $\Theta_r^s : \mathcal{K} \to \mathcal{K}$ by setting $t_{\Theta_r^s \gamma} = s - r$ and $\Theta_r^s \gamma(t) := \gamma(r + t)$ for $0 \le t \le t_{\Theta_r^s \gamma}$. Observe that $\Theta_r^s \gamma[0, t_{\Theta_r^s \gamma}] = \gamma[r, s]$. By definition, truncation undoes concatenation. If $\gamma_1,\ \gamma_2 \in \mathcal{K}$ with $\gamma_1(t_{\gamma_1}) = \gamma_2(0)$, then $\Theta_0^{t_{\gamma_1}} \gamma_1 \oplus \gamma_2(t) = \gamma_1(t)$, $0 \le t \le t_{\gamma_1}$, and $\Theta_{t_{\gamma_1}}^{t_{\gamma_1 \oplus \gamma_2}} \gamma_1 \oplus \gamma_2(t) = \gamma_2(t)$, $0 \le t \le t_{\gamma_2}$. It is easily seen that

$$d_{\mathcal{K}}(\Theta_r^s \gamma, \gamma) \le r + (t_\gamma - s) + \operatorname{diam}(\gamma[0, r]) + \operatorname{diam}(\gamma[s, t_\gamma]).$$

Therefore, if $r_n \to 0+$ and $s_n \to t_\gamma -$, then by Theorem 4.1.3, $\mathbb{d}(\Theta_{r_n}^{s_n} \gamma, \gamma) \to 0$.

4.1.9 Shift operators

Suppose that $\gamma \in \mathcal{K}$. If $a \in \mathbb{C}$, then we define the *space shift operator* $\vartheta^a : \mathcal{K} \to \mathcal{K}$ by setting $t_{\vartheta^a \gamma} = t_\gamma$ and $\vartheta^a \gamma(t) := \gamma(t) + a$ for $0 \le t \le t_{\vartheta^a \gamma}$. If $r > -t_\gamma$, then we define the *time shift operator* $\vartheta_r : \mathcal{K} \to \mathcal{K}$ by setting $t_{\vartheta_r \gamma} = t_\gamma + r$ and $\vartheta_r \gamma(t) := \gamma(\frac{t_\gamma}{t_\gamma + r}\, t)$ for $0 \le t \le t_{\vartheta_r \gamma}$. Finally, we define the *space-time shift operator* $\vartheta_r^a := \vartheta^a \circ \vartheta_r : \mathcal{K} \to \mathcal{K}$

by setting $t_{\vartheta_r^a\gamma} = t_{\vartheta_r\gamma} = t_\gamma + r$ and for $0 \le t \le t_{\vartheta_r^a\gamma}$,

$$\vartheta_r^a\gamma(t) := \gamma\left(\frac{t_\gamma}{t_\gamma + r}\,t\right) + a$$

Note that $\mathbb{d}(\gamma, \vartheta^a\gamma) = |a|$, $\mathbb{d}(\gamma, \vartheta_r\gamma) = |r|$, and $\mathbb{d}(\gamma, \vartheta_r^a\gamma) = |a| + |r|$.

4.2 Measures on metric spaces

Throughout this section, suppose that (Ξ, ρ) is a metric space. Let $\mathcal{B}_\rho := \mathcal{B}_\rho(\Xi)$, the Borel σ-algebra associated to the topology induced by ρ, so that $(\Xi, \mathcal{B}_\rho)$ is a measurable space. A measure on (Ξ, ρ) will always be a σ-finite measure on $(\Xi, \mathcal{B}_\rho)$. If m is such a measure, we denote its *total mass* by $|m| := m(\Xi)$. If $|m| < \infty$, then m is a *finite measure*; otherwise it is an *infinite measure*. Denote the space of finite measures on $(\Xi, \mathcal{B}_\rho)$ by $\mathcal{M}(\Xi)$, and the space of probability measures on $(\Xi, \mathcal{B}_\rho)$ by $\mathcal{PM}(\Xi)$. If $m \in \mathcal{M}(\Xi)$ with $|m| > 0$, we write $m^\# := m/|m|$ for m normalized to be a probability measure; thus, $\mathcal{PM}(\Xi) := \{m/|m| : m \in \mathcal{M}(\Xi), |m| > 0\}$. Recall (see [9]) that every finite measure m on $(\Xi, \mathcal{B}_\rho)$ is *regular*; i.e., if $V \in \mathcal{B}_\rho$ and $\varepsilon > 0$, then there exist a closed set F and an open set G such that $F \subseteq V \subseteq G$ and $m(G \setminus F) < \varepsilon$. If $V \in \mathcal{B}_\rho$, then we say that m is *concentrated on* V if $m(\Xi \setminus V) = 0$. Observe that V need not be closed.

4.2.1 The Prohorov metric

Definition 4.2.1. If m_1, $m_2 \in \mathcal{M} = \mathcal{M}(\Xi)$, let $\wp : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ denote the *Prohorov metric* given by $\wp(m_1, m_2) := \inf\{\varepsilon > 0 : m_1(F) \le m_2(F^{(\varepsilon)}) + \varepsilon,\ \ m_2(F) \le m_1(F^{(\varepsilon)}) + \varepsilon\ \ \forall\ F \in \mathcal{B}_\rho\}$ where $F^{(\varepsilon)} = \{x \in \Xi : \rho(x, y) < \varepsilon,\ \text{some } y \in F\}$.

It is easily checked that $(\mathcal{M}(\Xi), \wp)$ is itself a metric space. Observe that $F^{(\varepsilon)}$ is Borel, and that symmetry follows since $((F^{(\varepsilon)})^c)^{(\varepsilon)} \subseteq F^c$. If m_1, $m_2 \in \mathcal{PM}(\Xi)$, then an equivalent definition of $\wp$ is given by $\wp(m_1, m_2) = \inf\{\varepsilon > 0 : m_1(F) \leq m_2(F^{(\varepsilon)}) + \varepsilon$ for every closed $F \in \mathcal{B}_\rho\}$. It is proved in [11, Theorem 2.4.2] that both metrics on $\mathcal{PM}(\Xi)$ are equivalent and consistent with the Prohorov topology. Also note that $|\,|m_1| - |m_2|\,| \leq \wp(m_1, m_2) \leq \max\{|m_1|, |m_2|\}$; for more details, see [9], [11], or [19]. The importance of the Prohorov metric is that when (Ξ, ρ) is complete and separable, the Prohorov metric space $(\mathcal{M}(\Xi), \wp)$ is also complete and separable. Recall that probability measures m_n *converge weakly* to m, written $m_n \Rightarrow m$ weakly, if for every bounded, continuous function $h : \Xi \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_\Xi h(x)\, m_n(\mathrm{d}x) = \int_\Xi h(x)\, m(\mathrm{d}x).$$

The following theorem is worth stating explicitly; see [19, pages 363–366].

Theorem 4.2.2. *If (Ξ, ρ) is a complete and separable metric space, then the metric space $(\mathcal{PM}(\Xi), \wp)$ is also complete and separable, where $\wp$ is the Prohorov metric as in Definition 4.2.1. Furthermore, if m_n, $m \in \mathcal{PM}(\Xi)$, then as $n \to \infty$, $\wp(m_n, m) \to 0$ if and only if $m_n \Rightarrow m$ weakly.*

Important Remark. The Portmanteau theorem [11, Theorem 2.1.1] gives several other conditions equivalent to weak convergence; hence, the Prohorov topology is known as the topology of weak convergence. ***Whenever we say a sequence of measures converges, it will be with respect to the Prohorov metric.***

As noted in [11, page 29], Strassen proved another equivalent definition of $\wp$

consistent with the Prohorov topology is given by

$$\wp(m_1, m_2) = \inf_{\mathfrak{M}} \left[\inf\{\varepsilon > 0 : \mathbb{P}(\rho(X_1, X_2) \geq \varepsilon) \leq \varepsilon\} \right],$$

where $\mathfrak{M}$ is the set of all $\Xi \times \Xi$-valued random variables (X, Y) with $\mathcal{L}(X) = m_1$ and $\mathcal{L}(Y) = m_2$ where $\mathcal{L}$ denotes law. Thus, to show $\wp(m_1, m_2) \leq \varepsilon$, it suffices to find a measure m on $\Xi \times \Xi$ whose first marginal is m_1, whose second marginal is m_2, and $m\{(x, y) : \rho(x, y) \geq \varepsilon\} \leq \varepsilon$.

Proposition 4.2.3. *Let (Ξ, ρ) be a metric space, and let X_i be (Ξ, ρ)-valued random variables with $\mathcal{L}(X_i) = m_i$, $i = 1, 2$. If $\mathbb{P}(\rho(X_1, X_2) \geq \varepsilon) \leq \varepsilon$, then $\wp(m_1, m_2) \leq \varepsilon$.*

Proof. Observe that $\wp(m_1, m_2) \leq \varepsilon$ since

$$\begin{aligned} m_1(F) = \mathbb{P}\{X_1 \in F\} &= \mathbb{P}\{X_1 \in F, \rho(X_1, X_2) < \varepsilon\} + \mathbb{P}\{X_1 \in F, \rho(X_1, X_2) \geq \varepsilon\} \\ &\leq \mathbb{P}\{X_2 \in F^{(\varepsilon)}\} + \mathbb{P}(\rho(X_1, X_2) \geq \varepsilon) \\ &\leq \mathbb{P}\{X_2 \in F^{(\varepsilon)}\} + \varepsilon. \end{aligned}$$

□

Remark. If (Ξ, ρ) is a complete and separable metric space, then to show a sequence of non-zero finite measures $m_n \in \mathcal{M}(\Xi)$ converges to $m \in \mathcal{M}(\Xi)$, it suffices to show that both $|m_n| \to |m|$ and $\wp(m_n^{\#}, m^{\#}) \to 0$, as $n \to \infty$.

Lemma 4.2.4. *Suppose that (Ξ, ρ) is a complete, separable metric space, and that m_1, $m_2 \in \mathcal{M}(\Xi)$. If $C > 0$, then $\wp(Cm_1, Cm_2) \leq (C \vee 1)\, \wp(m_1, m_2)$.*

Proof. Suppose that $\wp(m_1, m_2) = \varepsilon$. To begin, let $C > 1$. Then since $m_1(F) \leq m_2(F^{(\varepsilon)}) + \varepsilon$ for every F Borel, we have $Cm_1(F) \leq Cm_2(F^{(\varepsilon)}) + C\varepsilon$. Since $C\varepsilon > \varepsilon$, we have $F^{(\varepsilon)} \subset F^{(C\varepsilon)}$. Hence, $Cm_1(F) \leq Cm_2(F^{(C\varepsilon)}) + C\varepsilon$. Interchanging m_1 and

m_2 gives $\wp(Cm_1, Cm_2) \leq C\varepsilon$. Suppose instead that $C < 1$. Then since $m_1(F) \leq m_2(F^{(\varepsilon)}) + \varepsilon$, and $C\varepsilon < \varepsilon$, we have $m_1(F) \leq m_2(F^{(\varepsilon)}) + \varepsilon/C$. Multiplying by C gives $Cm_1(F) \leq Cm_2(F^{(\varepsilon)}) + \varepsilon$. Interchanging m_1 and m_2 yields $\wp(Cm_1, Cm_2) \leq \varepsilon$. Thus, we conclude $\wp(Cm_1, Cm_2) \leq (C \vee 1)\, \wp(m_1, m_2)$. □

4.2.2 Transformation of a measure

We begin with the following easily proved change-of-variables result. Note, however, that a rather abstract version is proved in [8, page 58].

Proposition 4.2.5. *Suppose that f is a continuous mapping of the metric space (Ξ, ρ) into the metric space (Ξ', ρ'). Then a measure m on $(\Xi, \mathcal{B}_\rho)$ determines a measure m' on $(\Xi', \mathcal{B}_{\rho'})$ such that*

$$f \circ m(V') = m'(V') = m(f^{-1}(V')) \tag{4.6}$$

for any $V' \in \mathcal{B}_{\rho'}$. Furthermore,

$$\int_{\Xi'} h(x')\, m'(\mathrm{d}x') = \int_{\Xi} h(f(x))\, m(\mathrm{d}x)$$

for any bounded, continuous function $h : \Xi' \to \mathbb{R}$.

Remark. $f \circ m \in \mathcal{M}(\Xi')$ is given explicitly by $f \circ m(V') := m(\{x \in \Xi : f(x) \in V'\})$ for any $V' \in \mathcal{B}_{\rho'}(\Xi')$.

Proposition 4.2.6. *Suppose that (Ξ, ρ) is a metric space. If $f : (\Xi, \rho) \to (\Xi, \rho)$ is an isometry and a bijection, then for every $V \subset \Xi$ and $\varepsilon > 0$, $f^{-1}(V)^{(\varepsilon)} = f^{-1}(V^{(\varepsilon)})$.*

Proof. Let $x \in f^{-1}(V)^{(\varepsilon)}$. Then there exists $y \in f^{-1}(V)$ such that $\rho(x, y) < \varepsilon$ implies $\rho(f(x), f(y)) < \varepsilon$. Thus, $f(x) \in V^{(\varepsilon)}$ so that $x \in f^{-1}(V^{(\varepsilon)})$. Conversely, suppose that $x \in f^{-1}(V^{(\varepsilon)})$. Then, $f(x) \in V^{(\varepsilon)}$. Therefore there exists $y \in V$ such that $\rho(f(x), y) < \varepsilon$ and so $\rho(x, f^{-1}(y)) < \varepsilon$. Thus, $x \in f^{-1}(V)^{(\varepsilon)}$. □

Proposition 4.2.7. *If $f : (\Xi, \rho) \to (\Xi', \rho')$ is continuous, $m \in \mathcal{M}(\Xi)$, and C is a constant, then*

$$f \circ (Cm) = C(f \circ m). \tag{4.7}$$

Proof. Since $f \circ (Cm)(V') = (Cm)(f^{-1}(V')) = C\left[m(f^{-1}(V'))\right] = C\left[f \circ m(V')\right]$ for any $V' \in \mathcal{B}_{\rho'}(\Xi')$ the result follows. □

Proposition 4.2.8. *Suppose that (Ξ, ρ) is a complete, separable metric space, and let $m \in \mathcal{M}(\Xi)$. If $f_n : (\Xi, \rho) \to (\Xi, \rho)$; $f : (\Xi, \rho) \to (\Xi, \rho)$; f_n, f, are continuous; and $f_n \to f$ uniformly, then $\wp(f_n \circ m, f \circ m) \to 0$.*

Proof. Assume first that $m \in \mathcal{PM}(\Xi)$. If $\mu_n := f_n \circ m$ and $\mu := f \circ m$, then by Theorem 4.2.2, it suffices to show that $\mu_n \Rightarrow \mu$ weakly. Suppose that $h : \Xi \to \mathbb{R}$ is a bounded, continuous function. Hence, by Proposition 4.2.5, we conclude that

$$\int_\Xi h(x)\, \mu_n(\mathrm{d}x) = \int_\Xi h \circ f_n(x)\, \mu(\mathrm{d}x) \to \int_\Xi h \circ f(x)\, \mu(\mathrm{d}x) = \int_\Xi h(x)\, m(\mathrm{d}x)$$

since $f_n \to f$ uniformly. We next consider $m \in \mathcal{M}(\Xi)$. If m is the zero measure, the result is trivial. If $|m| > 0$, then by (4.7) and Proposition 4.2.4, $\wp(f_n \circ m, f \circ m) = \wp(|m|\,(f_n \circ m^{\#}), |m|\,(f \circ m^{\#})) = (|m| \vee 1)\, \wp(f_n \circ m^{\#}, f \circ m^{\#}) \to 0$. □

Corollary 4.2.9. *Under the same assumptions as Proposition 4.2.8, if $m_2 \in \mathcal{M}(\Xi)$, and $\wp(f_n \circ m_1, f \circ m_1) \to 0$ as $n \to \infty$, then $\wp(f_n \circ m_1, m_2) \to \wp(f \circ m_1, m_2)$.*

Proof. Since $\wp(\cdot, \cdot)$ is a metric, we have by the triangle inequality $\wp(f_n \circ m_1, m_2) \leq$

$\wp(f_n \circ m_1, f \circ m_1) + \wp(f \circ m_1, m_2)$, so that

$$\limsup_{n\to\infty} \wp(f_n \circ m_1, m_2) \le \wp(f \circ m_1, m_2). \tag{4.8}$$

However, the reverse inequality tells us that $\wp(f \circ m_1, m_2) \le \wp(f \circ m_1, f_n \circ m_1) + \wp(f_n \circ m_1, m_2)$, so that

$$\liminf_{n\to\infty} \wp(f_n \circ m_1, m_2) \ge \wp(f \circ m_1, m_2). \tag{4.9}$$

By combining (4.8) and (4.9), the result follows. □

Essentially the same proof as Proposition 4.2.8 yields the next result.

Proposition 4.2.10. *Suppose that m_n, $m \in \mathcal{M}(\Xi)$. If $f : (\Xi, \rho) \to (\Xi', \rho')$ is continuous and $\wp(m_n, m) \to 0$, then $\wp(f \circ m_n, f \circ m) \to 0$; that is,*

$$f \circ m = f \circ \left(\lim_{n\to\infty} m_n\right) = \lim_{n\to\infty} f \circ m_n. \tag{4.10}$$

4.2.3 Integrating measures

We now define a measure by integration which will be a Riemann integral with respect to arc length measure. Just as in elementary calculus, it can be computed as the appropriate limit of Riemann sums. Note, however, that this is a weak limit since we are considering convergence of measures in the Prohorov topology. For integrating measures in more generality consult, for example, [48].

Suppose that (Ξ, ρ) is a metric space (not necessarily complete and separable), and $\mathcal{M}(\Xi)$ is the space of finite measures on $(\Xi, \mathcal{B}_\rho)$. Let $\Gamma \subset \mathbb{C}$ be an analytic arc that is parameterized by $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $t_\gamma < \infty$. Since Γ is analytic and

$t_\gamma < \infty$, it is necessarily rectifiable [2], so write ℓ_Γ for the length of γ. The function $\Gamma \to \mathcal{M}(\Xi)$ given by $z \mapsto \mu(z, \cdot)$ is called a *measure-valued function*. Our goal in this section is to define the measure

$$\mu(\cdot) := \int_\Gamma \mu(z, \cdot)\, |\mathrm{d}z| \tag{4.11}$$

Consider the measures $\{\mu(z, \cdot) : z \in \Gamma\} \subset \mathcal{M}(\Xi)$, and let $\{\gamma(0) = z_0, z_1, \ldots, z_n = \gamma(t_\gamma)\}$ partition Γ. Let $z_i^* \in [z_{i-1}, z_i]$, $|\Delta z_i| = |z_i - z_{i-1}|$, $i = 1, \ldots n$, and set

$$\mu_n(\cdot) := \sum_{i=1}^n \mu(z_i^*, \cdot)\, |\Delta z_i|.$$

Note that $\mu_n(\cdot) \in \mathcal{M}(\Xi)$ for each n. If $\lim_{n\to\infty} \mu_n(\cdot)$ exists in $\mathcal{M}(\Xi)$, with convergence in the Prohorov metric $\wp$, then we define the Riemann integral of the measure-valued function $z \in \Gamma \mapsto \mu(z, \cdot) \in \mathcal{M}(\Xi)$ to be this limiting value; that is,

$$\mu(\cdot) := \int_\Gamma \mu(z, \cdot)\, |\mathrm{d}z| := \lim_{n\to\infty} \mu_n(\cdot).$$

We now give some conditions which guarantee the existence of the Riemann integral.

Proposition 4.2.11. *If (Ξ, ρ) is a complete and separable metric space, and $\{\mu_n(\cdot)\}$ is a Cauchy sequence in $\mathcal{M}(\Xi)$, then $\lim \mu_n(\cdot)$ exists in $\mathcal{M}(\Xi)$.*

Proof. Since (Ξ, ρ) is complete and separable, $(\mathcal{M}(\Xi), \wp)$ is also complete and separable. Thus a Cauchy sequence in $\mathcal{M}(\Xi)$ necessarily has a limit. □

Proposition 4.2.12. *If $\{\mu_n(\cdot)\}$ is a Cauchy sequence in $\mathcal{M}(\Xi)$, and $\{\mu_n(\cdot)\}$ is relatively compact, then $\lim \mu_n(\cdot)$ exists in $\mathcal{M}(\Xi)$.*

Proof. Recall that a subset Π of $\mathcal{M}(\Xi)$ is relatively compact if every sequence of elements from Π has a weakly convergent subsequence. Thus, if $\{\mu_n(\cdot)\}$ is a Cauchy sequence and has a weakly convergent subsequence, it converges in $\mathcal{M}(\Xi)$. $\square$

Recall that $\{\mu(z,\cdot) : z \in \Gamma\}$ is called *tight* if for every $\varepsilon > 0$ there exists a compact K such that $\mu(z, \Xi \setminus K) < \varepsilon$ for each $z \in \Gamma$.

Proposition 4.2.13. *If the family* $\{\mu(z,\cdot) : z \in \Gamma\} \subset \mathcal{M}(\Xi)$ *is tight, then the family* $\{\mu_n(\cdot)\} \subset \mathcal{M}(\Xi)$ *is tight.*

Proof. Let $\varepsilon > 0$, and find K such that $\mu(z, \Xi \setminus K) < \varepsilon/\ell_\Gamma$ for all $z \in \Gamma$, then

$$\mu_n(\Xi \setminus K) = \sum_{i=1}^{n} \mu(z_i^*, \Xi \setminus K)\, |\Delta z_i| < \frac{\varepsilon}{\ell_\Gamma} \sum_{i=1}^{n} |\Delta z_i| \leq \varepsilon. \qquad \square$$

Definition 4.2.14. The function $z \in \Gamma \mapsto \mu(z,\cdot) \in \mathcal{M}(\Xi)$ is *continuous at* $z_0 \in \Gamma$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\wp(\mu(z_0,\cdot), \mu(z_1,\cdot)) < \varepsilon$ for all $z_1 \in \Gamma$ with $|z_0 - z_1| < \delta$.

Proposition 4.2.15. *If* $z \mapsto \mu(Z,\cdot)$ *is continuous at* z_0 *for all* $z_0 \in \Gamma$*, then* $\lim \mu_n(\cdot)$ *exists in* $\mathcal{M}(\Xi)$.

4.3 Measures on the metric space $(\mathcal{K}, \mathbb{d})$

We now apply the results of the previous section to the particular metric space $(\mathcal{K}, \mathbb{d})$. Since $(\mathcal{K}, \mathbb{d})$ is not complete, as in Section 4.1.2, we consider the separable Banach space $(\mathcal{X}, d_{\mathcal{X}})$, and identify $\mathcal{K}$ with $\iota(\mathcal{K}) =: \mathcal{X}^+ \subset \mathcal{X}$. Recall that if $(\gamma^*, t) \in \mathcal{X}^+$, then $t > 0$ so that $\iota^{-1}(\gamma^*, t) \in \mathcal{K}$; that is, $(\mathcal{X}^+, d_{\mathcal{X}}) \cong (\mathcal{K}, \mathbb{d})$ and $\mathcal{B}_{d_{\mathcal{X}}}(\mathcal{X}) \cong \mathcal{B}_{\mathbb{d}}(\mathcal{K})$.

Definition 4.3.1. A *measure* μ *on* $\mathcal{K}$ is defined to be a σ-finite measure on the measurable space $(\mathcal{X}, \mathcal{B}_{d_{\mathcal{X}}})$ concentrated on $\mathcal{X}^+$.

Thus, if we discuss the measure space $(\mathcal{K}, \mathcal{B}_{\mathbb{d}}(\mathcal{K}), \mu)$, it is really shorthand for the induced measure space $(\mathcal{X}, \mathcal{B}_{d_{\mathcal{X}}}(\mathcal{X}), \mu)$ under the above identification. By Theorem 4.1.3, $\mathbb{d}$ and $d_{\mathcal{K}}$ generate the same topology on $\mathcal{K}$ so that $\mathcal{B}_{\mathbb{d}}(\mathcal{K}) = \mathcal{B}_{d_{\mathcal{K}}}(\mathcal{K}) =: \mathcal{B}(\mathcal{K})$. Therefore, we will write $(\mathcal{K}, \mathcal{B}(\mathcal{K}), \mu) \cong (\mathcal{X}, \mathcal{B}_{d_{\mathcal{X}}}(\mathcal{X}), \mu)$ and use whichever metric on $\mathcal{K}$ is most convenient in the given context. Suppose that D, $D' \in \mathcal{D}$ and $f \in \mathcal{T}(D, D')$. If $\gamma \in \mathcal{K}(D)$, then $f \circ \gamma$ is well-defined as discussed in Section 4.1.5, so that f induces a continuous map $(\mathcal{K}(D), \mathbb{d}) \mapsto (\mathcal{K}(D'), \mathbb{d})$, which we also denote by f. By Proposition 4.2.5, if $\mu \in \mathcal{M}(\mathcal{K}(D))$ it makes sense to consider the measure $f \circ \mu \in \mathcal{M}(\mathcal{K}(D'))$ which is given by $f \circ \mu(V') := \mu(\{\gamma \in \mathcal{K}(D) : f \circ \gamma \in V'\})$ for any $V' \in \mathcal{B}_{\mathbb{d}}(\mathcal{K}(D'))$. Finally, observe that Proposition 4.2.3 gives the following.

Proposition 4.3.2. *Let γ_n, γ be $\mathcal{K}$-valued random variables with $\mathcal{L}(\gamma_n) = \mu_n$ and $\mathcal{L}(\gamma) = \mu$, respectively. If $\mathbb{P}(\mathbb{d}(\gamma_n, \gamma) \geq \varepsilon) \leq \varepsilon$, then $\wp(\mu_n, \mu) \leq \varepsilon$.*

4.4 Interior-to-boundary excursion measure

We now begin our investigation of excursion measure. The interior-to-boundary excursion measure is essentially Wiener measure, and is constructed formally using Doob's h-path transform. The presentation, however, is made with an eye to the boundary-to-boundary excursion measure construction of the next section. We limit ourselves to domains $D \in \mathcal{D}$, those bounded, simply connected domains containing the origin with piecewise analytic boundary, and relegate extensions to Section 4.6.

Notation. We write measures of the form $m_E(x, y)$ to indicate that there are three parameters associated with the measure, namely E, x, and y, and that $m_E(x, y)(\cdot) = m_{E,x,y}(\cdot)$. As our primary concern is with the measures themselves, and exclusively in the topology of weak convergence, we choose $m_E(x, y)$ over $m_{E,x,y}(\cdot)$.

4.4.1 Definition of interior-to-boundary excursion measure

Suppose B_t is a Brownian motion with $B_0 = z$, and let $T_D := \inf\{t : B_t \notin D\}$ be its exit time from D. The process $X_t := B_{t\wedge T_D}$, $t \geq 0$, is Brownian motion killed on exiting D. Let $D \in \mathcal{D}$ and suppose $w \in \partial D$ so that the Poisson kernel $H_D(z,w)$ is well-defined. Define the continuous, positive martingale M by $M_t := H_D(X_t, w)/H_D(z,w)$, and the probability $\mathbb{P}^z_w$ by $\mathbb{P}^z_w(A) := \mathbb{E}^z[M_t; A]$ for $A \in \mathcal{F}_t$.

Remark. As noted in [5], the law of the process X_t under $\mathbb{P}^z_w$ is that of Brownian motion conditioned to exit D at w. This is formalized by the following proposition; see [5, Proposition (I.6.7), Proposition (III.2.7)].

Proposition 4.4.1. $(\mathbb{P}^z_w, X_t)$ *is a strong Markov process, and if* $A \in \mathcal{F}_{T_D}$, *then*

$$\int_B \mathbb{P}^z_w(A)\, \mathbb{P}^z\{X_{T_D} \in \mathrm{d}w\} = \mathbb{P}^z\{X_{T_D} \in B; A\}.$$

where $\mathbb{P}^z\{X_{T_D} \in \mathrm{d}w\}$ *is harmonic measure in* D *from* z *as in (2.21).*

Definition 4.4.2. Suppose that $D \in \mathcal{D}$. The *interior-to-boundary excursion measure from z to w in D*, written $\mu_D(z,w)$, is defined to be $\mu_D(z,w) := H_D(z,w)\cdot \mathbb{P}^z_w$, and the *interior-to-boundary excursion measure from z in D*, written $\mu_D(z)$, is defined by

$$\mu_D(z) := \int_{\partial D} \mu_D(z,w)\, |\mathrm{d}w|. \tag{4.12}$$

The measure on paths $\mu_D(z)$ is what is generally called *Wiener measure*. In other words, if $D \in \mathcal{D}$ with $z \in D$, and B is a standard Brownian motion with $B_0 = z$ stopped at T_D, then $\mu_D(z)$ is the law of $\{B_t, 0 \leq t \leq T_D\}$. Observe that $\mu_D(z)$ is a measure on $\mathcal{K}$ concentrated on $\mathcal{K}_z(D)$, via the identification in Section 4.3. We also remark here that if ∂D is piecewise analytic, then we can decompose it as a finite disjoint union $\partial D = \bigcup_{i=1}^{n} \Gamma_i$, where each Γ_i is analytic, so that

$$\int_{\partial D} \mu_D(z,w)\,|\mathrm{d}w| = \sum_{i=1}^{n} \int_{\Gamma_i} \mu_D(z,w)\,|\mathrm{d}w|. \tag{4.13}$$

As mentioned in Section 2.6, for any simply connected domain $D \in \mathbb{C}$ it is well-known that every point of ∂D is regular for D^c so that both harmonic measure and the Poisson kernel exist for D. Since we are assuming piecewise analyticity, if $D \in \mathcal{D}$, then the integral in (4.12) makes sense (interpreted as in (4.13)).

Remark. By definition, $\mu_D(z,w)$ is a finite measure with mass $|\mu_D(z,w)| = H_D(z,w)$. As such we can consider the normalized probability measure

$$\mu_D^{\#}(z,w) := \frac{\mu_D(z,w)}{|\mu_D(z,w)|} = \frac{\mu_D(z,w)}{H_D(z,w)} := \mathbb{P}_w^z. \tag{4.14}$$

For further work on conditioned Brownian motion in planar domains, see [6] or [13].

4.4.2 Behaviour under conformal transformation

It is well-known that two-dimensional Brownian motion is conformally invariant, and consequently so too is Wiener measure. We express this as follows. If D, $D' \in \mathcal{D}$, $z \in D$, and $f \in \mathcal{T}(D, D')$, then $f \circ \mu_D(z) = \mu_{D'}(f(z))$. This definition is independent of the choice of $f \in \mathcal{T}(D, D')$; indeed, if f_1, $f_2 \in \mathcal{T}(D, D')$ with $f_1(z) = f_2(z) = z'$, then

$$f_1 \circ \mu_D(z) = \mu_{D'}(f_1(z)) = \mu_{D'}(z') = \mu_{D'}(f_2(z)) = f_2 \circ \mu_D(z).$$

Recall from Proposition 2.6.3 that the Poisson kernel satisfies the conformal covariance property $H_D(z,w) = |f'(w)|\, H_{D'}(f(z), f(w))$, provided ∂D, $\partial D'$ are locally analytic at w, $f(w)$, respectively.

Proposition 4.4.3. *Suppose that D, $D' \in \mathcal{D}$, and $z \in D$, $w \in \partial D$ with ∂D locally analytic at w. If $f \in \mathcal{T}(D, D')$, and $\partial D'$ is locally analytic at $f(w)$, then*

$$f \circ \mu_D(z, w) = |f'(w)|\, \mu_{D'}(f(z), f(w)).$$

Proof. By definition we have that $\mu_D(z) = \int_{\partial D} \mu_D(z, w)\, |\mathrm{d}w|$. Since $f \circ \mu_D(z) = \mu_{D'}(f(z))$ we have on the one side that

$$f \circ \mu_D(z) = \int_{\partial D} f \circ \mu_D(z, w)\, |\mathrm{d}w|, \tag{4.15}$$

and on the other that

$$\mu_{D'}(f(z)) = \int_{\partial D'} \mu_{D'}(f(z), w')\, |\mathrm{d}w'|. \tag{4.16}$$

But by changing variables in (4.15), we have

$$\int_{\partial D} f \circ \mu_D(z, w)\, |\mathrm{d}w| = \int_{\partial D'} \frac{f \circ \mu_D(z, w)}{|f'(w)|}\, |\mathrm{d}w'|. \tag{4.17}$$

Equating the integrands in (4.16) and (4.17) and noting that since $f \in \mathcal{T}(D, D')$ it can be extended to a homeomorphism of ∂D onto $\partial D'$ so that $f(w) = w'$, we have

$$\frac{f \circ \mu_D(z, w)}{|f'(w)|} = \mu_{D'}(f(z), f(w)). \qquad \square$$

Proposition 4.4.4. *Suppose that D, $D' \in \mathcal{D}$, and $z \in D$, $w \in \partial D$ with ∂D locally analytic at w. If $f \in \mathcal{T}(D, D')$, and $\partial D'$ is locally analytic at $f(w)$, then*

$$f \circ \mu_D^{\#}(z, w) = \mu_{D'}^{\#}(f(z), f(w)).$$

Proof. By definition we have that

$$f \circ \mu_D^{\#}(z,w) = f \circ \left(\frac{\mu_D(z,w)}{H_D(z,w)} \right) = \frac{1}{H_D(z,w)} \, f \circ \mu_D(z,w)$$

using Proposition 4.2.7. But by Proposition 4.4.3 and (2.25) it follows that

$$\frac{1}{H_D(z,w)} \, f \circ \mu_D(z,w) = \frac{|f'(w)| \, \mu_{D'}(f(z), f(w))}{|f'(w)| \, H_{D'}(f(z), f(w))} = \mu_{D'}^{\#}(f(z), f(w)). \qquad \square$$

4.5 Boundary-to-boundary excursion measure

4.5.1 Construction of excursion measure in $\mathbb{D}$

Definition 4.5.1. If $x,\ y \in \partial\mathbb{D},\ x \neq y$, then *normalized excursion measure on excursions from x to y in $\mathbb{D}$* is the measure on $\mathcal{K}$, concentrated on $\mathcal{K}_x^y(\mathbb{D})$, defined by

$$\lim_{\varepsilon \to 0+} \mu_{\mathbb{D}}^{\#}((1-\varepsilon)x, y) =: \overline{\mu}_{\partial\mathbb{D}}(x,y), \tag{4.18}$$

where $\mu_{\mathbb{D}}(z,y) = H_{\mathbb{D}}(z,y) \cdot \mu_{\mathbb{D}}^{\#}(z,y)$ for $z \in \mathbb{D}$, $y \in \partial\mathbb{D}$ as in Section 4.4.1.

Proposition 4.5.2. *The limit in (4.18) exists.*

Proof. We prove this limit exists using Proposition 4.2.3. Let $\gamma \in \mathcal{K}(\mathbb{D})$ with $\gamma(0) = 0$, $\gamma(t_\gamma) = 1$. As in Section 2.9, let $f_\alpha(z) = \frac{z-\alpha}{1-\alpha z}$ for $\alpha \in (-1,1)$ so that $f_\alpha \in \mathcal{T}(\mathbb{D},\mathbb{D})$, $f_\alpha(0) = -\alpha$, and both 1 and -1 are fixed points of f_α. Using the exact form of the Möbius transformation f_α, a straightforward computation shows that $\lim_{\alpha \to 1} f_\alpha \circ \gamma$ exists in the metric space $(\mathcal{K}, \mathbb{d})$ where $f_\alpha \circ \gamma$ is defined as in Section 4.1.5. In particular, this shows $\lim_{\varepsilon \to 0+} \mu_{\mathbb{D}}^{\#}(-(1-\varepsilon), 1) =: \overline{\mu}_{\partial\mathbb{D}}(-1,1)$ exists. For other x and y, use a composition of Möbius transformations as in the example on page 29. $\square$

Definition 4.5.3. We define *excursion measure on excursions from x to y in $\mathbb{D}$* to be the measure on $\mathcal{K}$, concentrated on $\mathcal{K}_x^y(\mathbb{D})$, defined by $\mu_{\partial\mathbb{D}}(x,y) = H_{\partial\mathbb{D}}(x,y)\cdot\overline{\mu}_{\partial\mathbb{D}}(x,y)$ where $H_{\partial\mathbb{D}}$ is the excursion Poisson kernel as in (2.27).

Remark. By definition, the mass of excursion measure $\mu_\partial(x,y)$ is *defined* to be $|\mu_{\partial\mathbb{D}}(x,y)| = H_{\partial\mathbb{D}}(x,y)$. Hence,

$$\mu^{\#}_{\partial\mathbb{D}}(x,y) := \frac{\mu_{\partial\mathbb{D}}(x,y)}{|\mu_{\partial\mathbb{D}}(x,y)|} = \frac{\mu_{\partial\mathbb{D}}(x,y)}{H_{\partial\mathbb{D}}(x,y)} = \overline{\mu}_{\partial\mathbb{D}}(x,y).$$

Proposition 4.5.4. *If $f \in \mathcal{T}(\mathbb{D},\mathbb{D})$, x, $y \in \partial\mathbb{D}$, then $f\circ\mu^{\#}_{\partial\mathbb{D}}(x,y) = \mu^{\#}_{\partial\mathbb{D}}(f(x),f(y))$.*

Proof. From Proposition 4.2.10 and Proposition 4.4.4, and writing $\varepsilon_1 := \varepsilon|f'(x)|$,

$$\begin{aligned} f\circ\mu^{\#}_{\partial\mathbb{D}}(x,y) &= \lim_{\varepsilon\to0+} f\circ\mu^{\#}_{\mathbb{D}}((1-\varepsilon)x,y) = \lim_{\varepsilon\to0+}\mu^{\#}_{\mathbb{D}}(f((1-\varepsilon)x),f(y)) \\ &= \lim_{\varepsilon\to0+}\mu^{\#}_{\mathbb{D}}(f(x)+\varepsilon|f'(x)|\mathbf{n}_{f(x)}+o(\varepsilon),f(y)) \\ &= \lim_{\varepsilon_1\to0+}\mu^{\#}_{\mathbb{D}}(f(x)+\varepsilon_1\mathbf{n}_{f(x)}+o(\varepsilon_1),f(y)) \\ &= \mu^{\#}_{\partial\mathbb{D}}(f(x),f(y)). \end{aligned}$$

□

Proposition 4.5.5. *If $f \in \mathcal{T}(\mathbb{D},\mathbb{D})$ and x, $y \in \partial\mathbb{D}$ with $x \neq y$, then*

$$f\circ\mu_{\partial\mathbb{D}}(x,y) = |f'(x)|\,|f'(y)|\,\mu_{\partial\mathbb{D}}(f(x),f(y)). \tag{4.19}$$

Proof. From Proposition 4.2.7, $f\circ\mu_{\partial\mathbb{D}}(x,y) = H_{\partial\mathbb{D}}(x,y)\,(\,f\circ\mu^{\#}_{\partial\mathbb{D}}(x,y)\,)$. But by Proposition 4.5.4, and Proposition 2.7.7, we have

$$\begin{aligned} H_{\partial\mathbb{D}}(x,y)\,(\,f\circ\mu^{\#}_{\partial\mathbb{D}}(x,y)\,) &= |f'(x)|\,|f'(y)|\,H_{\partial\mathbb{D}}(f(x),f(y))\,\mu^{\#}_{\partial\mathbb{D}}(f(x),f(y)) \\ &= |f'(x)|\,|f'(y)|\,\mu_{\partial\mathbb{D}}(f(x),f(y)). \end{aligned}$$

□

Proposition 4.5.6. *If $f_1,\ f_2 \in \mathcal{T}(\mathbb{D},\mathbb{D})$, then $f_1 \circ \mu_{\partial\mathbb{D}}(x,y) = f_2 \circ \mu_{\partial\mathbb{D}}(x,y)$ so that (4.19) is independent of the choice of map.*

Proof. If $x,\ y \in \partial\mathbb{D}$ and $f_1,\ f_2 \in \mathcal{T}(\mathbb{D},\mathbb{D})$ with $f_1(x) = f_2(x) = x'$ and $f_1(y) = f_2(y) = y'$, then $f_1 \circ \mu_{\partial\mathbb{D}}(x,y) = |f_1'(x)|\,|f_2'(y)|\,\mu_{\partial\mathbb{D}}(x',y')$ and $f_2 \circ \mu_{\partial\mathbb{D}}(x,y) = |f_2'(x)|\,|f_2'(y)|\,\mu_{\partial\mathbb{D}}(x',y')$, so that

$$\frac{f_1 \circ \mu_{\partial\mathbb{D}}(x,y)}{f_2 \circ \mu_{\partial\mathbb{D}}(x,y)} = \frac{|f_2'(x)|\,|f_2'(y)|}{|f_1'(x)|\,|f_1'(y)|}.$$

Thus if $f_3 := f_1^{-1} \circ f_2$, then $f_3 \in \mathcal{T}(\mathbb{D},\mathbb{D})$ with $f_3(x) = x$ and $f_3(y) = y$ so that by Proposition 2.9.2, $|f_3'(x)|\,|f_3'(y)| = 1$. But by the chain rule, $|f_3'(x)| = |f_2'(x)|/|f_1'(x)|$ and similarly for $|f_3'(y)|$ so that

$$\frac{f_1 \circ \mu_{\partial\mathbb{D}}(x,y)}{f_2 \circ \mu_{\partial\mathbb{D}}(x,y)} = 1,$$

or $f_1 \circ \mu_{\partial\mathbb{D}}(x,y) = f_2 \circ \mu_{\partial\mathbb{D}}(x,y)$, as required. □

Proposition 4.5.7. *Excursion measure $\mu_{\partial\mathbb{D}}(x,y)$ satisfies*

$$\mu_{\partial\mathbb{D}}(x,y) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\mu_{\mathbb{D}}((1-\varepsilon)x,y). \tag{4.20}$$

Proof. Recall from the remark on page 68 that to show $m_n \Rightarrow m$ weakly for finite measures, it suffices to show both $|m_n| \to |m|$ and $\wp(m_n^{\#}, m^{\#}) \to 0$. Thus,

$$\begin{aligned}\mu_{\partial\mathbb{D}}(x,y) = H_{\partial\mathbb{D}}(x,y)\cdot\overline{\mu}_{\partial\mathbb{D}}(x,y) &= \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}H_{\mathbb{D}}((1-\varepsilon)x,y)\cdot\lim_{\varepsilon\to 0+}\mu_{\mathbb{D}}^{\#}((1-\varepsilon)x,y)\\ &= \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}H_{\mathbb{D}}((1-\varepsilon)x,y)\cdot\mu_{\mathbb{D}}^{\#}((1-\varepsilon)x,y)\\ &= \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\mu_{\mathbb{D}}((1-\varepsilon)x,y).\end{aligned}$$

□

4.5.2 Construction of excursion measure in $D \in \mathcal{D}$

Definition 4.5.8. Suppose that $D \in \mathcal{D}$, and z, $w \in \partial D$ with ∂D locally analytic at both z and w. Let $h \in \mathcal{T}(\mathbb{D}, D)$. *Excursion measure from z to w in D* is defined by

$$\mu_{\partial D}(z,w) := \frac{1}{|h'(h^{-1}(z))|\,|h'(h^{-1}(w))|}\, h \circ \mu_{\partial \mathbb{D}}(h^{-1}(z), h^{-1}(w)). \tag{4.21}$$

Proposition 4.5.9. *The definition of $\mu_{\partial D}(z,w)$ given by (4.21) does not depend on the choice of $h \in \mathcal{T}(\mathbb{D}, D)$.*

Proof. Suppose that h_1, $h_2 \in \mathcal{T}(\mathbb{D}, D)$ so that

$$\mu_{\partial \mathbb{D}}(h_1^{-1}(z), h_1^{-1}(w)) = \frac{|h_1'(h_1^{-1}(z))|\,|h_1'(h_1^{-1}(w))|}{|h_2'(h_2^{-1}(z))|\,|h_2'(h_2^{-1}(w))|}\,(h_1^{-1} \circ h_2) \circ \mu_{\partial \mathbb{D}}(h_2^{-1}(z), h_2^{-1}(w)).$$

Define $h_3 := h_1^{-1} \circ h_2 \in \mathcal{T}(\mathbb{D}, \mathbb{D})$ and note that $h_3^{-1} = h_2^{-1} \circ h_1$. Let $x := h_1^{-1}(z) \in \partial \mathbb{D}$ and $y := h_1^{-1}(w) \in \partial \mathbb{D}$ so that $h_3^{-1}(x) = h_2^{-1}(z)$ and $h_3^{-1}(y) = h_2^{-1}(w)$. Then,

$$\mu_{\partial \mathbb{D}}(x,y) = \frac{|h_1'(x)|\,|h_1'(y)|}{|h_2'(h_3^{-1}(x))|\,|h_2'(h_3^{-1}(y))|}\, h_3 \circ \mu_{\partial \mathbb{D}}(h_3^{-1}(x), h_3^{-1}(y)).$$

By the chain rule,

$$h_3'(h_3^{-1}(x)) = (h_1^{-1})'(h_2(h_3^{-1}(x))) \cdot h_2'(h_3^{-1}(x)) = \frac{h_2'(h_3^{-1}(x))}{h_1'(h_1^{-1}(h_2(h_3^{-1}(x))))} = \frac{h_2'(h_3^{-1}(x))}{h_1'(x)},$$

and similarly for y. Thus we conclude that

$$h_3 \circ \mu_{\partial \mathbb{D}}(h_3^{-1}(x), h_3^{-1}(y)) = |h_3'(h_3^{-1}(x))|\,|h_3'(h_3^{-1}(y))|\,\mu_{\partial \mathbb{D}}(x,y).$$

Finally, if $h_1^{-1}(z) = h_2^{-1}(z)$ and $h_1^{-1}(w) = h_2^{-1}(w)$, then $h_3 \in \mathcal{T}(\mathbb{D}, \mathbb{D})$ with $h_3(x) = x$

and $h_3(y) = y$ so that $h_3 \circ \mu_{\partial\mathbb{D}}(x,y) = \mu_{\partial\mathbb{D}}(x,y)$, or $h_1 \circ \mu_{\partial\mathbb{D}}(x,y) = h_2 \circ \mu_{\partial\mathbb{D}}(x,y)$, by Proposition 2.9.2. □

Proposition 4.5.10. *Let D, $D' \in \mathcal{D}$, and let z, $w \in \partial D$ with ∂D locally analytic at both z, w. If $f \in \mathcal{T}(D, D')$, and $\partial D'$ is locally analytic at both $f(z)$, $f(w)$, then*

$$f \circ \mu_{\partial D}(z,w) = |f'(z)|\,|f'(w)|\,\mu_{\partial D'}(f(z), f(w)). \tag{4.22}$$

If f_1, $f_2 \in \mathcal{T}(D, D')$, then $f_1 \circ \mu_{\partial D}(z,w) = f_2 \circ \mu_{\partial D}(z,w)$ so (4.22) is independent of the choice of map. In particular, combining (4.22) with Proposition 4.5.7 yields

$$\mu_{\partial D}(z,w) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \mu_D(z + \varepsilon \mathbf{n}_z, w).$$

Proof. Write $f(z) = z'$ and $f(w) = w'$. Let $h_1 \in \mathcal{T}(\mathbb{D}, D)$, $h_2 \in \mathcal{T}(\mathbb{D}, D')$ with $h_1^{-1}(z) = h_2^{-1}(z')$, $h_1^{-1}(w) = h_2^{-1}(w')$ so that $f = h_2 \circ h_1^{-1}$; thus, by Definition 4.5.8,

$$\begin{aligned} f \circ \mu_{\partial D}(z,w) &= \frac{1}{|h_1'(h_1^{-1}(z))|\,|h_1'(h_1^{-1}(w))|}\, h_2 \circ \mu_{\partial\mathbb{D}}(h_2^{-1}(z'), h_2^{-1}(w')) \\ &= \frac{|h_2'(h_2^{-1}(z'))|\,|h_2'(h_2^{-1}(w'))|}{|h_1'(h_1^{-1}(z))|\,|h_1'(h_1^{-1}(w))|}\, \mu_{\partial D'}(z', w'). \end{aligned}$$

But by the chain rule, since $h_1^{-1}(z) = h_2^{-1}(z')$,

$$|f'(z)| = |h_2'(h_1^{-1}(z))|\,|(h_1^{-1})'(z)| = \frac{|h_2'(h_2^{-1}(z'))|}{|h_1'(h_1^{-1}(z))|}$$

and similarly for $|f'(w)|$ so that $f \circ \mu_{\partial D}(z,w) = |f'(z)|\,|f'(w)|\,\mu_{\partial D'}(z', w')$. □

Suppose that Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$. Recall that $\mathcal{K}_\Gamma^\Upsilon(D)$ is the set of curves $\gamma : [0, t_\gamma] \to \mathbb{C}$ in D with $\gamma(0) \in \Gamma$ and $\gamma(t_\gamma) \in \Upsilon$.

Definition 4.5.11. Let $\mu_{\partial D}(\Gamma, \Upsilon)$ be the measure concentrated on $\mathcal{K}_{\Gamma}^{\Upsilon}(D)$ given by

$$\mu_{\partial D}(\Gamma, \Upsilon) := \int_{\Gamma} \int_{\Upsilon} \mu_{\partial D}(x, y) \, |\mathrm{d}y| \, |\mathrm{d}x|. \tag{4.23}$$

In order to define $\mu_{\partial D}(x, y)$ we required in Definition 4.5.8 that ∂D be locally analytic at both x and y. Since $D \in \mathcal{D}$, there are only finitely many points at which ∂D is not analytic. If we write $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$ and $\Upsilon = \bigcup_{j=1}^{m} \Upsilon_j$ with each Γ_i and each Υ_j analytic, then there is no problem in defining

$$\int_{\Gamma} \int_{\Upsilon} \mu_{\partial D}(x, y) \, |\mathrm{d}y| \, |\mathrm{d}x| := \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{\Gamma_i} \int_{\Upsilon_j} \mu_{\partial D}(x, y) \, |\mathrm{d}y| \, |\mathrm{d}x|.$$

Proposition 4.5.12 (Conformal Invariance of Excursion Measure). *Suppose that D, $D' \in \mathcal{D}$ and $f \in \mathcal{T}(D, D')$. Let Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$ be analytic open boundary arcs, and write Γ', Υ' for the images under f of Γ, Υ, respectively. Then, $f \circ \mu_{\partial D}(\Gamma, \Upsilon) = \mu_{\partial D'}(\Gamma', \Upsilon')$.*

4.5.3 σ-finite excursion measures

Having defined excursion measure on excursions from x to y in D for $D \in \mathcal{D}$, we now define two infinite, but σ-finite, measures on excursions by integrating along ∂D.

Definition 4.5.13. Suppose that $D \in \mathcal{D}$ and $z \in \partial D$ with ∂D locally analytic at z. *Excursion measure from z in D* is the measure concentrated on $\mathcal{K}_z(D)$ defined by

$$\mu_{\partial D}(z) := \int_{\partial D} \mu_{\partial D}(z, w) \, |\mathrm{d}w|.$$

As in Section 4.4.1, if ∂D is piecewise analytic, then we can decompose it as a finite disjoint union $\partial D = \bigcup_{i=1}^{n} \Gamma_i$, where each Γ_i is analytic, so that

$$\int_{\partial D} \mu_{\partial D}(z,w)\,|\mathrm{d}w| = \sum_{i=1}^{n} \int_{\Gamma_i} \mu_{\partial D}(z,w)\,|\mathrm{d}w|.$$

As a corollary to Proposition 4.5.10, we have the following.

Proposition 4.5.14. *Suppose that D, $D' \in \mathcal{D}$, $f \in \mathcal{T}(D,D')$ and $z \in D$ so that $f(z) \in \partial D'$. If ∂D and $\partial D'$ are locally analytic at z and $f(z)$, respectively, then $f \circ \mu_{\partial D}(z) = |f'(z)|\,\mu_{\partial D'}(f(z))$.*

Proof. By Definition 4.5.13 and Proposition 4.5.10, writing $f(w) = w'$, it follows that

$$\begin{aligned} f \circ \mu_{\partial D}(z) = \int_{\partial D} f \circ \mu_{\partial D}(z,w)\;|\mathrm{d}w| &= \int_{\partial D} |f'(z)|\,|f'(w)|\,\mu_{\partial D'}(f(z),f(w))\;|\mathrm{d}w| \\ &= |f'(z)| \int_{\partial D'} \mu_{\partial D'}(f(z),w')\;|\mathrm{d}w'| \\ &= |f'(z)|\,\mu_{\partial D'}(f(z)). \end{aligned}$$

□

Definition 4.5.15. Suppose that $D \in \mathcal{D}$. *Excursion measure in D* is defined by

$$\mu_{\partial D} := \int_{\partial D} \mu_{\partial D}(z)\;|\mathrm{d}z|.$$

Theorem 4.5.16 (Conformal Invariance of Excursion Measure). *If D, $D' \in \mathcal{D}$ and $f \in \mathcal{T}(D,D')$, then $f \circ \mu_{\partial D} = \mu_{\partial D'}$.*

Proof. By Definition 4.5.15 and Proposition 4.5.14, writing $f(z) = z'$, it follows that

$$\begin{aligned} f \circ \mu_{\partial D} = \int_{\partial D} f \circ \mu_{\partial D}(z)\;|\mathrm{d}z| = \int_{\partial D} |f'(z)|\,\mu_{\partial D'}(f(z))\;|\mathrm{d}z| &= \int_{\partial D'} \mu_{\partial D'}(z')\;|\mathrm{d}z'| \\ &= \mu_{\partial D'}. \end{aligned}$$

□

4.6 Extension of $\mu_{\partial D}$ to general $D \in \mathcal{D}^*$

We now extend excursion measure to domains more general than $D \in \mathcal{D}$. Recall that $\mu_{\partial\mathbb{D}}(x, y)$, excursion measure from x to y in $\mathbb{D}$, was constructed directly, and then used to define $\mu_{\partial D}(x', y')$ for $D \in \mathcal{D}$ via conformal transformation (Definition 4.5.8 and Proposition 4.5.9). By Proposition 4.5.10 if D, $D' \in \mathcal{D}$ and $f \in \mathcal{T}(D, D')$, then $f \circ \mu_{\partial D}(z, w) = |f'(z)|\,|f'(w)|\,\mu_{\partial D'}(f(z), f(w))$, provided z, $w \in \partial D$ with ∂D locally analytic at both z and w, and $\partial D'$ locally analytic at both $f(z)$ and $f(w)$. It was necessary to restrict to D, $D' \in \mathcal{D}$ and to require that ∂D, $\partial D'$ be locally analytic at x, y and $f(x)$, $f(y)$, respectively, in order to guarantee that for $f \in \mathcal{T}(D, D')$, both $|f'(x)|$ and $|f'(y)|$ would exist in $(0, \infty)$. Excursion measure in D for $D \in \mathcal{D}$ was then defined by

$$\mu_{\partial D} = \int_{\partial D} \int_{\partial D} \mu_{\partial D}(z, w)\,|\mathrm{d}z|\,|\mathrm{d}w|$$

and it was proved in Theorem 4.5.16 that excursion measure is conformally invariant.

Definition 4.6.1. Suppose that $D \in \mathcal{D}^*$ and $f \in \mathcal{T}(\mathbb{D}, D)$. *Excursion measure in* D is defined by

$$\mu_{\partial D} := f \circ \mu_{\partial\mathbb{D}}. \tag{4.24}$$

Of course, this definition holds for *any* simply connected domain $D \subset \mathbb{C}$ conformally equivalent to $\mathbb{D}$ provided that ∂D is a Jordan curve. A Jordan boundary is required in order to extend f (Theorem 2.1.2). We restrict to $D \in \mathcal{D}^*$ so that excursions $\gamma \in \mathcal{K}(D)$ will necessarily have $t_\gamma < \infty$. For example, excursions from 0 to ∞ in $\mathbb{H} := \{z \in \mathbb{C} : \mathit{Im}(z) > 0\}$ do not satisfy this finite lifetime condition.

Definition 4.6.2. If $D \in \mathcal{D}^*$ and Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$, define $\mu_{\partial D}(\Gamma, \Upsilon)$ to be the measure $\mu_{\partial D}$ (as in (4.24)) restricted to those excursions $\gamma \in \mathcal{K}_\Gamma^\Upsilon(D)$.

We observe that if $D \in \mathcal{D}$, then the definition of $\mu_{\partial D}(\Gamma, \Upsilon)$ above agrees with that given by (4.23). A consequence of these definitions is that restricted excursion measure is conformally invariant.

Proposition 4.6.3. *If D, $D' \in \mathcal{D}^*$; $f \in \mathcal{T}(D, D')$; Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$; and Γ', Υ' are the images under f of Γ, Υ, respectively, then $f \circ \mu_{\partial D}(\Gamma, \Upsilon) = \mu_{\partial D'}(\Gamma', \Upsilon')$.*

4.7 Extension of $H_{\partial D}$ to general $D \in \mathcal{D}^*$

In the present section, we extend the excursion Poisson kernel $H_{\partial D}$ to include $D \in \mathcal{D}^*$. Recall that if $D \in \mathcal{D}$, and Γ, $\Upsilon \subset \partial D$ are analytic open boundary arcs with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$, then for $x \in \Gamma$ and $y \in \Upsilon$, the excursion Poisson kernel $H_{\partial D}(x, y)$ exists so that

$$H_{\partial D}(\Gamma, \Upsilon) := \int_{\Upsilon} \int_{\Gamma} H_{\partial D}(x, y) \, |\mathrm{d}x| \, |\mathrm{d}y| \tag{4.25}$$

is well-defined. As such, if $D' \in \mathcal{D}$ and $f \in \mathcal{T}(D, D')$, then from the conformal covariance of the excursion Poisson kernel (Proposition 2.7.7) we conclude that $H_{\partial D}(\Gamma, \Upsilon) = H_{\partial D'}(\Gamma', \Upsilon')$ where Γ', Υ' are the images under f of Γ, Υ, respectively. However, if $D \in \mathcal{D}^* \setminus \mathcal{D}$, then the above integration is not valid. In Definition 4.6.2 $\mu_{\partial D}(\Gamma, \Upsilon)$ was defined to be the restriction of $\mu_{\partial D}$ to those excursions $\gamma \in \mathcal{K}_{\Gamma}^{\Upsilon}(D)$. For $D \in \mathcal{D}$ this definition of $\mu_{\partial D}(\Gamma, \Upsilon)$ is equivalent to defining

$$\mu_{\partial D}(\Gamma, \Upsilon) := \int_{\Upsilon} \int_{\Gamma} \mu_{\partial D}(x, y) \, |\mathrm{d}x| \, |\mathrm{d}y|. \tag{4.26}$$

In particular, if $D \in \mathcal{D}$, then (4.25) and (4.26) imply

$$|\mu_{\partial D}(\Gamma, \Upsilon)| = \int_{\Upsilon} \int_{\Gamma} H_{\partial D}(x, y) \, |\mathrm{d}x| \, |\mathrm{d}y|.$$

Definition 4.7.1. Suppose that $D \in \mathcal{D}^*$ and $\Gamma,\ \Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$. Let $\mu_{\partial D}(\Gamma, \Upsilon)$ be defined as in Definition 4.6.2. The *excursion Poisson kernel* $H_{\partial D}(\Gamma, \Upsilon)$ is defined to be the mass of $\mu_{\partial D}(\Gamma, \Upsilon)$; that is, $H_{\partial D}(\Gamma, \Upsilon) := |\mu_{\partial D}(\Gamma, \Upsilon)|$.

From Proposition 4.6.3 excursion measure $\mu_{\partial D}(\Gamma, \Upsilon)$ is conformally invariant. Thus, we have the following extended version of Proposition 2.7.9.

Proposition 4.7.2. *Suppose that D, $D' \in \mathcal{D}^*$ and $f \in \mathcal{T}(D, D')$. If Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$, and Γ', Υ' are the images under f of Γ, Υ, respectively, then $H_{\partial D}(\Gamma, \Upsilon) = H_{\partial D'}(\Gamma', \Upsilon')$.*

4.8 Discrete excursions

In this section, we define a discrete excursion and formulate the discrete analogues of the definitions in Section 4.1.4. Since random walk has positive probability of exiting a set at a fixed point, we will be conditioning on events of non-zero probability. Thus, the subtleties that arise in the Brownian case with probability zero events are not an issue here. Let $A \in \mathcal{A}$, and throughout this section, assume w, $z \in A$; x, $y \in \partial A$. If S_j is a simple random walk with $S_0 = w$, denote the one-step transition probability $p(w, z) := \mathbb{P}^w\{S_1 = z\}$. As in [15, §3.1], if $\tau_A := \min\{j > 0 : S_j \notin A\}$, then

$$q(w, z; y) := P^w\{S_1 = z | S_{\tau_A} = y\} = p(w, z)\frac{h_A(z, y)}{h_A(w, y)} \tag{4.27}$$

defines the one-step transition probabilities of simple random walk conditioned to exit A at y, where h_A is the discrete Poisson kernel as in Definition 2.11.1. Similarly, if $\Gamma \subset \partial A$, then

$$q(w, z; \Gamma) := p(w, z)\frac{h_A(z, \Gamma)}{h_A(w, \Gamma)} \tag{4.28}$$

is simple random walk conditioned to exit A in Γ. Note that h_A is discrete harmonic, and both (4.27) and (4.28) are examples of discrete h-transforms.

Recall from Section 2.11 that for x, $y \in \partial A$, the discrete excursion Poisson kernel $h_{\partial A}(x, y)$ is given by $h_{\partial A}(x, y) := \mathbb{P}^x\{S_{\tau_A} = y, S_1 \in A\}$. We now define a *discrete excursion*.

Definition 4.8.1. A *discrete excursion in A* is a path $\omega = [\omega_0, \omega_1, \ldots, \omega_k]$ where $\omega_0 \in \partial A$, $\omega_k \in \partial A$, $|\omega_i - \omega_{i-1}| = 1$ for $i = 1, \ldots, k$, and $\omega_i \in A$ for $i = 1, \ldots, k-1$. It is implicit that $2 \leq k < \infty$. If $\omega = [\omega_0, \omega_1, \ldots, \omega_k]$, define the *length* of ω, written $|\omega|$, to be k.

Definition 4.8.2. If ω is a discrete excursion in A with $\omega_0 = x$ and $\omega_k = y$, then ω is called a *discrete excursion from x to y in A*.

Definition 4.8.3. Suppose that Γ, $\Upsilon \subset \partial A$ with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. If ω is a discrete excursion in A with $\omega_0 \in \Gamma$ and $\omega_k \in \Upsilon$, then w is called a *discrete excursion from Γ to Υ in A*, or a *(Γ, Υ)-discrete excursion in A*.

4.9 Discrete excursion measure

Throughout this section, suppose that $A \in \mathcal{A}$; x, $y \in \partial A$; and Γ, $\Upsilon \subset \partial A$ with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. Discrete excursions can be generated by starting a simple random walk S_n at $x \in \partial A$, conditioning it to take its first step into A, and stopping it at $\tau_A := \min\{j > 0 : S_j \in \partial A\}$. As such, we define *discrete excursion measure* to be the measure that assigns weight $4^{-|\omega|}$ to each discrete excursion ω. Write this measure as $\mu^{\mathsf{rw}}_{\partial A}(\cdot)$ so that

$$\mu^{\mathsf{rw}}_{\partial A}(\omega) := \left(\frac{1}{4}\right)^{|\omega|}.$$

Similarly, $\mu^{\mathsf{rw}}_{\partial A}(x,y)$ denotes the measure on discrete excursions from x to y in A, where we write $\mu^{\mathsf{rw}}_{\partial A}(x,y)$ for the measure $\mu^{\mathsf{rw}}_{\partial A,x,y}(\cdot)$ using the notation from Section 4.4. Finally, let

$$\mu^{\mathsf{rw}}_{\partial A}(\Gamma,\Upsilon) := \sum_{x\in\Gamma}\sum_{y\in\Upsilon}\mu^{\mathsf{rw}}_{\partial A}(x,y) \tag{4.29}$$

denote the measure on (Γ,Υ)-discrete excursions in A. Note that the mass of excursion measure $\mu^{\mathsf{rw}}_{\partial A}(x,y)$ is $\mathbb{P}^x\{S_{\tau_A}=y, S_1\in A\}$, namely the discrete excursion Poisson kernel; that is, $|\mu^{\mathsf{rw}}_{\partial A}(x,y)| = h_{\partial A}(x,y)$. Similarly, $\mu^{\mathsf{rw}}_{\partial A}(\Gamma,\Upsilon) = h_{\partial A}(\Gamma,\Upsilon)$ where

$$h_{\partial A}(\Gamma,\Upsilon) := \sum_{x\in\Gamma}\sum_{y\in\Upsilon} h_{\partial A}(x,y).$$

Remark. Lawler and Werner defined $\mu^{\mathsf{rw}}_{\partial A}(\omega) := (2\pi\, 4^{|\omega|})^{-1}$ in [39]; this difference only affects things up to a constant.

We want both discrete excursion measure and Brownian excursion measure to be measures on the metric space $(\mathcal{K},\mathbb{d})$. Consequently, we need to associate to each discrete excursion ω a curve $\tilde\omega\in\mathcal{K}$. Suppose that ω is a discrete excursion in A, and let $\mathrm{cl}(A) := A\cup\partial A$ with associated domain $\widetilde{\mathrm{cl}(A)}\subset\mathbb{C}$. We associate to ω a curve $\tilde\omega\in\mathcal{K}\left(\widetilde{\mathrm{cl}(A)}\right)$ by setting $t_{\tilde\omega} := 2|\omega|$, and

$$\tilde\omega(t) := \omega_{\lfloor t/2\rfloor} + \frac{1}{2}(t-\lfloor t\rfloor)(\omega_{\lfloor t/2\rfloor+1}-\omega_{\lfloor t/2\rfloor}), \quad 0\le t\le t_{\tilde\omega}. \tag{4.30}$$

Thus, we join the lattice points in order with line segments parallel to the coordinate axes in $\mathbb{Z}^2$, with each segment taking time 2 to traverse. Note that $\tilde\omega(0)=\omega_0$ and $\tilde\omega(t_{\tilde\omega}) = \omega_{|\omega|}$. Using this identification, if ω is an excursion from x to y in A, then $\mu^{\mathsf{rw}}_{\partial A}(x,y)\in\mathcal{M}(\mathcal{K})$ and $\mu^{\mathsf{rw}}_{\partial A,x,y}(\tilde\omega) = 4^{-t_{\tilde\omega}}$.

Remark. Recall from Theorem 3.1.2 that $|B_t - S_{2t}| = O(\log t)$. Thus, it is simply a matter of æsthetics that a random walk path of $|\omega|$ steps take time $2|\omega|$ to traverse: if γ is Brownian curve and $\tilde{\omega}$ is as above with $\gamma(0) = \tilde{\omega}(0)$, then $|\gamma(t) - \tilde{\omega}(t)| = O(\log t)$.

Remark. In Section 5.4, we will consider scaling excursions as the mesh of the lattice becomes finer. As in the case of simple random walk converging to Brownian motion, we will have to apply the usual *Brownian scaling* to our discrete paths.

Definition 4.9.1. Suppose that $A \in \mathcal{A}$ and $x,y \in \partial A$. *Discrete excursion measure* $\mu^{\mathsf{rw}}_{\partial A}(x,y)$ is defined to be the measure on $(\mathcal{K}, \mathbb{d})$, concentrated on $V = V(x,y;A) := \{\gamma \in \mathcal{K} : \mathbb{d}(\gamma, \tilde{\omega}) = 0$ for some discrete excursion ω from x to y in $A\}$ given by

$$\mu^{\mathsf{rw}}_{\partial A}(x,y)(\gamma) := \left(\frac{1}{4}\right)^{t_\gamma}$$

for $\gamma \in V$. Note that $\mu^{\mathsf{rw}}_{\partial A}(x,y)(V) = h_{\partial A}(x,y)$.

Chapter 5

Approximating $D \in \mathcal{D}^*$ and the Main Convergence Arguments

In this chapter we consolidate our earlier work, and complete the proof that discrete excursion measure converges in the scaling limit to Brownian excursion measure. In particular, we carefully formulate what this convergence means; indeed, one of the difficulties is constructing an appropriate discrete approximation to $D \in \mathcal{D}^*$ on which to define the necessary random walk excursions. We begin in Section 5.1 by recalling the definition of Carathéodory convergence of domains, and in Section 5.2, we introduce the notation that will be used throughout this chapter. Section 5.3 is devoted to the proof that our approximate domains $\tilde{D}_N \in \mathcal{D}$ converge to $D \in \mathcal{D}^*$. The following section translates the Green's function estimates of Chapter 3, which were originally proved for $A \in \mathcal{A}^N$, into the corresponding results for the $1/N$-scale approximation D_N. In Section 5.5, we are finally able to precisely state our principal convergence result, namely Theorem 5.5.1. The proof is spread over the final three sections of this chapter. Specifically, we first show that the discrete excursion Poisson kernel $h_{\partial D_N}$ converges to the excursion Poisson kernel $H_{\partial D}$ for $D \in \mathcal{D}^*$. We then show that Brownian excursion measure on $\tilde{D}_N$ converges to Brownian excursion measure on D, and finally we establish estimates in the Prohorov metric $\wp$ relating discrete excursion measure on D_N with Brownian excursion measure on $\tilde{D}_N \in \mathcal{D}$.

5.1 Carathéodory convergence

Although we will be discussing convergence of domains in $\mathbb{C}$, the usual topological notion of convergence will not serve us, and instead we define what it means for sets to converge in the sense of Carathéodory. The main tool is the classic result of Carathéodory which roughly states that the convergence of domains is equivalent to the uniform convergence of the appropriate Riemann maps.

Definition 5.1.1. Fix $r > 0$. Suppose that D_n is a sequence of domains with $D_n \in \mathcal{D}^r$ for each n. The *kernel* of D_n, written $\ker(\{D_n\})$, is the largest domain D containing the origin and having the property that each compact subset of D lies in all but a finite number of the domains D_n. Suppose that $\ker(\{D_n\}) = D$. The sequence D_n *converges in the Carathéodory sense* to D, written $D_n \overset{\text{Cara}}{\to} D$, if every subsequence D_{n_j} of D_n has $\ker(\{D_{n_j}\}) = D$.

Remark. For completeness, recall that if D_n is a sequence of domains, then

$$D^* := \limsup_{n\to\infty} D_n := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} D_n \quad \text{and} \quad D_* := \liminf_{n\to\infty} D_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} D_n.$$

If $D^* = D_* =: D$, then $D_n \to D$ *topologically*.

Definition 5.1.2. A sequence of functions f_n on a domain D converges to a function f *uniformly on compacta of* D if for each compact $K \subset D$, $f_n \to f$ uniformly on K.

A proof of the following theorem may be found in [14, Theorem 3.1].

Theorem 5.1.3 (Carathéodory Convergence Theorem). *Suppose that D_n is a sequence of domains with $D_n \in \mathcal{D}^*$ for each n, and let $f_n \in \mathcal{T}(\mathbb{D}, D_n)$ with $f_n(0) = 0$, $f_n'(0) > 0$. Suppose further that $D \in \mathcal{D}$ and $f \in \mathcal{T}(\mathbb{D}, D)$ with $f(0) = 0$, $f'(0) > 0$. Then $f_n \to f$ uniformly on compacta of $\mathbb{D}$ if and only if $D_n \overset{\text{Cara}}{\to} D$.*

Proposition 5.1.4. *Suppose that $D_n \overset{Cara}{\to} D$ with D_n, $D \in \mathcal{D}^*$. Suppose further that there exists an $E \in \mathcal{D}^*$ with $D_n \subset E$ for all n, and $D \subseteq E$. If $F : E \to \mathbb{D}$ is the conformal transformation with $F(0) = 0$, $F'(0) > 0$, then $F(D_n) \overset{Cara}{\to} F(D)$.*

Proof. Let $f_n : \mathbb{D} \to D_n$ and let $f : \mathbb{D} \to D$ be conformal transformations mapping 0 to 0 with positive derivative at the origin. By Theorem 5.1.3, the convergence of D_n to D is equivalent to the uniform convergence of f_n to f on compacta of $\mathbb{D}$. Set $h_n := F \circ f_n$ and $h := F \circ f$, and let K be a compact subset of $\mathbb{D}$. If $z \in K$, then $|h_n(z) - h(z)| = |F(f_n(z)) - F(f(z))| \to 0$ uniformly as $n \to \infty$. □

We now make two observations about increasing sequences. The first is that if $D_n \subset E$ is an increasing sequence and $f \in \mathcal{T}(E, \mathbb{D})$ with $f(0) = 0$, $f'(0) > 0$, then $f(D_n) \subset \mathbb{D}$ is also an increasing sequence. The second is that if D_n is an increasing sequence, then $D_n \overset{\text{Cara}}{\to} D$ if and only if $D_n \to D$ topologically. Indeed, if D_n is increasing, then it is clear that $\ker(\{D_n\}) = \bigcup_{n=1}^{\infty} D_n =: D$ and $D^* = D_* = D$.

Having defined what it means to converge uniformly on compacta, we present the following results. Recall that the metric space $(\mathcal{K}, \mathbb{d})$ was considered in Section 4.1.

Proposition 5.1.5. *Suppose that F_n, F are conformal mappings of $\mathbb{D}$. Let $D := F(\mathbb{D})$. If $F_n \to F$ uniformly on compacta of $\mathbb{D}$, then $F_n \circ F^{-1} \to I$ uniformly on compacta of D, where $I : D \to D$ is the identity map $I(z) = z$.*

Proof. Let $K' \subset D$ be compact. Let $\varepsilon > 0$ be given. Let $K = F^{-1}(K') \subset \mathbb{D}$ which is clearly compact. By uniform convergence, there exists $N = N(\varepsilon, K)$ such that $|F_n(x) - F(x)| < \varepsilon$ for all $n > N$, $x \in K$. If $y \in K'$, then $y = F(x)$ for some $x \in K$. Hence, if $n > N$, then $|F_n \circ F^{-1}(y) - I(y)| = |F_n(x) - F(x)| < \varepsilon$, and the proof is complete. □

Proposition 5.1.6. *Suppose that F_n, F are conformal mappings of the unit disk $\mathbb{D}$, and that $F_n \to F$ uniformly on compacta of $\mathbb{D}$. If $\gamma \in \mathcal{K}(\mathbb{D})$ with $|\gamma(0)| < 1$ and $|\gamma(t_\gamma)| < 1$, then $\mathrm{d}(F_n \circ \gamma, F \circ \gamma) \to 0$ as $n \to \infty$.*

Proof. Suppose that $\gamma \in \mathcal{K}(\mathbb{D})$ with $|\gamma(0)| < 1$ and $|\gamma(t_\gamma)| < 1$. Note that γ is *not* an excursion in $\mathbb{D}$. Therefore, there necessarily exists a compact set $K \subset \mathbb{D}$ such that $\gamma \in \mathcal{K}(K)$. Consider $t_{F_n \circ \gamma} = A^n_{t_\gamma} = \int_0^{t_\gamma} |F_n'(\gamma(r))|^2 \,\mathrm{d}r$ and $t_{F\circ\gamma} = A_{t_\gamma} = \int_0^{t_\gamma} |F'(\gamma(r))|^2 \,\mathrm{d}r$. Since $F_n \to F$ uniformly on compacta of $\mathbb{D}$, we necessarily have that $F_n \to F$ uniformly on K. Hence, it follows that $t_{F_n\circ\gamma} \to t_{F\circ\gamma}$. Furthermore,

$$\begin{aligned}
&\sup_{0\le s\le 1} |F \circ \gamma(t_{F\circ\gamma} s) - F_n \circ \gamma(t_{F_n\circ\gamma} s)| \\
&\quad \le \sup_{0\le s\le 1} |F \circ \gamma(t_{F\circ\gamma} s) - F \circ \gamma(t_{F_n\circ\gamma} s)| + |F \circ \gamma(t_{F_n\circ\gamma} s) - F_n \circ \gamma(t_{F_n\circ\gamma} s)| \to 0.
\end{aligned}$$

Taken together, these imply the result. □

5.2 Construction of approximate domains $\tilde{D}_N$

Suppose that $D \in \mathcal{D}^*$ with $\mathrm{inrad}(D) = 1$, and let

$$D_N'' := \{x \in \frac{1}{N}\mathbb{Z}^2 \cap D : \frac{1}{N}\mathcal{S}_x \subset D\},$$

where $\mathcal{S}_x := x + (\,[-1/2, 1/2] \times [-1/2, 1/2]\,)$ as in (2.1). Let D_N' be the connected component of D_N'' containing the origin, and set $D_N := D_N' \setminus \partial_i D_N'$. As in Section 2.1, take $\tilde{D}_N \subset \mathbb{C}$ to be the interior of the union of the scaled squares centred at those $x \in D_N$. We call D_N the $1/N$*-scale discrete approximation to* D (with respect to the origin), and we informally refer to $\tilde{D}_N$ as the associated "union of squares" domain;

that is,

$$\tilde{D}_N = \operatorname{int}\left(\bigcup_{x \in D_N} \frac{1}{N}\,\mathcal{S}_x\right) \quad \text{and} \quad \operatorname{cl}(\tilde{D}_N) := \tilde{D}_N \cup \partial \tilde{D}_N = \bigcup_{x \in D_N} \frac{1}{N}\,\mathcal{S}_x.$$

Let $f \in \mathcal{T}(\mathbb{D}, D)$ with $f(0) = 0$, $f'(0) > 0$. Let $\Gamma_{\mathbb{D}}$, $\Upsilon_{\mathbb{D}} \subset \partial\mathbb{D}$ be (open) boundary arcs with $\overline{\Gamma_{\mathbb{D}}} \cap \overline{\Upsilon_{\mathbb{D}}} = \emptyset$; that is,

$$\Gamma_{\mathbb{D}} := \{e^{i\theta} : \theta_1 < \theta < \theta_2\} \quad \text{and} \quad \Upsilon_{\mathbb{D}} := \{e^{i\theta} : \theta_3 < \theta < \theta_4\},$$

for some $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_1 + 2\pi$. Define $\Gamma \subset \partial D$ to be the image of $\Gamma_{\mathbb{D}}$ under f, and similarly, let $\Upsilon \subset \partial D$ be the image of $\Upsilon_{\mathbb{D}}$ under f. Let $s := \operatorname{sep}(\Gamma, \Upsilon)$ as in Definition 2.9, and let N be sufficiently large so that $s \ge \varepsilon_n := n^{-1/48} \log^{2/3} n$ if $n \ge N$. If $f_N \in \mathcal{T}(\mathbb{D}, \tilde{D}_N)$ with $f_N(0) = 0$, $f_N'(0) > 0$, then define $\tilde{\Gamma}_N$ to be the image of $\Gamma_{\mathbb{D}}$ under f_N, with $\tilde{\Upsilon}_N$ defined similarly. In Theorem 5.3.4, we prove $f_N \to f$ uniformly on compacta of $\mathbb{D}$ showing $\tilde{D}_N \overset{\text{Cara}}{\to} D$.

We now define our approximating discrete boundary arcs. If $\tilde{\Gamma}_N \subset \partial \tilde{D}_N$, then associate to $\tilde{\Gamma}_N$ the set $\Gamma_N \subset \partial D_N$ as follows. Let $\Gamma_N' := \{x \in \partial_i D_N : \frac{1}{N}\,\mathcal{S}_x \cap \tilde{\Gamma}_N \ne \emptyset\}$, and then take

$$\Gamma_N := \{y \in \partial D_N : (x, y) \in \partial_e D_N \text{ with } x \in \Gamma_N' \text{ and } \frac{1}{N}\,\ell_{x,y} \subset \tilde{\Gamma}_N\}.$$

Similarly, let Υ_N be the discrete boundary arc associated to $\tilde{\Upsilon}_N$.

Our notation is summarized in the following table.

$\mathbb{D} \subset \mathbb{C}$	$D \subset \mathbb{C}$, $D \in \mathcal{D}^*$	$\tilde{D}_N \subset \mathbb{C}$, $\tilde{D}_N \in \mathcal{D}$	$D_N \subset \frac{1}{N}\,\mathbb{Z}^2$, $2ND_N \in \mathcal{A}^N$
$\Gamma_{\mathbb{D}}, \Upsilon_{\mathbb{D}} \subset \partial\mathbb{D}$	$\Gamma, \Upsilon \subset \partial D$	$\tilde{\Gamma}_N, \tilde{\Upsilon}_N \subset \partial \tilde{D}_N$	$\Gamma_N, \Upsilon_N \subset \partial D_N$

Remark. By conformal invariance, it is equivalent to specify either Γ, $\Upsilon \subset \partial D$, or $\Gamma_{\mathbb{D}}$, $\Upsilon_{\mathbb{D}} \subset \partial\mathbb{D}$. We have (arbitrarily) chosen the latter.

5.3 Convergence of domains $\tilde{D}_N$ to D

Suppose that $D \in \mathcal{D}^*$ with $\text{inrad}(D) = 1$ and let D_N be the $1/N$-scale discrete approximation to D with associated "union of squares" domain $\tilde{D}_N$ as in Section 5.2. The following facts are an immediate consequence of those definitions.

Lemma 5.3.1. *For each N, $\tilde{D}_N \in \mathcal{D}$ with $\text{cl}(\tilde{D}_N) \subset D$. That is, $\tilde{D}_N$ is a simply connected proper subset of D with piecewise analytic boundary. Furthermore, the lattice* $\text{cl}(D_N) := D_N \cup \partial D_N \subset D$.

Lemma 5.3.2. *If $x \in \partial_i D_N$, $y \in \partial D_N$, and $z \in \partial \tilde{D}_N$, then* $\text{dist}(x, \partial D) \le c_1 N^{-1}$, $\text{dist}(y, \partial D) \le c_2 N^{-1}$, *and* $\text{dist}(z, \partial D) \le c_3 N^{-1}$ *where $c_1 = 2\sqrt{2} + 1/\sqrt{2}$, $c_2 = \sqrt{2} + 1/\sqrt{2}$, and $c_3 = 2\sqrt{2}$.*

Proposition 5.3.3. *If $x \in \partial_i D_N$ and $f \in \mathcal{T}(D, \mathbb{D})$ with $f(0) = 0$, $f'(0) > 0$, then there exists a constant C such that*

$$\text{dist}(f(x), \partial \mathbb{D}) \le \frac{C}{\sqrt{N}} \quad \textit{and} \quad f(\tilde{D}_N) \supseteq \{|z| \le 1 - \frac{C}{\sqrt{N}}\}.$$

Proof. This is an immediate application of the Beurling estimate; see [32, Proposition 3.8.10]. □

We can now state the main result of this section.

Theorem 5.3.4. *The sets $\tilde{D}_N$ converge to D in the Carathéodory sense.*

The proof of this theorem requires two lemmas. The first is a simple power series estimate, while the second gives good bounds on the difference of the image of a point under two different maps: the identity map from $\mathbb{D}$ to $\mathbb{D}$, and a map which is "almost the identity."

Lemma 5.3.5. *If* $0 \le |z| \le 1/2$*, then* $|\log(1+z) - z| \le |z|/2$*.*

Proof. Since

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} z^n,$$

we have

$$|\log(1+z) - z| \le \sum_{n=2}^{\infty} \frac{1}{n} |z|^n \le \frac{1}{2} |z| \sum_{n=1}^{\infty} |z|^n \le \frac{1}{2} |z|$$

provided that $0 \le |z| \le 1/2$. □

Lemma 5.3.6. *For* $N > 4C^2$*, where* C *is the constant in Lemma 5.3.3, suppose that* E_N *is a domain with* $\{|z| \le 1 - \frac{C}{\sqrt{N}}\} \subseteq E_N \subseteq \{|z| \le 1 + \frac{C}{\sqrt{N}}\}$*. Let* $h_N : \mathbb{D} \to E_N$ *be the conformal transformation with* $h_N(0) = 0$ *and* $h_N'(0) > 0$*. Then, there exists a constant* C' *such that for* $|z| \le 1 - \frac{C}{\sqrt{N}}$*,*

$$|h_N(z) - z| \le \frac{C' \log N}{\sqrt{N}}.$$

Proof. Without loss of generality, assume that h_N may be extended to an analytic function in a neighbourhood of $\overline{\mathbb{D}}$. For if this is not the case, we may approximate h_N by $h_{N,r}(z) := r^{-1} h_N(rz)$ and take the limit as $r \to 1-$. From the Schwarz lemma [2, page 135], we can immediately see that $1 - \frac{C}{\sqrt{N}} \le h_N'(0) \le 1 + \frac{C}{\sqrt{N}}$. Let

$$\kappa_N(z) := \log \left[\frac{h_N(z)}{z} \right]$$

so that $\kappa_N = u_N + iv_N$ is analytic on $\mathbb{D}$ with $|u_N(z)| \le (3/2)CN^{-1/2}$ for $|z| = 1$ using the estimate from Lemma 5.3.5. Thus, the maximum principle for harmonic functions tells us that $|u_N(z)| \le (3/2)CN^{-1/2}$ for all $|z| \le 1$. We therefore conclude that the partial derivatives of u_N at z are bounded by an absolute constant times

$N^{-1/2}\,\mathrm{dist}(z,\partial\mathbb{D})^{-1}$; whence $|\kappa_N'(z)| \le C_1 N^{-1/2}(1-|z|)^{-1}$. Writing

$$\left|\log\left[1+\frac{h_N(z)-z}{z}\right]\right| = |\kappa_N(z)| = \left|\kappa_N(0)+\int_0^z \kappa_N'(w)\,\mathrm{d}w\right| \le \frac{C_2}{\sqrt{N}}\left[1+\log\frac{1}{1-|z|}\right]$$

with $C_2 = \max\{C, C_1\}$, we see that if $\varepsilon > 0$ is such that

$$\left|\frac{h_N(z)-z}{z}\right| \le \frac{1}{2}, \qquad |z| \le \varepsilon, \tag{5.1}$$

then

$$\left|\frac{h_N(z)-z}{z}\right| \le 2\left|\log\left[1+\frac{h_N(z)-z}{z}\right]\right| \le 2C_2N^{-1/2}\left[1+\log\frac{1}{1-|z|}\right]. \tag{5.2}$$

Since (5.1) holds for some $\varepsilon > 0$, we can iterate (5.2) to see that (5.2) must hold for all $|z|$ such that the right side of (5.2) is less than 1/2. For N sufficiently large, this includes all $|z| \le 1 - CN^{-1/2}$. □

Proof of Theorem 5.3.4. Suppose that $f : D \to \mathbb{D}$ is the conformal transformation with $f(0) = 0$, $f'(0) > 0$, and let $\tilde{f}_N : f(\tilde{D}_N) \to \mathbb{D}$ be the conformal transformation with $\tilde{f}_N(0) = 0$, $\tilde{f}_N'(0) > 0$. Let $F_N : \mathbb{D} \to \tilde{D}_N$ and $F : \mathbb{D} \to D$ be the conformal transformations with $F_N(0) = 0$, $F_N'(0) > 0$, and $F(0) = 0$, $F'(0) > 0$, respectively, which are defined by setting $F_N = (\tilde{f}_N \circ f)^{-1}$ and $F = f^{-1} = (I \circ f)^{-1}$ where $I(z) = z$ is the identity map from $\mathbb{D}$ to $\mathbb{D}$. Finally, let $z \in \mathbb{D}$, and let $w = \tilde{f}_N^{-1}(z)$ so that $F_N(z) = F(w)$.

We prove that $\tilde{D}_N \overset{\text{Cara}}{\to} D$ by applying Theorem 5.1.3 which states that it is sufficient to show $F_N \to F$ uniformly on each compact subset of $\mathbb{D}$. Equivalently, we will show that for each $\delta > 0$ sufficiently small, $F_N \to F$ uniformly for all $|z| \le 1-\delta$. Fix $0 < \delta < 1/2$ and choose M so that $M > (3C'\delta^{-1})^3$ where C' is the constant in

Lemma 5.3.6. Let $N > M$. Then by Lemma 5.3.6, we have that for $|z| \le 1-\delta$,

$$|w-z| \le \frac{C' \log N}{\sqrt{N}} |z| \le \left(\frac{C' \log N}{\sqrt{N}} \cdot \frac{1-\delta}{\delta} \right) \delta.$$

Our choice of M guarantees that $\left(\frac{C' \log N}{\sqrt{N}} \cdot \frac{1-\delta}{\delta} \right) < 1$ for $N > M$. By [32, Corollary 3.2.9], if for some $0 < r < 1$, $|w-z| \le r \operatorname{dist}(z, \partial\mathbb{D})$, then

$$|F(w) - F(z)| \le \frac{4 \operatorname{dist}(F(z), \partial D)}{1-r^2} |w-z|.$$

Hence, we conclude

$$|F_N(z) - F(z)| = |F(w) - F(z)| \le \left(\frac{4RC'(1-\delta)}{1 - \left(\frac{C' \log N}{\sqrt{N}} \cdot \frac{1-\delta}{\delta} \right)^2} \right) \cdot \frac{\log N}{\sqrt{N}}$$

where $R = \operatorname{rad}(D)$ so that $F_N \to F$ uniformly; whence $\tilde{D}_N \overset{\text{Cara}}{\to} D$. □

Corollary 5.3.7. *If $F \in \mathcal{T}(D, \mathbb{D})$ with $F(0) = 0$, $F'(0) > 0$, then $F(\tilde{D}_N) \overset{Cara}{\to} \mathbb{D}$.*

Proof. By Lemma 5.3.1, $\tilde{D}_N \subset D$, so Proposition 5.1.4 yields the result. □

5.4 Applying results for $A \in \mathcal{A}^N$ to D_N

Suppose that $D \in \mathcal{D}^*$ with $\operatorname{inrad}(D) = 1$. In this section, we combine our construction of D_N with Proposition 3.2.3 and Corollary 3.5.3 to restate those results for random walk on D_N. The most difficult part of this section is keeping track of the notation.

We begin by mentioning several scaling relationships that will be needed throughout. If S_n is a random walk on $\mathbb{Z}^2$, then for any $r > 0$ there is an associated random walk (which we will also denote by S_n) on the lattice $r\mathbb{Z}^2$. In other words, there is a one-to-one correspondence between paths from x to y in A on $\mathbb{Z}^2$, and paths from rx to ry in rA on $r\mathbb{Z}^2$. Hence if $A \subset \mathbb{Z}^2$ and $r > 0$, then $G_{rA}(rx, ry) = G_A(x, y)$, where the Green's function on the left side is for random walk on the lattice $r\mathbb{Z}^2$, and the Green's function on the right side is for random walk on $\mathbb{Z}^2$. Similarly, we have $h_{rA}(rx, ry) = h_A(x, y)$ for the discrete Poisson kernel, and $h_{\partial rA}(rx, ry) = h_{\partial A}(x, y)$ for the discrete excursion Poisson kernel.

From the conformal invariance of the Green's function for Brownian motion (Proposition 2.2.1), it follows that if $D \in \mathcal{D}^*$ and $r > 0$, then $g_{rD}(rx, ry) = g_D(x, y)$. However, from the conformal covariance of the Poisson kernel (Proposition 2.6.3) and the excursion Poisson kernel (Proposition 2.7.7), it follows that

$$rH_{rD}(rx, ry) = H_D(x, y) \quad \text{and} \quad r^2 H_{\partial rD}(rx, ry) = H_{\partial D}(x, y).$$

Note that a random walk on D_N is taking steps of size $1/N$. Therefore, let $A_N := 2ND_N$ so that $A_N \in \mathcal{A}^N$, and $\tilde{A}_N := (\widetilde{2ND_N}) = 2N\tilde{D}_N \in \mathcal{D}$. Hence, $z' \in A_N$ if and only if $z := z'/2N \in D_N$. Suppose $x' = 2Nx \in A^N$ with $x \in D_N$ and $y' = 2Ny \in A^N$ with $y \in D_N$. Thus, when the above scaling is applied to $\tilde{A}_N$, we conclude that

$$g_{A_N}(x', y') = g_{2ND_N}(2Nx, 2Ny) = g_{D_N}(x, y). \tag{5.3}$$

Recall that we write $g_{D_N} := g_{\tilde{D}_N}$, as in Section 2.2. In particular, if $f_{D_N} \in \mathcal{T}(\tilde{D}_N, \mathbb{D})$ with $f_{D_N}(0) = 0$, $f'_{D_N}(0) > 0$, and $f_{A_N} \in \mathcal{T}(\tilde{A}_N, \mathbb{D})$ with $f_{A_N}(0) = 0$, $f'_{A_N}(0) > 0$, then since $f_{A_N}(x') = f_{D_N}(x)$, $g_{A_N}(x') = g_{D_N}(x)$, and we can write $f_{A_N}(x') = \exp\{-g_{A_N}(x') + i\theta_{A_N}(x')\}$ and $f_{D_N}(x) = \exp\{-g_{D_N}(x) + i\theta_{D_N}(x)\}$, it follows that $\theta_{A_N}(x') = \theta_{2ND_N}(2Nx) = \theta_{D_N}(x)$. Furthermore, in the random walk case,

$$G_{A_N}(x', y') = G_{2ND_N}(2Nx, 2Ny) = G_{D_N}(x, y),$$

and for $x' = 2Nx \in \partial A_N$ with $x \in \partial D_N$, we have

$$h_{\partial A_N}(x', y') = h_{\partial 2ND_N}(2Nx, 2Ny) = h_{\partial D_N}(x, y);$$

similarly, $h_{A_N}(0, x') = h_{D_N}(0, x)$, and $h_{A_N}(0, y') = h_{D_N}(0, y)$.

For $A_N \in \mathcal{A}^N$, let $A_N^* = \{x' \in A_N : g_{A_N}(x') \geq N^{-1/16}\}$ which is consistent with the usage in (3.2). If $x' \in A_N^*$, $y' \in A_N$, then Proposition 3.2.3 implies that

$$G_{A_N}(x', y') = \frac{2}{\pi} g_{A_N}(x', y') + k_{y'-x'} + O(N^{-7/24} \log N).$$

With the above notation in hand, we are finally able to state the following corollaries to Proposition 3.2.3.

Corollary 5.4.1. *Let $x \in D_N$ be such that $x' := 2Nx \in (2ND_N)^{*,N} =: A_N^*$, and let $y \in D_N$ with $y' = 2Ny \in A_N$. Then,*

$$G_{D_N}(x, y) = (2/\pi)\, g_{D_N}(x, y) + k_{y'-x'} + O(N^{-7/24} \log N)$$

where k_z is as in Proposition 3.2.2.

Note that $k_{y'-x'} \leq cN^{-3/2}|x-y|^{-3/2}$. Thus, if $|x-y| \geq N^{-29/36}$, then $k_{y'-x'} = O(N^{-7/24})$, and we have a refined version of the previous corollary.

Corollary 5.4.2. *If $x \in D_N$ with $x' := 2Nx \in A_N^*$, $y \in D_N$ with $y' = 2Ny \in A_N$, and $|x-y| \geq N^{-29/36}$, then*

$$G_{D_N}(x, y) = \frac{2}{\pi} g_{D_N}(x, y) + O(N^{-7/24} \log N).$$

We also have the following corollary to Corollary 3.5.3.

Corollary 5.4.3. *There exists a decreasing sequence $\varepsilon_N \downarrow 0$ such that if $D \in \mathcal{D}^*$ with $\mathrm{inrad}(D) = 1$, D_N is the $1/N$-scale discrete approximation to D, and $x,\, y \in \partial D_N$ with $|\theta_{D_N}(x) - \theta_{D_N}(y)| \ge \varepsilon_N$, then*

$$h_{\partial D_N}(x,y) = \frac{(\pi/2)\; h_{D_N}(0,x)\; h_{D_N}(0,y)}{1-\cos(\theta_{D_N}(x)-\theta_{D_N}(y))}\;[1 + O(\frac{\varepsilon_N^3}{|\theta_{D_N}(x)-\theta_{D_N}(y)|})].$$

We now make several observations regarding excursion measure. Suppose $x,\, y \in \partial\tilde{D}_N$ so that $x' := 2Nx,\ y' := 2Ny \in \partial\tilde{A}_N$ as above. If $f(z) = 2Nz$, then $f \in \mathcal{T}(\tilde{D}_N, \tilde{A}_N)$ with $f(0) = 0$ and $f'(z) = 2N$ for all z. Since excursion measure is conformally covariant/invariant, we are able to conclude that

$$\mu_{\partial\tilde{D}_N}(x,y) = 4N^2 \mu_{\partial\tilde{A}_N}(x',y'),\ \ \mu_{\partial\tilde{D}_N}(x) = 2N\mu_{\partial\tilde{A}_N}(x'), \quad \text{and} \quad \mu_{\partial\tilde{D}_N} = \mu_{\partial\tilde{A}_N}.$$

Recall from Chapter 1 that simple random walk converges in the scaling limit to Brownian motion provided we scale space and time appropriately. In order to prove discrete excursion measure converges to Brownian excursion measure, we will need to apply a similar scaling. Recall from (4.30) that if ω is a discrete excursion then we can associate to it a curve $\tilde{\omega} \in \mathcal{K}$. Furthermore, recall that the Brownian scaling map $\mathfrak{T}_a$ was defined in the example on page 63. For $N \in \mathbb{N}$, write $\Phi_N := \mathfrak{T}_{1/(2N)}$ so that

$$\Phi_N\tilde{\omega}(t) = \frac{1}{2N}\,\tilde{\omega}(4N^2 t) \quad \text{for} \quad 0 \le t \le t_{\Phi_N\tilde{\omega}} = \frac{t_{\tilde{\omega}}}{4N^2} = \frac{|\omega|}{2N^2}. \tag{5.4}$$

Lemma 5.4.4. *Suppose that $\gamma,\, \gamma' \in \mathcal{K}$. If $\Phi_N := \mathfrak{T}_{1/(2N)}$ is the Brownian scaling map defined above, then*

$$\frac{1}{4N^2}\,\mathbb{d}(\gamma,\gamma') \le \mathbb{d}(\Phi_N\gamma, \Phi_N\gamma') \le \frac{1}{2N}\,\mathbb{d}(\gamma,\gamma').$$

Proof. From the definitions of $\mathbb{d}$ and Φ_N we conclude that

$$\begin{aligned}\mathbb{d}(\Phi_N\gamma, \Phi_N\gamma') &= \sup_{0\le s\le 1} |\Phi_N\gamma(st_{\Phi_N\gamma}) - \Phi_N\gamma'(st_{\Phi_N\gamma'})| + |t_{\Phi_N\gamma} - t_{\Phi_N\gamma'}| \\ &= \sup_{0\le s\le 1} |\frac{1}{2N}\gamma(st_\gamma) - \frac{1}{2N}\gamma'(st_{\gamma'})| + \frac{1}{4N^2}|t_\gamma - t_{\gamma'}|\end{aligned}$$

so the result follows. □

Definition 5.4.5. Suppose that $D \in \mathcal{D}^*$ with $\operatorname{inrad}(D) = 1$, D_N is the $1/N$-scale discrete approximation to D, and $x,\ y \in \partial D_N$. The *$1/N$-scale discrete excursion measure* $\mu^{\mathsf{rw}}_{\partial D_N}(x,y)$ is defined to be the measure on $(\mathcal{K}, \mathbb{d})$, concentrated on $V_N = V_N(x,y;D) := \{\gamma \in \mathcal{K} : \mathbb{d}(\gamma, \Phi_N\tilde{\omega}) = 0$ for some discrete excursion ω from $2Nx$ to $2Ny$ in $2ND_N\}$ given by $\mu^{\mathsf{rw}}_{\partial D_N}(x,y)(\gamma) := 4^{-4N^2 t_\gamma} = 4^{-|\omega|}$ for $\gamma \in V_N$.

As in (4.29), if $\Gamma_N,\ \Upsilon_N \subset \partial D_N$ with $\overline{\Gamma_N} \cap \overline{\Upsilon_N} = \emptyset$, then

$$\mu^{\mathsf{rw}}_{\partial D_N}(\Gamma_N, \Upsilon_N) := \sum_{x\in\Gamma_N}\sum_{y\in\Upsilon_N} \mu^{\mathsf{rw}}_{\partial D_N}(x,y).$$

5.5 The principal theorem

We now state the main result of this dissertation, namely that simple random walk excursion measure converges to Brownian excursion measure for $D \in \mathcal{D}^*$.

Note that the limit in **(a)** is a limit of real numbers, while the limit in **(b)** is taken in the Prohorov metric of Section 4.2.1. The proof of this result will be spread over the next several sections: in Section 5.6 we prove Theorem 5.6.3 establishing **(a)**, in Section 5.7 we prove Theorem 5.7.1 establishing **(b)**, and finally in Section 5.8 we prove Theorem 5.8.1 establishing **(c)**.

Theorem 5.5.1. *Suppose $D \in \mathcal{D}^*$ with* $\operatorname{inrad}(D) = 1$*, and let Γ, $\Upsilon \subset \partial D$ be open boundary arcs with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. For every $\varepsilon > 0$, there exists an N such that*

(a) $$\left| h_{\partial D_N}(\Gamma_N, \Upsilon_N) - \frac{1}{4} H_{\partial D}(\Gamma, \Upsilon) \right| \le \varepsilon,$$

(b) $$\wp(\, \mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N),\ \mu^{\#}_{\partial D}(\Gamma, \Upsilon)\,) \le \varepsilon, \text{ and}$$

(c) $$\wp(\, \mu^{\mathsf{rw},\#}_{\partial D_N}(\Gamma_N, \Upsilon_N),\ \mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)\,) \le \varepsilon,$$

where D_N is the $1/N$-scale discrete approximation to D, $\tilde{D}_N \in \mathcal{D}$ is the "union of squares" domain associated to D_N, and Γ_N, $\Upsilon_N \subset D_N$ are the corresponding discrete boundary arcs with associated boundary arcs $\tilde{\Gamma}_N$, $\tilde{\Upsilon}_N \subset \partial \tilde{D}_N$, respectively. In particular, **(a)**, **(b)**, *and* **(c)** *imply that*

$$\lim_{N \to \infty} \wp(\, 4\, \mu^{\mathsf{rw}}_{\partial D_N}(\Gamma_N, \Upsilon_N),\ \mu_{\partial D}(\Gamma, \Upsilon)\,) = 0. \tag{5.5}$$

Remark. Notice that in **(a)** we will prove directly that $4 h_{\partial D_N}(\Gamma_N, \Upsilon_N) \to H_{\partial D}(\Gamma, \Upsilon)$ without establishing any estimates relating $h_{\partial D_N}(\Gamma_N, \Upsilon_N)$ to $H_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)$. This is in contrast to **(b)** and **(c)** where it is vital to establish this second kind of estimate. It is a direct consequence of the definitions of Section 5.2 and Proposition 4.7.2 that $H_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N) = H_{\partial D}(\Gamma, \Upsilon)$, thus eliminating the need for the intermediate step.

Although excursion measure $\mu_{\partial D}$ is an infinite measure, its restriction $\mu_{\partial D}(\Gamma, \Upsilon)$ to any pair of disjoint boundary arcs Γ, Υ is finite. Theorem 5.5.1 implies that *for any* such pair $\wp(4\mu^{\mathsf{rw}}_{\partial D_N}(\Gamma_N, \Upsilon_N), \mu_{\partial D}(\Gamma, \Upsilon)) \to 0$ which is, in fact, a reasonable way to *define* Prohorov convergence of the measures $\mu^{\mathsf{rw}}_{\partial D_N}$ to the infinite (but σ-finite) measure $\mu_{\partial D}$. Hence, the conclusion (5.5) can be re-formulated as follows.

Theorem 5.5.2. *If $D \in \mathcal{D}^*$ with* $\operatorname{inrad}(D) = 1$ *and D_N is the $1/N$-scale discrete approximation to D, then $\wp(\, 4\, \mu^{\mathsf{rw}}_{\partial D_N},\ \mu_{\partial D}\,) \to 0$.*

5.6 Convergence of $4h_{\partial D_N}(\Gamma_N, \Upsilon_N)$ to $H_{\partial D}(\Gamma, \Upsilon)$

The goal of the present section is to prove that if $D \in \mathcal{D}^*$ with $\text{inrad}(D) = 1$, and Γ, $\Upsilon \subset \partial D$ are disjoint open boundary arcs, then $4h_{\partial D_N}(\Gamma_N, \Upsilon_N) \to H_{\partial D}(\Gamma, \Upsilon)$ using the notation from Section 5.2, therefore establishing Theorem 5.5.1 **(a)**. We begin with the following extension of Proposition 2.7.8.

Lemma 5.6.1. *If $D \in \mathcal{D}^*$ and Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$, then*

$$\frac{2\pi H_D(0,\Gamma)\, H_D(0,\Upsilon)}{1 - \cos(\text{spr}(\Gamma, \Upsilon))} \leq H_{\partial D}(\Gamma, \Upsilon) \leq \frac{2\pi H_D(0,\Gamma)\, H_D(0,\Upsilon)}{1 - \cos(\text{sep}(\Gamma, \Upsilon))}$$

where $H_{\partial D}(\Gamma, \Upsilon)$ is as in (2.30), and $H_D(0, \Gamma)$, $H_D(0, \Upsilon)$ are as in (2.28).

Proof. Suppose first that $D \in \mathcal{D}$, and that Γ, Υ are analytic open boundary arcs. Then from Proposition 2.7.8, we conclude that for all $x \in \Gamma$ and for all $y \in \Upsilon$,

$$\frac{2\pi H_D(0,x) H_D(0,y)}{1 - \cos(\text{spr}(\Gamma, \Upsilon))} \leq H_{\partial D}(x, y) \leq \frac{2\pi H_D(0,x) H_D(0,y)}{1 - \cos(\text{sep}(\Gamma, \Upsilon))}.$$

Since $D \in \mathcal{D}$, and Γ, Υ are analytic there is no difficulty integrating along Γ, Υ. Hence the conformal covariance of the excursion Poisson kernel (Proposition 2.7.7) implies

$$\frac{2\pi H_D(0,\Gamma)\, H_D(0,\Upsilon)}{1 - \cos(\text{spr}(\Gamma, \Upsilon))} \leq H_{\partial D}(\Gamma, \Upsilon) \leq \frac{2\pi H_D(0,\Gamma)\, H_D(0,\Upsilon)}{1 - \cos(\text{sep}(\Gamma, \Upsilon))}.$$

Now, suppose $D' \in \mathcal{D}^*$, and let $f \in \mathcal{T}(D, D')$. Write Γ', Υ' for the images under f of Γ, Υ, respectively. The conformal invariance of harmonic measure yields $H_D(0, \Gamma) = H_{D'}(0, \Gamma')$ and $H_D(0, \Upsilon) = H_{D'}(0, \Upsilon')$. (Indeed this holds for all domains $D \in \mathcal{D}^*$ since ∂D is regular.) From Proposition 4.7.2, we know that $H_{\partial D}(\Gamma, \Upsilon) = H_{\partial D'}(\Gamma', \Upsilon')$. Combining this with the previous part yields the result. $\square$

Let $f \in \mathcal{T}(\mathbb{D}, D)$ with $f(0) = 0$, $f'(0) > 0$. Analogous to Section 5.2, by rotating[1] if necessary, it is possible to find $0 \leq \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$ such that Γ, Υ, are the images under f of $\Gamma_{\mathbb{D}}$, $\Upsilon_{\mathbb{D}}$, respectively, where $\Gamma_{\mathbb{D}} := \{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ and $\Upsilon_{\mathbb{D}} := \{e^{i\theta'} : \theta_3 < \theta' < \theta_4\}$. Define the *length of* Γ, written ℓ_Γ, to be length of $\Gamma_{\mathbb{D}}$ so that $\ell_\Gamma := \theta_2 - \theta_1$. Note that while Γ may not even be rectifiable, we are defining length in the unit disk, where the length of a boundary arc *is* well-defined. Similarly define $\ell_\Upsilon := \theta_4 - \theta_3$.

The notion of length that we are using is really harmonic measure. The length of $\Gamma \subset \partial D$ is the probability that a Brownian motion exits D at Γ; that is, the harmonic measure of Γ. For the unit disk, harmonic measure and Lebesgue (arc length) measure correspond on $\partial\mathbb{D}$.

It then follows from Proposition 2.7.4 that

$$\frac{1-\cos(\theta_3-\theta_2)}{1-\cos(\theta_4-\theta_1)} = 1 + \frac{(\theta_2-\theta_1)^2+(\theta_4-\theta_3)^2}{(\theta_4-\theta_1)^2} + O\left(\frac{\theta_4-\theta_3}{\theta_4-\theta_1}\right) + O\left(\frac{\theta_2-\theta_1}{\theta_4-\theta_1}\right) + O\left(\frac{(\theta_4-\theta_3)(\theta_2-\theta_1)}{(\theta_4-\theta_1)^2}\right),$$

and so if $(\theta_3 - \theta_2)$, $(\theta_4 - \theta_1)$ are fixed, we conclude that

$$\frac{1-\cos(\theta_3-\theta_2)}{1-\cos(\theta_4-\theta_1)} = 1 + O(\theta_4-\theta_3) + O(\theta_2-\theta_1)$$

as $(\theta_4 - \theta_3) \to 0$, $(\theta_2 - \theta_1) \to 0$. In particular, this shows that as $\ell_\Upsilon \to 0$, $\ell_\Gamma \to 0$,

$$\frac{1-\cos(\operatorname{sep}(\Gamma,\Upsilon))}{1-\cos(\operatorname{spr}(\Gamma,\Upsilon))} = 1 + O(\ell_\Upsilon) + O(\ell_\Gamma). \tag{5.6}$$

Thus, we have proved the following lemma.

[1] Both the excursion Poisson kernel for $\mathbb{D}$ and excursion measure in $\mathbb{D}$ are rotationally invariant.

Lemma 5.6.2. *If $D \in \mathcal{D}^*$, then for any $\eta > 0$ there exist open boundary arcs Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$ such that*

$$1 \leq \frac{1 - \cos(\mathrm{spr}(\Gamma, \Upsilon))}{1 - \cos(\mathrm{sep}(\Gamma, \Upsilon))} \leq 1 + \eta.$$

Note that the lower bound holds automatically by the definitions of separation and spread.

Remark. Lemma 5.6.1 and (5.6) together imply that as $\ell_\Upsilon \to 0$, $\ell_\Gamma \to 0$,

$$H_{\partial D}(\Gamma, \Upsilon) = \frac{2\pi H_D(0, \Gamma) H_D(0, \Upsilon)}{1 - \cos(\mathrm{sep}(\Gamma, \Upsilon))} [1 + O(\ell_\Upsilon) + O(\ell_\Gamma)].$$

Theorem 5.6.3. *For every $D \in \mathcal{D}^*$ with* $\mathrm{inrad}(D) = 1$, *and for every pair of open boundary arcs Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$, if D_N is the $1/N$-scale discrete approximation to D, and Γ_N, Υ_N are the discrete approximations to Γ, Υ, respectively, as in Section 5.2, then $4h_{\partial D_N}(\Gamma_N, \Upsilon_N) \to H_{\partial D}(\Gamma, \Upsilon)$.*

Proof. Consider $D \in \mathcal{D}^*$, and let Γ, $\Upsilon \subset \partial D$ be (open) boundary arcs with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$. Find M so that $\mathrm{sep}(\Gamma, \Upsilon) \geq \varepsilon_N := N^{-1/48} \log^{2/3} N$ for $N \geq M$. Throughout this section, let $N \geq M$. Let D_N be the $1/N$-scale discrete approximation to D with associated "union of squares" domain $\tilde{D}_N$, and let $\tilde{\Gamma}_N$, $\tilde{\Upsilon}_N \subset \partial \tilde{D}_N$ with associated discrete boundary arcs Γ_N, $\Upsilon_N \subset \partial D_N$. From the definitions of separation and spread, and from Corollary 5.4.3, since Γ and Υ are fixed so that $\mathrm{sep}(\Gamma, \Upsilon) = O(1)$, it follows that

$$\begin{aligned} &\frac{(\pi/2)\, h_{D_N}(0, x)\, h_{D_N}(0, y)}{1 - \cos(\mathrm{spr}(\Gamma, \Upsilon))} [1 + O(\varepsilon_N^3)] \\ &\qquad \leq h_{\partial D_N}(x, y) \leq \frac{(\pi/2)\, h_{D_N}(0, x)\, h_{D_N}(0, y)}{1 - \cos(\mathrm{sep}(\Gamma, \Upsilon))} [1 + O(\varepsilon_N^3)]. \end{aligned}$$

Summing over all $x \in \Gamma_N$ and all $y \in \Upsilon_N$ yields

$$\frac{h_{D_N}(0,\Gamma_N)\; h_{D_N}(0,\Upsilon_N)}{1-\cos(\mathrm{spr}(\Gamma,\Upsilon))}\;[1+O(\varepsilon_N^3)]$$
$$\le (2/\pi)\; h_{\partial D_N}(\Gamma_N,\Upsilon_N) \le \frac{h_{D_N}(0,\Gamma_N)\; h_{D_N}(0,\Upsilon_N)}{1-\cos(\mathrm{sep}(\Gamma,\Upsilon))}\;[1+O(\varepsilon_N^3)].$$

where we have written $h_{\partial D_N}(\Gamma_N,\Upsilon_N) := \sum_{x\in\Gamma_N}\sum_{y\in\Upsilon_N} h_{\partial D_N}(x,y)$ and similarly for $h_{D_N}(0,\Gamma_N)$ and $h_{D_N}(0,\Upsilon_N)$. However, from Proposition 3.1.6,

$$h_{D_N}(0,\Gamma_N)\, h_{D_N}(0,\Upsilon_N) = H_{\tilde D_N}(0,\tilde\Gamma_N)\, H_{\tilde D_N}(0,\tilde\Upsilon_N) + O(\delta_N),$$

where $\delta_N := N^{-7/8}\log N$, so that we conclude

$$[\frac{H_{\tilde D_N}(0,\tilde\Gamma_N)\, H_{\tilde D_N}(0,\tilde\Upsilon_N)}{1-\cos(\mathrm{spr}(\Gamma,\Upsilon))} + O(\delta_N)]\;[1+O(\varepsilon_N^3)]$$
$$\le (2/\pi)\; h_{\partial D_N}(\Gamma_N,\Upsilon_N) \le [\frac{H_{\tilde D_N}(0,\tilde\Gamma_N)\, H_{\tilde D_N}(0,\tilde\Upsilon_N)}{1-\cos(\mathrm{sep}(\Gamma,\Upsilon))} + O(\delta_N)]\;[1+O(\varepsilon_N^3)].$$

Now, as we let $N\to\infty$, it follows that

$$\frac{H_D(0,\Gamma)\, H_D(0,\Upsilon)}{1-\cos(\mathrm{spr}(\Gamma,\Upsilon))} \le (2/\pi)\;\liminf_{N\to\infty} h_{\partial D_N}(\Gamma_N,\Upsilon_N)$$
$$\le (2/\pi)\;\limsup_{N\to\infty} h_{\partial D_N}(\Gamma_N,\Upsilon_N) \le \frac{H_D(0,\Gamma)\, H_D(0,\Upsilon)}{1-\cos(\mathrm{sep}(\Gamma,\Upsilon))}.$$

However, Lemma 5.6.1 implies that

$$\frac{1-\cos(\mathrm{sep}(\Gamma,\Upsilon))}{1-\cos(\mathrm{spr}(\Gamma,\Upsilon))} H_{\partial D}(\Gamma,\Upsilon) \le 4\;\liminf_{N\to\infty} h_{\partial D_N}(\Gamma_N,\Upsilon_N)$$
$$\le 4\;\limsup_{N\to\infty} h_{\partial D_N}(\Gamma_N,\Upsilon_N) \le \frac{1-\cos(\mathrm{spr}(\Gamma,\Upsilon))}{1-\cos(\mathrm{sep}(\Gamma,\Upsilon))} H_{\partial D}(\Gamma,\Upsilon).$$

For any $\eta > 0$, let $\{\Gamma_i\}$, $\{\Upsilon_j\}$ be finite partitions of Γ, Υ, respectively, with

$$1 \leq \frac{1-\cos(\mathrm{spr}(\Gamma_i,\Upsilon_j))}{1-\cos(\mathrm{sep}(\Gamma_i,\Upsilon_j))} \leq 1+\eta.$$

Note that such a partitioning is possible by Lemma 5.6.2. Hence, the equation above becomes

$$\begin{aligned}\frac{1-\cos(\mathrm{sep}(\Gamma_i,\Upsilon_j))}{1-\cos(\mathrm{spr}(\Gamma_i,\Upsilon_j))} H_{\partial D}(\Gamma_i,\Upsilon_j) &\leq 4 \liminf_{N\to\infty} h_{\partial D_N}(\Gamma_{N,i},\Upsilon_{N,j}) \\ &\leq 4 \limsup_{N\to\infty} h_{\partial D_N}(\Gamma_{N,i},\Upsilon_{N,j}) \leq \frac{1-\cos(\mathrm{spr}(\Gamma_i,\Upsilon_j))}{1-\cos(\mathrm{sep}(\Gamma_i,\Upsilon_j))} H_{\partial D}(\Gamma_i,\Upsilon_j).\end{aligned}$$

Summing over i and j and noting that

$$\sum_i \sum_j H_{\partial D}(\Gamma_i,\Upsilon_j) = H_{\partial D}(\Gamma,\Upsilon), \quad \text{and} \quad \sum_i \sum_j h_{\partial D_N}(\Gamma_{N,i},\Upsilon_{N,j}) = h_{\partial D_N}(\Gamma_N,\Upsilon_N)$$

since $\{\Gamma_{N,i}\}$, $\{\Upsilon_{N,j}\}$ partition $\{\Gamma_N\}$, $\{\Upsilon_N\}$, respectively, gives

$$\begin{aligned}(1+\eta)^{-1} H_{\partial D}(\Gamma,\Upsilon) &\leq 4 \liminf_{N\to\infty} h_{\partial D_N}(\Gamma_N,\Upsilon_N) \\ &\leq 4 \limsup_{N\to\infty} h_{\partial D_N}(\Gamma_N,\Upsilon_N) \leq (1+\eta) H_{\partial D}(\Gamma,\Upsilon).\end{aligned}$$

Since $\eta > 0$ was arbitrary, we conclude that $4h_{\partial D_N}(\Gamma_N,\Upsilon_N) \to H_{\partial D}(\Gamma,\Upsilon)$ as $N \to \infty$, as required. □

5.7 Convergence of $\mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)$ to $\mu^{\#}_{\partial D}(\Gamma,\Upsilon)$

We now prove Theorem 5.5.1 **(b)** via a result which basically says that an excursion in D can be thought of as an excursion in $\tilde{D}_N$ with Brownian tails.

Theorem 5.7.1. *For every* $D \in \mathcal{D}^*$ *with* $\text{inrad}(D) = 1$*, and for every pair of open boundary arcs* Γ*,* $\Upsilon \subset \partial D$ *with* $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$*,*

$$\lim_{N\to\infty} \wp(\, \mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N),\, \mu^{\#}_{\partial D}(\Gamma, \Upsilon)\,) = 0 \tag{5.7}$$

where D_N *is the* $1/N$ *discrete approximation to* D *with associated domain* $\tilde{D}_N \in \mathcal{D}$*, and corresponding boundary arcs* $\tilde{\Gamma}_N$*,* $\tilde{\Upsilon}_N \subset \partial \tilde{D}_N$ *as in Section 5.2.*

By conformal invariance, we can define excursion measure $\mu^{\#}_{\partial D}(\Gamma, \Upsilon)$ to be either the measure $f \circ \mu^{\#}_{\partial \mathbb{D}}(\Gamma_{\mathbb{D}}, \Upsilon_{\mathbb{D}})$ for $f \in \mathcal{T}(\mathbb{D}, D)$, or $\mu_{\partial D}$ restricted to those excursions $\gamma \in \mathcal{K}^{\Upsilon}_{\Gamma}(D)$ (and normalized by $H_{\partial D}(\Gamma, \Upsilon)$). Also using conformal invariance, we have $\mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N) = f_N \circ \mu^{\#}_{\partial \mathbb{D}}(\Gamma_{\mathbb{D}}, \Upsilon_{\mathbb{D}})$ for $f_N \in \mathcal{T}(\mathbb{D}, \tilde{D}_N)$, so that we conclude

$$\mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N) = (f_N \circ f^{-1}) \circ \mu^{\#}_{\partial D}(\Gamma, \Upsilon). \tag{5.8}$$

As noted in the remark on page 104, in order to show the convergence of the masses $h_{\partial D_N}(\Gamma_N, \Upsilon_N)$ to $H_{\partial D}(\Gamma, \Upsilon)$, the intermediate step of showing

$$\lim_{N\to\infty} H_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N) = H_{\partial D}(\Gamma, \Upsilon) \tag{5.9}$$

is unnecessary because, as explicitly seen in (5.8), of the conformal invariance of the excursion Poisson kernel: $H_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N) = H_{\partial D}(\Gamma, \Upsilon)$. However, in contrast to the excursion Poisson kernel, it is not simply a matter of applying the conformal invariance of excursion measure to conclude that (cf. Proposition 5.1.6)

$$\wp(\, (f_N \circ f^{-1}) \circ \mu^{\#}_{\partial D}(\Gamma, \Upsilon)\,,\, \mu^{\#}_{\partial D}(\Gamma, \Upsilon)\,) \to 0. \tag{5.10}$$

Proof of Theorem 5.7.1. Suppose that $D \in \mathcal{D}^*$ with $\operatorname{inrad}(D) = 1$, and associated "union of squares" domain $\tilde{D}_N$. As mentioned in Lemma 5.3.2, if $z \in \partial \tilde{D}_N$, then $\operatorname{dist}(z, \partial D) \le 2\sqrt{2}\, N^{-1}$. It follows from the Beurling estimates that Brownian motion started at z is likely to exit D quickly and nearby; that is,

$$\mathbb{P}^z\{\operatorname{diam} B[0,T_D] \ge N^{-1/2}\} \le CN^{-1/4} \quad \text{and} \quad \mathbb{P}^z\{T_D \ge N^{-1/2}\} \le CN^{-3/8}. \quad (5.11)$$

Unfortunately, if $z \in \tilde{\Gamma}_N$, it may be extremely unlikely that $\{B_{T_D} \in \Gamma\}$. This will be the case, for example, if z and Γ are on opposite sides of a "channel" (or "fjord"). However, since $\tilde{D}_N \overset{\text{Cara}}{\to} D$ by Theorem 5.3.4, for *fixed* $D \in \mathcal{D}^*$, fixed disjoint open boundary arcs Γ, Υ, and for every $\varepsilon > 0$, there exists an N such that $\max\{\operatorname{dist}(\tilde{\Gamma}_N, \Gamma), \operatorname{dist}(\tilde{\Upsilon}_N, \Upsilon)\} < \varepsilon$. The following is then a consequence of (5.11) and easy bounds on the Poisson kernel.

Lemma 5.7.2. *For every $\varepsilon > 0$, there exists an N such that for all $z \in \tilde{\Gamma}_N$,*

$$\mathbb{P}^z\{T_D \ge \varepsilon \;\; or \;\; \operatorname{diam} B[0,T_D] \ge \varepsilon \;\; or \;\; B_{T_D} \notin \Gamma_\varepsilon\} \le \varepsilon \quad (5.12)$$

where $\Gamma_\varepsilon := \{z \in \partial D : \operatorname{dist}(z, \Gamma) \le \varepsilon\}$.

Suppose that $\gamma : [0, t_\gamma] \to \mathbb{C}$ is a $(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)$-excursion in $\tilde{D}_N$. Let $b_2 : [0, t_{b_2}] \to \mathbb{C}$ be a Brownian motion started at $\gamma(t_\gamma)$ and stopped at $t_{b_2} := \inf\{t : b_2(t) \in D\}$, its hitting time of ∂D. Let $b' : [0, t_{b'}] \to \mathbb{C}$ be an independent Brownian motion started at $\gamma(0)$, stopped at $t_{b'} := \inf\{t : b'(t) \in D\}$, and set $b_1(t) := b'(t_{b'} - t)$. If $\zeta := b_1 \oplus \gamma \oplus b_2$, then by construction $\zeta : [0, t_\zeta] \to \mathbb{C}$ has $\zeta(0) \in \partial D$, $\zeta(t_\zeta) \in \partial D$, $0 < t_\zeta < \infty$, and $\zeta(0, t_\zeta) \subset D$. In other words, ζ is an excursion in D. Unfortunately, ζ is not necessarily a (Γ, Υ)-excursion in D, but with high probability is very close to one. Indeed, if we denote by $\nu_{\partial\tilde{D}_N}(\Gamma, \Upsilon)$ the probability measure on paths obtained by this $(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)$-*excursion in $\tilde{D}_N$ plus Brownian tails* procedure, then it follows from (5.12) and Proposition 4.3.2 that for every $\varepsilon > 0$ there exists an N such that

$$\mathbb{P}(\mathbb{d}(\zeta,\gamma) \geq \varepsilon) \leq \varepsilon \quad \text{and therefore} \quad \wp(\nu_{\partial\tilde{D}_N}(\Gamma,\Upsilon), \mu^{\#}_{\partial\tilde{D}_N}(\tilde{\Gamma}_N,\tilde{\Upsilon}_N) \leq \varepsilon.$$

The proof is completed by noting that $\wp(\nu_{\partial\tilde{D}_N}(\Gamma,\Upsilon), \mu^{\#}_{\partial D}(\Gamma,\Upsilon)) \to 0$ as a consequence of Proposition 4.5.10: (Γ,Υ)-Brownian excursions in D are generated by starting ε from Γ inside D and conditioning the Brownian motion to exit D at Υ. $\square$

As in the discussion preceding Theorem 5.5.2, we can use (5.7) and (5.9) to define the convergence of the infinite measures $\mu_{\partial\tilde{D}_N}$ to $\mu_{\partial D}$.

Theorem 5.7.3. *If $D \in \mathcal{D}^*$ with* $\operatorname{inrad}(D) = 1$*, then* $\wp(\mu_{\partial\tilde{D}_N}, \mu_{\partial D}) \to 0$ *where D_N is the $1/N$-scale discrete approximation to D with associated domain $\tilde{D}_N$.*

Remark. It must be noted, however, that by Theorem 4.5.16 and Definition 4.6.1, we *define* excursion measure μ_D for $D \in D^*$ by conformal invariance. Let $f_N \in \mathcal{T}(\mathbb{D}, \tilde{D}_N)$ as above, and also suppose that $f \in \mathcal{T}(\mathbb{D}, D)$. Hence, $\mu_{\partial\tilde{D}_N} := f_N \circ \mu_{\partial\mathbb{D}}$ and $\mu_{\partial D} := f \circ \mu_{\partial\mathbb{D}}$ so that $\mu_{\partial\tilde{D}_N} = (f_N \circ f^{-1}) \circ \mu_{\partial D}$ as in (5.8). Thus, we can rephrase the conclusion of Theorem 5.7.3 as $\wp(\,(f_N \circ f^{-1}) \circ \mu_{\partial D},\ \mu_{\partial D}\,) \to 0$; compare this with (5.10).

5.8 Estimating $\wp(\,\mu^{\mathsf{rw},\#}_{\partial D_N}(\Gamma_N,\Upsilon_N),\ \mu^{\#}_{\partial\tilde{D}_N}(\tilde{\Gamma}_N,\tilde{\Upsilon}_N)\,)$

In this section we establish Theorem 5.5.1 **(c)** by proving the following result.

Theorem 5.8.1. *For every $D \in \mathcal{D}^*$ with $\operatorname{inrad}(D) = 1$, for every pair of open boundary arcs Γ, $\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} \neq \emptyset$, and for every $\varepsilon > 0$, there exists an N such that*

$$\wp(\mu^{\mathsf{rw},\#}_{\partial D_N}(\Gamma_N, \Upsilon_N), \mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)) \leq \varepsilon \tag{5.13}$$

where D_N is the $1/N$ discrete approximation to D with associated domain $\tilde{D}_N \in \mathcal{D}$ and corresponding boundary arcs Γ_N, $\Upsilon_N \subset \partial D_N$; $\tilde{\Gamma}_N$, $\tilde{\Upsilon}_N \subset \partial \tilde{D}_N$ as in Section 5.2.

In order to prove (5.13), it will be necessary to use Theorem 3.1.2 and the strong approximation of Proposition 3.1.5. Hence, let $A_N := 2ND_N$ so that $A_N \in \mathcal{A}^N$, and write $\Gamma_{N,A} := 2N\Gamma_N$, $\Upsilon_{N,A} := 2N\Upsilon_N \subset \partial A_N$ for the corresponding boundary arcs. Suppose further that N is chosen large enough so that $\operatorname{dist}(\Gamma_{N,A}, \Upsilon_{N,A}) \geq N^{15/16}$. Since $D \in \mathcal{D}^*$, it follows that A_N is necessarily bounded so that $\operatorname{rad}(A_N) \asymp \operatorname{inrad}(A_N) \asymp N$, and furthermore, $|\Upsilon_{N,A}| \asymp |\Gamma_{N,A}| \asymp N$ where all of the constants may depend on D.

Proof. Suppose that $x \in A_N^* := \{x \in A_N : g_{A_N}(x) \geq N^{-1/16}\}$, and let S be a simple random walk with $S_0 = x$. As in the proof of Proposition 3.2.3, it follows from the Beurling estimate that $\operatorname{dist}(x, \partial A) \geq CN^{7/8}$. Hence, a straightforward gambler's ruin estimate shows that $\mathbb{P}^x\{S_\tau \in \Upsilon_{N,A}\} \asymp N^{-1/16}$ where $\tau = \tau_{A_N} := \min\{j : S_j \in \partial A\}$. The coupling of Brownian motion and random walk provided by Corollary 3.1.3 is so strong that even conditioning on the rare event $\{S_\tau \in \Upsilon_{N,A}\}$ does not uncouple the processes. Hence, there exists a Brownian motion B, a simple random walk S with $B_0 = S_0 = x$, and a constant C such that

$$\mathbb{P}^x\left(\sup_{0 \leq t \leq \tau}\left|\frac{1}{\sqrt{2}}\,B_t - S_t\right| \geq C\log N \,\middle|\, S_\tau \in \Upsilon_{N,A}\right) \leq CN^{-8}. \tag{5.14}$$

Our strong approximation (Proposition 3.1.5) allows us to conclude that conditioned on $\{S_\tau \in \Upsilon_{N,A}\}$, Brownian motion and simple random walk starting $N^{7/8}$ away from the boundary still exit near each other; that is,

$$\mathbb{P}^x \left(|B_T - S_\tau| \geq CN^{1/4} \log N \,|\, S_\tau \in \Upsilon_{N,A} \right) \leq CN^{-1/16} \tag{5.15}$$

where $T = T_{A_N} := \inf\{t : B_t \in \partial \tilde{A}_N\}$. The time version of the Beurling estimate (Corollary 2.5.3) says that $\mathbb{P}^x\{|T - \tau| \geq r^2 \operatorname{dist}(x, \partial\tilde{A})^2\} \leq Cr^{-1/2}$. Hence,

$$\mathbb{P}^x \left(|T - \tau| \geq CN^{1/2} \log^2 N \,|\, S_\tau \in \Upsilon_{N,A} \right) \leq CN^{-1/16}. \tag{5.16}$$

We can now use Proposition 4.3.2 to deduce statements about convergence in $\wp$ from statements about convergence in $\mathbb{d}$. In particular, let $\gamma : [0, t_\gamma] \to \mathbb{C}$ be given by $t_\gamma := T$, $\gamma(t) := B_t$, $0 \leq t \leq t_\gamma$, and associate to the random walk S the curve $\tilde{\omega} : [0, t_{\tilde{\omega}}] \to \mathbb{C}$ as in (4.30), so that from (5.14), (5.15), and (5.16), we conclude that $\mathbb{P}\left(\mathbb{d}(\gamma, \tilde{\omega}) \geq CN^{1/2}\log^2 N\right) \leq CN^{-1/16}$, and using Lemma 5.4.4, we can scale our results to D_N:

$$\mathbb{P}\left(\mathbb{d}(\Phi_N\gamma, \Phi_N\tilde{\omega}) \geq CN^{-1/2}\log^2 N\right) \leq \mathbb{P}\left(\mathbb{d}(\gamma, \tilde{\omega}) \geq CN^{1/2}\log^2 N\right) \leq N^{-1/16} \tag{5.17}$$

where $\Phi_N := \mathfrak{T}_{1/(2N)}$ is the Brownian scaling map as in (5.4). Let $V_{N,A}$ be the set $V_{N,A} := \{x \in \partial A_N : \operatorname{dist}(x, \Upsilon_{N,A}) \leq CN^{1/4}\log N\}$, let $\tilde{V}_{N,A}$ be the associated subset of $\partial\tilde{A}_N$, and let $2N\tilde{V}_N = \tilde{V}_{N,A}$. It then follows that $\mathcal{L}(\Phi_N\tilde{\omega}) = \mu^{\mathsf{rw},\#}_{D_N}(x, \Upsilon_N)$ and $\mathcal{L}(\Phi_N\gamma) = \mu^{\#}_{\tilde{D}_N}(x, \tilde{V}_N)$. Since $N^{-1/2}\log N \ll N^{-1/16}$, Proposition 4.3.2 and (5.17) yield

$$\wp(\mu^{\mathsf{rw},\#}_{D_N}(x, \Upsilon_N), \mu^{\#}_{\tilde{D}_N}(x, \tilde{V}_N)) \leq CN^{-1/16}. \tag{5.18}$$

As in the proof of Proposition 3.1.6, $H_{\tilde{D}_N}(x, \tilde{V}_N) = H_{\tilde{D}_N}(x, \tilde{\Upsilon}_N) + O(N^{-3/4} \log N)$, so it follows that

$$\wp(\mu^{\#}_{\tilde{D}_N}(x, \tilde{\Upsilon}_N), \mu^{\#}_{\tilde{D}_N}(x, \tilde{V}_N)) \leq C N^{-3/4} \log N. \tag{5.19}$$

Combining (5.18) and (5.19) then yields $\wp(\mu^{\mathsf{rw},\#}_{D_N}(x, \Upsilon_N), \mu^{\#}_{\tilde{D}_N}(x, \tilde{\Upsilon}_N)) \leq C N^{-1/16}$, and, in particular, if $y \in \mathcal{A}^*_N$ with $|x - y| \leq C \log N$, then

$$\wp(\mu^{\mathsf{rw},\#}_{D_N}(x, \Upsilon_N), \mu^{\#}_{\tilde{D}_N}(y, \tilde{\Upsilon}_N)) \leq C N^{-1/16}. \tag{5.20}$$

To complete the proof, suppose that S' is a simple random walk on the scaled lattice $\frac{1}{2N}\,\mathbb{Z}^2$, and let $D^*_N := \frac{1}{2N}\,A^*_N$ so that $D^*_N = \{z \in D_N : g_{D_N}(z) \geq N^{-1/16}\}$ by (5.3) where g_{D_N} is the Green's function for Brownian motion on $\tilde{D}_N$. Also recall from Theorem 5.3.4 that $\tilde{D}_N \overset{\text{Cara}}{\to} D$. Hence, if $\eta_N = \eta(D, N) := \min\{j \geq 0 : S'_j \in D^*_N \cup D^c_N\}$ as in Lemma 3.4.2 and $x \in D_N \setminus D^*_N$, then for every $\varepsilon > 0$, there exists an N such that

$$\mathbb{P}^x\left(\eta_N \geq \varepsilon \,\middle|\, S'_{\eta_N} \in D^*_N\right) \leq \varepsilon. \tag{5.21}$$

Furthermore, using Lemma 3.4.2 again, we can find constants C, α such that

$$\mathbb{P}^x\{\max_{0 \leq j \leq \eta - 1} |f_{D_N}(S'_j) - f_{D_N}(x)| \geq N^{-1/16} \log N\} \leq C\, N^{-\alpha}, \tag{5.22}$$

and

$$\mathbb{P}^x\left(|f_{D_N}(S'_\eta) - f_{D_N}(x)| \geq N^{-1/16} \log N \,\middle|\, S'_\eta \in D^*_N\right) \leq C\, N^{-\alpha}. \tag{5.23}$$

Suppose further that $\tilde{B}$ is a Brownian motion started at $x \in D_N \setminus D^*_N$. As in Lemma 5.7.2, if $\tilde{\eta}_N = \tilde{\eta}(D, N) := \inf\{t \geq 0 : \tilde{B}_t \in \tilde{D}^*_N \cup \tilde{D}^c_N\}$, then for every $\varepsilon > 0$, there exists an N such that

$$\mathbb{P}^x \left(\tilde{\eta}_N \geq \varepsilon \quad \text{or} \quad \operatorname{diam} B[0, \tilde{\eta_N}] \geq \varepsilon \,\middle|\, B_{\tilde{\eta}_N} \in \tilde{D}_N^* \right) \leq \varepsilon. \tag{5.24}$$

If we let $\tilde{\gamma} : [0, t_{\tilde{\gamma}}] \to \mathbb{C}$ be given by $t_{\tilde{\gamma}} := \tilde{\eta}_N$, $\tilde{\gamma}(t) := \tilde{B}_t$, $0 \leq t \leq t_{\tilde{\gamma}}$, and associate to the (scaled) random walk S' the (scaled) curve $\tilde{\omega}' : [0, t_{\tilde{\omega}'}] \to \mathbb{C}$ as in (5.4) (i.e., Brownian scaled in both time and space), then letting $\underset{\sim}{\gamma} := \tilde{\gamma} \oplus \Phi_N \gamma$ and $\underset{\sim}{\omega} := \tilde{\omega}' \oplus \Phi_N \tilde{\omega}$ we see that $\mathcal{L}(\underset{\sim}{\gamma}) = \mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)$ and $\mathcal{L}(\underset{\sim}{\omega}) = \mu^{\mathsf{rw},\#}_{\partial D_N}(\Gamma_N, \Upsilon_N)$. Hence, by combining (5.21), (5.22), (5.23), and (5.24) with (5.20), we conclude that for every $\varepsilon > 0$, there exists an N with

$$\wp(\mu^{\mathsf{rw},\#}_{\partial D_N}(\Gamma_N, \Upsilon_N), \mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)) \leq \varepsilon. \qquad \square$$

Chapter 6

Loop-Erased Random Walk and Fomin's Identity

In this chapter we extend a result due to S. Fomin [18] which relates a particular functional of loop-erased random walk to the determinant of the matrix of hitting probabilities of simple random walk. In fact, the motivation for this extension is the following quote which is taken from [18, page 3580]. We remark that the Theorem 7.5 referred to in the quote is reproduced below in Theorem 6.3.2.

> In order for the statement of Theorem 7.5 to make sense, the Markov process under consideration does not have to be discrete. . . . The proofs can be obtained by passing to a limit in the discrete approximation. The same limiting procedure can be used to justify the well-definedness of the quantities involved; notice that in order to define a continuous analogue of Theorem 7.5, we do not need the notion of loop-erased Brownian motion. Instead, we discretize the model, compute the probability, and then pass to the limit. One can further extend these results to densities of the corresponding hitting distributions. Technical details are omitted.

Since we have proved that discrete excursion measure converges to Brownian excursion measure (Theorem 5.5.1), we are now able to supply these "technical details." In the first section, we define our conformally invariant scaling limit which extends the results of Sections 2.7 and 4.7. We then review the definition of loop-erased random walk originally introduced in [27]. In Section 6.3, we summarize the results of Fomin from [18]. Finally, in the remaining section, we establish Theorem 6.4.2 and resolve the conjecture.

6.1 The excursion Poisson kernel determinant

We now extend the results of Sections 2.7 and 4.7 on the excursion Poisson kernel $H_{\partial D}$ to the case of the determinant of the matrix of excursion Poisson kernels. The conformally invariant $\mathcal{H}_{\partial D}$ will turn out to be the scaling limit in Fomin's conjecture.

Definition 6.1.1. Suppose that $D \in \mathcal{D}$, $x^i \in \partial D$, $i = 1, \ldots, k$, and $y^j \in \partial D$, $j = 1, \ldots, k$. Let $\mathbf{H}_{\partial D} = [H_{\partial D}(x^i, y^j)]_{1 \le i,j \le k}$ denote the $k \times k$ *hitting matrix*

$$\mathbf{H}_{\partial D} := \begin{bmatrix} H_{\partial D}(x^1, y^1) & \cdots & H_{\partial D}(x^1, y^k) \\ \vdots & \ddots & \vdots \\ H_{\partial D}(x^k, y^1) & \cdots & H_{\partial D}(x^k, y^k) \end{bmatrix}$$

where $H_{\partial D}(x^i, y^j)$ is the excursion Poisson kernel as in Definition 2.7.1.

A straightforward extension of Proposition 2.7.7 is that the determinant of the hitting matrix of excursion Poisson kernels is conformally covariant.

Proposition 6.1.2. *If D, $D' \in \mathcal{D}$; x^i, $y^j \in \partial D$; ∂D is locally analytic at x^i, y^j; $f : D \to D'$ is a conformal transformation of D onto D'; and $\partial D'$ is locally analytic at $f(x^i)$, $f(y^j)$, $i = 1, \ldots k$, $j = 1, \ldots k$, then*

$$\det \mathbf{H}_{\partial D} = \left(\prod_{j=1}^{k} |f'(x^j)| \, |f'(y^j)| \right) \det \left[H_{\partial D'}(f(x^i), f(y^j)) \right]_{1 \le i,j \le k}.$$

Proof. To define the determinant, we follow [47, page 232]. For any ordered k-tuple of integers $(j_1, \ldots, j_k)$, let $\sigma(j_1, \ldots, j_k) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$ where $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$, and $\operatorname{sgn}(x) = 0$ if $x = 0$. If $\mathbf{A}$ is the $k \times k$ matrix $\mathbf{A} = [a(x^i, y^j)]_{1 \le i,j \le k}$, then we define

$$\det \mathbf{A} = \sum_{\sigma} \sigma(j_1, \ldots, j_k) a(x^1, y^{j_1}) \cdots a(x^k, y^{j_k})$$

where the sum is over all ordered k-tuples of integers $(j_1, \ldots, j_k)$, $1 \le j_i \le k$. Thus,

$$\begin{aligned}
\det \mathbf{H}_{\partial D} &= \sum_{\sigma} \sigma(j_1, \ldots, j_k)\, H_{\partial D}(x^1, y^{j_1}) \cdots H_{\partial D}(x^k, y^{j_k}) \\
&= \sum_{\sigma} \sigma(j_1, \ldots, j_k) \prod_{j=1}^{k} |f'(x^j)| \prod_{i=1}^{k} |f'(y^{j_i})|\, H_{\partial D'}(f(x^1), f(y^{j_1})) \cdots H_{\partial D'}(f(x^k), f(y^{j_k})) \\
&= \left(\prod_{j=1}^{k} |f'(x^j)|\, |f'(y^j)| \right) \sum_{\sigma} \sigma(j_1, \ldots, j_k)\, H_{\partial D'}(f(x^1), f(y^{j_1})) \cdots H_{\partial D'}(f(x^k), f(y^{j_k})) \\
&= \left(\prod_{j=1}^{k} |f'(x^j)|\, |f'(y^j)| \right) \det \left[H_{\partial D'}(f(x^i), f(y^j)) \right]_{1 \le i,j \le k}.
\end{aligned}$$

□

The next result is immediate from Proposition 2.7.8, and elementary properties of the determinant.

Proposition 6.1.3. *If $D \in \mathcal{D}$, and x^i, $y^j \in \partial D$ with ∂D locally analytic at x^i, y^j, $1 \le i, j \le k$, then*

$$\det \mathbf{H}_{\partial D} = (2\pi)^k \left(\prod_{j=1}^{k} H_D(0, x^j)\, H_D(0, y^j) \right) \det \left[\frac{1}{1 - \cos(\theta_D(x^i) - \theta_D(y^j))} \right]_{1 \le i,j \le k}.$$

As an extension of Proposition 2.7.9, the integral of the determinant of the hitting matrix of excursion Poisson kernels over an appropriate set is conformally invariant.

Proposition 6.1.4 (Conformal Invariance of Integrated Excursion Poisson Kernel Determinant). *Suppose that $D \in \mathcal{D}$, and let Γ, $\Upsilon \subset \partial D$ be analytic open boundary arcs with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. Let $D' \in \mathcal{D}$, and suppose that $f \in \mathcal{T}(D, D')$ with $f(0) = 0$, $f'(0) > 0$. Write Γ', Υ' for the images under f of Γ, Υ, respectively. If*

$$\mathcal{H}_{\partial D}(\Gamma, \Upsilon; k) := \int_{V(\Upsilon;k)} \int_{V(\Gamma;k)} \det[H_{\partial D}(x^i, y^j)]_{1\le i,j\le k} \, |\mathrm{d}x^1| \cdots |\mathrm{d}x^k| \, |\mathrm{d}y^1| \cdots |\mathrm{d}y^k|$$

where $V(\Gamma; k) := \{(x^1, \ldots, x^k) : x^i \in \Gamma, x^1 < \cdots < x^k\}$ and similarly, $V(\Upsilon; k) := \{(y^1, \ldots, y^k) : y^i \in \Upsilon, y^1 < \cdots < y^k\}$, then

$$\mathcal{H}_{\partial D}(\Gamma, \Upsilon; k) = \mathcal{H}_{\partial D'}(\Gamma', \Upsilon'; k).$$

Proof. By definition,

$$\mathcal{H}_{\partial D}(\Gamma, \Upsilon; k) = \int_{V(\Upsilon;k)} \int_{V(\Gamma;k)} \det \mathbf{H}_{\partial D} \, |\mathrm{d}x| \, |\mathrm{d}y|$$

where $\mathbf{H}_{\partial D}$ is the $k \times k$ hitting matrix as in Definition 6.1.1. Thus,

$$\begin{aligned}
&\int_{V(\Upsilon;k)} \int_{V(\Gamma;k)} \det[H_{\partial D}(x^i, y^j)]_{1\le i,j\le k} \, |\mathrm{d}x| \, |\mathrm{d}y| \\
&\qquad = \int_{V(\Upsilon;k)} \int_{V(\Gamma;k)} \det[H_{\partial D'}(f(x^i), f(y^j))]_{1\le i,j\le k} \left(\prod_{j=1}^{k} |f'(x^j)||f'(y^j)| \right) |\mathrm{d}x| \, |\mathrm{d}y| \\
&\qquad = \int_{V(\Upsilon';k)} \int_{V(\Gamma';k)} \det[H_{\partial D'}(u^i, v^j)]_{1\le i,j\le k} \, |\mathrm{d}u^1| \cdots |\mathrm{d}u^k| \, |\mathrm{d}v^1| \cdots |\mathrm{d}v^k|
\end{aligned}$$

by changing variables, and the proof is complete. □

Notice that our proof that $\mathcal{H}_{\partial D}$ is conformally invariant relied on the fact $D \in \mathcal{D}$ so that integration along the boundary is justified. We now extend $\mathcal{H}_{\partial D}$ to $D \in \mathcal{D}^*$ by conformal invariance. This is exactly analogous to Section 4.7 where the integrated excursion Poisson kernel $H_{\partial D}(\Gamma, \Upsilon)$ was extended to $D \in \mathcal{D}^*$. We remark that this definition is independent of the choice of conformal transformation $f \in \mathcal{T}(\mathbb{D}, D)$.

Definition 6.1.5. Suppose that $D \in \mathcal{D}^*$ and $f \in \mathcal{T}(\mathbb{D}, D)$. Let $\Gamma_{\mathbb{D}}$, $\Upsilon_{\mathbb{D}} \subset \partial\mathbb{D}$ be open boundary arcs with $\overline{\Gamma_{\mathbb{D}}} \cap \overline{\Upsilon_{\mathbb{D}}} = \emptyset$, and write Γ, Υ for the images under f of $\Gamma_{\mathbb{D}}$, $\Upsilon_{\mathbb{D}}$, respectively. For integers $k \geq 2$, the *k-fold integrated excursion Poisson kernel determinant* $\mathcal{H}_{\partial D}$ is defined by

$$\mathcal{H}_{\partial D}(\Gamma, \Upsilon; k) := \mathcal{H}_{\partial \mathbb{D}}(\Gamma_{\mathbb{D}}, \Upsilon_{\mathbb{D}}; k). \tag{6.1}$$

6.2 Definition of loop-erased random walk

In this section, we briefly review the definition of the loop-erased random walk. The main reference for this material is [28, Chapter 7]. See also [29] for a more elementary overview. Since simple random walk in $\mathbb{Z}^2$ is recurrent, it is not possible to construct loop-erased random walk by erasing loops from an infinite walk. However, the loop-erasing procedure that we are about to describe makes perfect sense since it assigns to each finite simple random walk path a self-avoiding walk.

Let $S = S[0, m] = [S_0, S_1, \ldots, S_m]$ be a simple random walk path of length m. We construct $\Lambda(S)$, the loop-erased part of S, recursively as follows. If S is already self-avoiding, set $\Lambda(S) = S$. Otherwise, let $s_0 = \max\{j : S_j = S_0\}$, and for $i > 0$, let $s_i = \max\{j : S_j = S_{s_{j-1}+1}\}$. If we let $n = \min\{i : s_i = m\}$, then $\Lambda(S) = [S_{s_0}, S_{s_1}, \ldots, S_{s_n}]$.

Remark. The history of loop-erased random walk began when it was introduced by Lawler [27] in an attempt to analyze the usual self-avoiding random walk. It was later discovered that loop-erased random walk and self-avoiding random walk are in different *universality classes.*[1] It is proved in [28] that loop-erased random walk in dimension $d \geq 4$ converges in the scaling limit to Brownian motion (with a logarithmic correction in $d = 4$). The recent introduction, however, of the Schramm-Loewner evolution (SLE_κ) has led to a flurry of important and deep results. (See [46], [56, 57], and the forthcoming book [32] for details.) In particular, it has been proved by Lawler, Schramm, and Werner [38] that the scaling limit of loop-erased random walk in a simply connected domain $D \subset \mathbb{C}$ from an interior point to a boundary point converges to radial SLE_2. Specifically, the limit exists and is conformally invariant. Exciting work is currently being done by Beneš [7] to show that loop-erased random walk converges to chordal SLE_2 in the upper half plane. We also note that Schramm's study of loop-erased random walk is what led him to initially develop this process.

6.3 Fomin's result

Suppose that $A \in \mathcal{A}^n$, $x \in \partial A$, $y \in \partial A$. Recall from Definition 2.11.2 that $h_{\partial A}(x, y)$ is the discrete excursion Poisson kernel defined by $h_{\partial A}(x, y) = \mathbb{P}^x(S_{\tau_A} = y, S_1 \in A)$. The following definition is the discrete analogue of Definition 6.1.1.

Definition 6.3.1. Suppose that $A \in \mathcal{A}^n$, $x^i \in \partial A$, $i = 1, \ldots, k$, and $y^j \in \partial A$, $j = 1, \ldots, k$. Let $\mathbf{h}_{\partial A} = [h_{\partial A}(x^i, y^j)]_{1 \leq i,j \leq k}$ denote the $k \times k$ *discrete hitting matrix*

$$\mathbf{h}_{\partial A} := \begin{bmatrix} h_{\partial A}(x^1, y^1) & \cdots & h_{\partial A}(x^1, y^k) \\ \vdots & \ddots & \vdots \\ h_{\partial A}(x^k, y^1) & \cdots & h_{\partial A}(x^k, y^k) \end{bmatrix}.$$

[1] This notion has existed for many years in the physics literature with several different meanings. Here we use it to indicate that LERW and SAW have different continuum limits; see [30, 31] and [39].

For the remainder of this section, suppose that Γ, $\Upsilon \subset \partial A$ are discrete boundary arcs with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. Suppose further that $|\Gamma| \geq k$, $|\Upsilon| \geq k$, and let $x^1, x^2, \ldots, x^k \in \Gamma$. Let $S^1, S^2, \ldots, S^k$ be independent simple random walks starting at $x^1, x^2, \ldots, x^k$, respectively, and set $\tau_A^i := \min\{j > 0 : S_j^i \notin A\}$. Finally, let $y^1, y^2, \ldots, y^k \in \Upsilon$ be such that any random walk trajectory from x^i to y^j (with $S^i(0, \tau_A^i) \subset A$, $S_0^i = x^i$, $S_{\tau_A^i}^i = y^j$) intersects any random walk trajectory from $x^{i'}$, $i' > i$, to $y^{j'}$, $j' < j$. (Such an ordering of points is always possible for *any* disjoint pair of discrete boundary arcs.) Let $L^i = \Lambda(S^i)$ be the loop erasure of the path $[S_0^i = x^i, S_1^i, \ldots, S_{\tau_A^i}^i]$, and let $\mathcal{E} = \mathcal{E}(x^1, \ldots, x^k, y^1, \ldots, y^k; A)$ be the event that both

- $S_{\tau_A^i}^i = y^i, \quad i = 1, \ldots, k,$ and
- $S^i[0, \tau_A^i] \cap \{L^1 \cup \cdots \cup L^{i-1}\} = \emptyset, \quad i = 2, \ldots, k.$

The following is proved in [18, Theorem 7.5]. Notice it relates the determinant of the discrete hitting matrix to the event $\mathcal{E}$, a functional of loop-erased random walk.

Theorem 6.3.2 (Fomin). *If $\mathcal{E}$ and $\mathbf{h}_{\partial A}$ are defined as above, then $\mathbb{P}(\mathcal{E}) = \det \mathbf{h}_{\partial A}$.*

By scaling the lattice, it is possible to take $A = D_N$, the $1/N$-scale discrete approximation to $D \in \mathcal{D}^*$, where the random walk quantities are understood to be on the lattice $\frac{1}{N}\mathbb{Z}^2$ as in Section 5.4. This is precisely what we do in the following section in order to prove Fomin's conjecture.

6.4 A conformally invariant scaling limit

In this section, we prove Theorem 6.4.2 below which gives a satisfactory resolution to Fomin's conjecture. To begin, suppose that $D \in \mathcal{D}^*$ with $\text{inrad}(D) = 1$, and Γ,

$\Upsilon \subset \partial D$ with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. Recall that $\operatorname{sep}(\Gamma, \Upsilon)$ is defined in Definition 2.2.2. Choose N large enough so that $\operatorname{sep}(\Gamma, \Upsilon) \geq \varepsilon_N := N^{-1/48} \log^{2/3} N$, and let D_N be the $1/N$-scale discrete approximation to D with corresponding discrete boundary arcs Γ_N, $\Upsilon_N \subset \partial D_N$. It follows from Corollary 5.4.3 that for $x \in \Gamma_N$, $y \in \Upsilon_N$, we have

$$h_{\partial D_N}(x, y) = \frac{(\pi/2)\, h_{D_N}(0, x)\, h_{D_N}(0, y)}{1 - \cos(\theta_{D_N}(x) - \theta_{D_N}(y))}\, [1 + O(\varepsilon_N^3)].$$

Write

$$\varphi_N(x, y) = \varphi_N(x, y; D, \Gamma, \Upsilon) := \frac{(\pi/2)\, h_{D_N}(0, x)\, h_{D_N}(0, y)}{1 - \cos(\theta_{D_N}(x) - \theta_{D_N}(y))}$$

so that $h_{\partial D_N}(x, y) = \varphi_N(x, y)[1 + O(\varepsilon_N^3)]$. Let $x^1, \ldots, x^k \in \Gamma_N$, $y^1, \ldots, y^k \in \Upsilon_N$ be labelled as to satisfy the trajectory constraint defined in Section 6.3.

Lemma 6.4.1. *If φ_N is defined as above and $k \ll N$ is fixed, then*

$$\sum_{V(\Gamma_N;k)} \sum_{V(\Upsilon_N;k)} \det \mathbf{h}_{\partial D_N} = \sum_{V(\Gamma_N;k)} \sum_{V(\Upsilon_N;k)} \det[\varphi_N(x^i, y^j)]_{1 \leq i,j \leq 2} + O(\varepsilon_N^3)$$

where $\mathbf{h}_{\partial D_N}$ is the $k \times k$ discrete hitting matrix, and $V(\Gamma_N; k) := \{(x^1, \ldots, x^k) : x^i \in \Gamma_N,\ x^1 < \cdots < x^k\}$, $V(\Upsilon_N; k) := \{(y^1, \ldots, y^k) : y^i \in \Upsilon_N,\ y^1 < \cdots < y^k\}$.

Proof. We prove this result for $k = 2$; the notationally challenging general case is identical. To begin, note that $\theta_{D_N}(x) = \theta_D(x) + O(N^{-1/2})$ for $x \in \partial D_N$ as in (2.19). Thus, as in the proof of Theorem 5.6.3, for fixed D, Γ, Υ, there exist constants C_1, C_2, depending on D, Γ, Υ, such that

$$C_1\, h_{D_N}(0, x)\, h_{D_N}(0, y) \leq \varphi_N(x, y) \leq C_2\, h_{D_N}(0, x)\, h_{D_N}(0, y). \tag{6.2}$$

If x^1, $x^2 \in \Gamma_N$ with $x^1 < x^2$, and y^1, $y^2 \in \Upsilon_N$ with $y^1 < y^2$, then it follows that

$$\begin{aligned}
&\det[h_{\partial D_N}(x^i, y^j)]_{1\le i,j\le 2} \\
&\quad = \det[\varphi_N(x^i, y^j)]_{1\le i,j\le 2} + O(\varepsilon_N^3)\,[\,\varphi_N(x^1,y^1)\,\varphi_N(x^2,y^2) + \varphi_N(x^1,y^2)\,\varphi_N(x^2,y^1)\,].
\end{aligned}$$

so that by summing over $V(\Gamma_N) := V(\Gamma_N; 2)$, $V(\Upsilon_N) := V(\Upsilon_N; 2)$ we conclude

$$\begin{aligned}
&\left| \sum_{V(\Gamma_N)} \sum_{V(\Upsilon_N)} \det[h_{\partial D_N}(x^i, y^j)]_{1\le i,j\le 2} - \sum_{V(\Gamma_N)} \sum_{V(\Upsilon_N)} \det[\varphi_N(x^i, y^j)]_{1\le i,j\le 2} \right| \\
&\qquad \le O(\varepsilon_N^3) \sum_{V(\Gamma_N)} \sum_{V(\Upsilon_N)} [\,\varphi_N(x^1,y^1)\,\varphi_N(x^2,y^2) + \varphi_N(x^1,y^2)\,\varphi_N(x^2,y^1)\,] \\
&\qquad \le 2C_2 O(\varepsilon_N^3) \sum_{V(\Gamma_N)} \sum_{V(\Upsilon_N)} h_{D_N}(0,x^1)\, h_{D_N}(0,x^2)\, h_{D_N}(0,y^1)\, h_{D_N}(0,y^2).
\end{aligned}$$

where the last line comes from (6.2). However,

$$\begin{aligned}
&\sum_{V(\Gamma_N)} \sum_{V(\Upsilon_N)} h_{D_N}(0,x^1)\, h_{D_N}(0,x^2)\, h_{D_N}(0,y^1)\, h_{D_N}(0,y^2) \\
&\qquad \le \sum_{x^1\in\Gamma_N} \sum_{x^2\in\Gamma_N} \sum_{y^1\in\Upsilon_N} \sum_{y^2\in\Upsilon_N} h_{D_N}(0,x^1)\, h_{D_N}(0,x^2)\, h_{D_N}(0,y^1)\, h_{D_N}(0,y^2) \\
&\qquad = [\, h_{D_N}(0,\Gamma_N)\, h_{D_N}(0,\Upsilon_N)\,]^2 = O(1).
\end{aligned}$$

Hence, there exists a constant C_3 depending on D, Γ, Υ, such that

$$\left| \sum_{V(\Gamma_N)} \sum_{V(\Upsilon_N)} \det[h_{\partial D_N}(x^i, y^j)]_{1\le i,j\le 2} - \sum_{V(\Gamma_N)} \sum_{V(\Upsilon_N)} \det[\varphi_N(x^i, y^j)]_{1\le i,j\le 2} \right| \le C_3 \varepsilon_N^3$$

completing the proof. □

Notice that we can write

$$\sum_{x^1\in\Gamma_N}\sum_{x^2\in\Gamma_N}\sum_{y^1\in\Upsilon_N}\sum_{y^2\in\Upsilon_N} h_{D_N}(0,x^1)\,h_{D_N}(0,x^2)\,h_{D_N}(0,y^1)\,h_{D_N}(0,y^2)$$
$$= 2\sum_{V(\Gamma_N)}\sum_{V(\Upsilon_N)} h_{D_N}(0,x^1)\,h_{D_N}(0,x^2)\,h_{D_N}(0,y^1)\,h_{D_N}(0,y^2)$$
$$+ \sum_{W(\Gamma_N)}\sum_{W(\Upsilon_N)} h_{D_N}(0,x^1)\,h_{D_N}(0,x^2)\,h_{D_N}(0,y^1)\,h_{D_N}(0,y^2)$$

where $V(\Gamma_N) := V(\Gamma_N;2)$, $V(\Upsilon_N) := V(\Upsilon_N;2)$ as above, and $W(\Gamma_N) = W(\Gamma_N;2) := \{(x^1,x^2) : x^i \in \Gamma_N,\ x^1 = x^2\}$, $W(\Upsilon_N) = W(\Upsilon_N;2) := \{(y^1,y^2) : y^i \in \Upsilon_N,\ y^1 = y^2\}$. However,

$$\sum_{W(\Gamma_N)}\sum_{W(\Upsilon_N)} h_{D_N}(0,x^1)\,h_{D_N}(0,x^2)\,h_{D_N}(0,y^1)\,h_{D_N}(0,y^2)$$
$$= \sum_{x^1\in\Gamma_N}[\,h_{D_N}(0,x^1)\,]^2 \sum_{y^1\in\Upsilon_N}[\,h_{D_N}(0,y^1)\,]^2,$$

and from the discrete Beurling projection theorem we have $h_{D_N}(0,x^1) \le CN^{-1/2}$, and similarly $h_{D_N}(0,y^1) \le CN^{-1/2}$, so that

$$\sum_{x^1\in\Gamma_N}[\,h_{D_N}(0,x^1)\,]^2 \sum_{y^1\in\Upsilon_N}[\,h_{D_N}(0,y^1)\,]^2 \le CN^{-1}h_{D_N}(0,\Gamma_N)h_{D_N}(0,\Upsilon_N) = O(N^{-1}).$$

In the continuous case, from the Beurling estimate we have the analogous result:

$$\int_\Gamma [\,H_D(0,x)\,]^2\,|\mathrm{d}x| \int_\Upsilon [\,H_D(0,y)\,]^2\,|\mathrm{d}y| \le CN^{-1}H_D(0,\Gamma)H_D(0,\Upsilon) = O(N^{-1}).$$

In other words, this shows that the square terms do not contribute very much to the sum (integral). Hence, it follows from Proposition 3.1.6 (with $\delta_N := N^{-7/8}\log N$)

and from Proposition 6.1.3 that for $D \in \mathcal{D}$,

$$
\begin{aligned}
(4)^2 \sum_{V(\Gamma_N)} \sum_{V(\Upsilon_N)} & \det[\varphi_N(x^i, y^j)]_{1 \le i,j \le 2} \\
&= \int_{V(\Gamma)} \int_{V(\Upsilon)} \det[H_{\partial D}(x^i, y^j)]_{1 \le i,j \le 2} |\mathrm{d}y^1| \, |\mathrm{d}y^2| \, |\mathrm{d}x^1| \, |\mathrm{d}x^2| + O(\delta_N) + O(N^{-1/2}) \\
&= \mathcal{H}_{\partial D}(\Gamma, \Upsilon; 2) + O(N^{-7/8} \log N) + O(N^{-1/2})
\end{aligned}
$$

where $V(\Gamma) := V(\Gamma; 2)$, $V(\Upsilon) := V(\Upsilon; 2)$ as in Proposition 6.1.4. The extra factors of 4 arise exactly as in Theorem 5.6.3.

Recall from Proposition 6.1.4 and Definition 6.1.5 that $\mathcal{H}_{\partial D}(\Gamma, \Upsilon; 2)$ is, in fact, conformally invariant. Since the above result holds for $D \in \mathcal{D}$, we can use Proposition 6.1.4 to define $\mathcal{H}_{\partial D}(\Gamma, \Upsilon; 2)$ for general $D \in \mathcal{D}^*$. Thus, we have established the following theorem.

Theorem 6.4.2. *Suppose that $D \in \mathcal{D}^*$ and Γ, $\Upsilon \subset \partial D$ are open boundary arcs with $\overline{\Gamma} \cap \overline{\Upsilon} = \emptyset$. Let D_N be the $1/N$-scale discrete approximation to D with associated boundary arcs Γ_N, $\Upsilon_N \subset \partial D_N$ as in Section 5.2. If, for each N, the labelling of $x^1, \ldots, x^k \in \Gamma_N$, $y^1, \ldots, y^k \in \Upsilon_N$ is such that the trajectory constraint of Section 6.3 is satisfied, and if $\mathbf{h}_{\partial D_N} = [h_{\partial D_N}(x^i, y^j)]_{1 \le i,j \le k}$ is the $k \times k$ discrete hitting matrix, then*

$$\lim_{N \to \infty} \sum_{V(\Upsilon_N; k)} \sum_{V(\Gamma_N; k)} \det \mathbf{h}_{\partial D_N} = (1/4)^k \, \mathcal{H}_{\partial D}(\Gamma, \Upsilon; k),$$

where $\mathcal{H}_{\partial D}(\Gamma, \Upsilon; k)$ is defined in Definition 6.1.5, and $V(\Gamma_N; k)$, $V(\Upsilon_N; k)$ are as in Lemma 6.4.1. In particular, the limit exists and is conformally invariant.

References

[1] L. V. Ahlfors. *Conformal Invariants: Topics in Geometric Function Theory.* McGraw-Hill, New York, NY, 1973.

[2] L. V. Ahlfors. *Complex Analysis.* McGraw-Hill, New York, NY, third edition, 1979.

[3] M. Aizenman and A. Burchard. Hölder regularity and dimension bounds for random curves. *Duke Math. J.*, 99:419–453, 1999.

[4] P. Auer. Some hitting probabilities of random walks on Z^2. In I. Berkes, E. Csáki, and P. Révész, editors, *Limit Theorems in Probability and Statistics*, volume 57 of *Colloquia Mathematica Societatis János Bolyai*, pages 9–25, Budapest, Hungary, 1990. North-Holland.

[5] R. F. Bass. *Probabilistic Techniques in Analysis.* Springer-Verlag, New York, NY, 1995.

[6] R. F. Bass and K. Burdzy. Conditioned Brownian motion in planar domains. *Probab. Theory Related Fields*, 101:479–493, 1995.

[7] C. G. Beneš. *On some problems concerning planar random walks.* Ph.D. dissertation, Duke University, Durham, NC, 2004.

[8] H. Bergström. *Weak Convergence of Measures.* Academic Press, New York, NY, 1982.

[9] P. Billingsley. *Convergence of Probability Measures.* John Wiley & Sons, New York, NY, 1968.

[10] R. M. Blumenthal. *Excursions of Markov Processes.* Birkhäuser, Boston, MA, 1992.

[11] V. S. Borkar. *Probability Theory: An Advanced Course*. Springer-Verlag, New York, NY, 1995.

[12] K. Burdzy. *Multidimensional Brownian Excursions and Potential Theory*, volume 164 of *Pitman Research Notes in Mathematics*. Longman Scientific & Technical, Guildford, United Kingdom, 1987.

[13] B. Davis. Conditioned Brownian motion in planar domains. *Duke Math. J.*, 57:397–421, 1988.

[14] P. L. Duren. *Univalent Functions*, volume 259 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, NY, 1983.

[15] R. Durrett. *Brownian Motion and Martingales in Analysis*. Wadsworth, Belmont, CA, 1984.

[16] P. J. Flory. The configuration of real polymer chains. *J. Chem. Phys.*, 17:303–310, 1949.

[17] G. B. Folland. *Introduction to Partial Differential Equations*. Princeton University Press, Princeton, NJ, second edition, 1995.

[18] S. Fomin. Loop-erased walks and total positivity. *Trans. Amer. Math. Soc.*, 353:3563–3583, 2001.

[19] B. Fristedt and L. Gray. *A Modern Approach to Probability Theory*. Birkhäuser, Boston, MA, 1997.

[20] Y. Fukai. Hitting distribution to a quadrant of two-dimensional random walk. *Kodai Math. J.*, 23:35–80, 2000.

[21] Y. Fukai and K. Uchiyama. Potential kernel for two-dimensional random walk. *Ann. Probab.*, 24:1979–1992, 1996.

[22] E. P. Hsu and K.-T. Sturm. Maximal coupling of Euclidean Brownian motions. Sonderforschungsbereich SFB 611, Preprint No. 85, Universität Bonn, 2003.

[23] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, NY, second edition, 1991.

[24] J. Komlós, P. Major, and G. Tusnády. An approximation of partial sums of independent RV'-s, and the sample DF. I. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 32:111–131, 1975.

[25] J. Komlós, P. Major, and G. Tusnády. An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 34:33–58, 1976.

[26] M. J. Kozdron. A short proof of the continuity of the Poisson kernel and the excursion Poisson kernel. Unpublished, 2004.

[27] G. F. Lawler. A self-avoiding random walk. *Duke Math. J.*, 47:655–693, 1980.

[28] G. F. Lawler. *Intersections of Random Walks*. Birkhäuser, Boston, MA, 1991.

[29] G. F. Lawler. Loop-erased random walk. In R. Durrett and M. Bramson, editors, *Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten*, volume 44 of *Progress in Probability*, pages 197–217. Birkhäuser, Boston, MA, 1999.

[30] G. F. Lawler. Conformal invariance, universality, and the dimension of the Brownian frontier. In Li Tatsien, editor, *Proceedings of the International Congress of Mathematicians (Beijing 2002). Vol. III: Invited Lectures*, pages 63–72, Beijing, China, 2002. Higher Education Press.

[31] G. F. Lawler. The restriction property for conformally covariant measures. Preprint, 2003. To appear in *Proceedings of the XIV International Congress on Mathematical Physics (Lisbon 2003)*.

[32] G. F. Lawler. Conformally Invariant Processes in the Plane. In progress, 2004.

[33] G. F. Lawler and V. Limic. The Beurling estimate for a class of random walks. Preprint, 2004. Available online at `arXiv:math.PR/0403309`.

[34] G. F. Lawler, O. Schramm, and W. Werner. The dimension of the planar Brownian frontier is 4/3. *Math. Res. Lett.*, 8:401–411, 2001.

[35] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents, I: Half-plane exponents. *Acta Math.*, 187:237–273, 2001.

[36] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents, II: Plane exponents. *Acta Math.*, 187:275–308, 2001.

[37] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents III: Two-sided exponents. *Ann. Inst. H. Poincaré Probab. Statist.*, 38:109–123, 2002.

[38] G. F. Lawler, O. Schramm, and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32:939–995, 2004.

[39] G. F. Lawler and W. Werner. Universality for conformally invariant intersection exponents. *J. Eur. Math. Soc.*, 2:291–328, 2000.

[40] G. F. Lawler and W. Werner. The Brownian loop soup. *Probab. Theory Related Fields*, pages 565–588, 2004.

[41] T. Lindvall. *Lectures on the Coupling Method*. Dover, Mineola, NY, 2002.

[42] D. Mackenzie. Taking the measure of the wildest dance on Earth. *Science*, 290:1883–1884, 2000.

[43] P. E. Protter. *Stochastic Integration and Differential Equations*, volume 21 of *Applications of Mathematics*. Springer-Verlag, Berlin, Germany, second edition, 2004.

[44] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes, and Martingales. Volume 1: Foundations*. Cambridge University Press, Cambridge, United Kingdom, second edition, 2000.

[45] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes, and Martingales. Volume 2: Itô Calculus*. Cambridge University Press, Cambridge, United Kingdom, second edition, 2000.

[46] S. Rohde and O. Schramm. Basic properties of SLE. Preprint, 2004. Available online at `arXiv:math.PR/0106036`. To appear in *Ann. Math.*

[47] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, New York, NY, third edition, 1976.

[48] A. V. Skorohod. *Integration in Hilbert Space*, volume 79 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, Germany, 1974. Translated from the Russian by K. Wickwire.

[49] F. Spitzer. *Principles of Random Walk*, volume 34 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, NY, second edition, 2001.

[50] A. Stöhr. Über einige partielle Differenzengleichungen mit konstanten Koeffizienten. I. Allgemeiner Teil. *Math. Nachr.*, 3:208–242, 1950.

[51] A. Stöhr. Über einige partielle Differenzengleichungen mit konstanten Koeffizienten. II. Erstes Beispeil: Der operator $\nabla\Phi(y_1, y_2) = \Phi(y_1 + 1, y_2) + \Phi(y_1 - 1, y_2) + \Phi(y_1, y_2 + 1) + \Phi(y_1, y_2 - 1) - 4\kappa\Phi(y_1, y_2)$ mit $\kappa > 1$. *Math. Nachr.*, 3:295–315, 1950.

[52] A. Stöhr. Über einige partielle Differenzengleichungen mit konstanten Koeffizienten. III. Zweites Beispeil: Der operator $\nabla\Phi(y_1, y_2) = \Phi(y_1+1, y_2)+\Phi(y_1-1, y_2)+\Phi(y_1, y_2+1)+\Phi(y_1, y_2-1)-4\Phi(y_1, y_2)$. *Math. Nachr.*, 3:330–357, 1950.

[53] T. Szabados. Strong approximation of fractional Brownian motion by moving averages of simple random walks. *Stochastic Process. Appl.*, 92:31–60, 2001.

[54] B. Virág. Brownian beads. *Probab. Theory Related Fields*, 127:367–387, 2003.

[55] W. Werner. Critical exponents, conformal invariance and planar Brownian motion. In C. Casacuberta, R. M. Miró-Roig, J. Verdera, and S. Xambó-Descamps, editors, *European Congress of Mathematics (Barcelona, July 10–14, 2000). Volume II*, volume 202 of *Progress in Mathematics*, pages 87–103. Birkhäuser, Basel, Switzerland, 2001.

[56] W. Werner. Conformal restriction and related questions. Preprint, 2003. Available online at `arXiv:math.PR/0307353`.

[57] W. Werner. Random planar curves and Schramm-Loewner evolutions. In *Lectures on Probability Theory and Statistics: Ecole d'Eté de Probabilités de Saint-Flour XXXII-2002*, volume 1840 of *Lecture Notes in Mathematics*, pages 113–196. Springer-Verlag, Berlin, 2004.

[58] D. Williams. *Probability with Martingales.* Cambridge University Press, Cambridge, United Kingdom, 1991.

[59] S. Windwer. Polymer conformation and the excluded-volume problem. In G. Lowry, editor, *Markov Chains and Monte Carlo Calculations in Polymer Science*, pages 125–152. Marcel Dekker, New York, NY, 1970.

[60] A. Yu. Zaitsev. Estimates for the strong approximation in multidimensional central limit theorem. In Li Tatsien, editor, *Proceedings of the International Congress of Mathematicians (Beijing 2002). Vol. III: Invited Lectures*, pages 107–116, Beijing, China, 2002. Higher Education Press.

Biography

Michael John Kozdron was born on March 23, 1976, in Montréal, Québec, to Casey and Audrey Kozdron. In July 1979, one year after the birth of his sister, Allison, the family moved to British Columbia. He grew up in Burnaby, and in 1994 graduated from Notre Dame Regional Secondary School in Vancouver. During the next four years, Michael attended the University of British Columbia, living in residence there. As a first-year student he decided to follow the Honours Mathematics program, and in May 1998 he graduated with a B.Sc. (Hon.). In August 1998, Michael moved to Durham, North Carolina, to attend graduate school at Duke University. He was awarded his M.A. from Duke in December 1999. On August 5, 2000, he married Jessica Ann Burns, the love of his life, in Maple Ridge, B.C., and shortly thereafter she moved to Durham. From August–December 2001, he and Jessica enjoyed the hospitality of the Institut Mittag-Leffler in Djursholm, Sweden, while Michael participated in their program on probability and conformal mappings. In June 2002, Michael became a visiting non-degree student at Cornell University in Ithaca, New York, where Prof. Lawler had accepted a faculty position. Michael was appointed a member of the Summer Session Faculty at Duke University during 2001, and a Lecturer at Cornell University during Summer Session 2003. For teaching excellence, he received the L. P. and Barbara Smith Award from the Duke University Department of Mathematics for 2000–2001 and 2002–2003. Currently a member of the Institute of Mathematical Statistics, the American Statistical Association, the American Mathematical Society, and the TEX Users Group, Michael has accepted a tenure-track assistant professorship at the University of Regina in Regina, Saskatchewan, to begin on July 1, 2004.