The following is from O. Schramm [5]. The (. . .) are mine.

It is often the case that grid-based probabilistic models should be considered as a mere substitute, or simplification, of a continuous process. There are definite advantages for working in the discrete setting, where unpleasant technicalities can frequently be avoided, simulations are possible, and the setup is easier to comprehend. On the other hand, one is often required to pay some price for the simplification. When we adopt the grid-based world, we sacrifice rotational or conformal symmetries which the continuous model may enjoy, and often have to accept some arbitrariness in the formulation of the model. There are also numerous examples where the continuous process is easier to analyze than the discrete process, and in such situations the continuous may be a useful simplification of the discrete.

Understanding the connections between grid-based models and continuous processes is a project of fundamental importance (and so far has only limited success). As mathematicians, we should not content ourselves with the vague notion that the discrete and continuous models behave “essentially the same,” but strive to make the relations concrete and precise.

One reasonable way to define a continuous process, is by taking a **scaling limit** of a grid process. This means making sense of the limit of a sequence of grid processes on finer and finer grids.

Although, in general, the understanding of the connections between grid-based models and continuous models is lacking, there have been some successes. The classical and archetypical example is the relation between simple random walk and Brownian motion, which is well studied and quite well understood.

In the plane, the scaling limits are conjectured to possess conformal invariance, and this can serve as one justification for the special interest in dimension 2.

The most fundamental task in studying a scaling limit process is to set up a conceptual foundation for the scaling limit. This means answering the following two questions: what kind of object is the scaling limit, and what does it mean to be the scaling limit?

For example, the scaling limit of simple random walk in $\mathbb{Z}$ is usually defined as a probability measure on $C(\mathbb{R})$, the space of continuous, real-valued functions on $\mathbb{R}$.

After the conceptual framework is fixed, the next natural question is the existence of the scaling limit.

But it is not less important to prove properties of the scaling limit.

Complex Brownian Motion

The process $\{B_t, t \geq 0\}$ is a 1-D Brownian motion (starting at $x \in \mathbb{R}$) if

- $\mathbb{P}(B_0 = x) = 1$ and the function $t \mapsto B_t$ is continuous (wp1),
- $B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_{n-1}} - B_{t_n}$ are independent $\forall\, t_0 < t_1 < \ldots < t_n$,
- for any s, $t \geq 0$, the increment $B_{t+s} - B_s \sim \mathcal{N}(0, t)$ is normally distributed.

If B^1, B^2 are independent 1-D Brownian motions, then $B = (B^1, B^2)$ is a 2-D Brownian motion started at $z = (B^1_0, B^2_0)$.

Write $B[0, T] = \{z \in \mathbb{C} : B_s = z$ some $0 \leq s \leq T\}$.

Suppose that $D \subset \mathbb{C}$.

Let $T_D = \inf\{t : B_t \notin D\}$.

We call T_D the **exit time** of Brownian motion from D (or the hitting time of D^c).

Simple random walk excursion measure in the plane

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Motivation and History

Interested in some physical system; model using discrete or lattice approximation

criticality: temperature at which there is a phase transition [liquid $\to$ gas]

1828: Botanist R. Brown observed erratic motion of pollen in oil

1905: A. Einstein showed "Brownian motion" satisfies heat equation: $u_t = \frac{1}{2}\, u_{xx}$

1923: N. Wiener "mathematicized" BM ($\exists$ a process ...)

1951: Donsker proved invariance principle that SRW converges in law to BM

early 1980's: Physicist M. Aizenman showed intersection exponents for SRW related to critical exponents in statistical physics

early 1990's: G. Lawler-K. Burdzy showed SRW and BM exponents same

2000: G. Lawler-O. Schramm-W. Werner: compute exponent for BM using SLE, showing "intersection exponents are conformally invariant"

2001: S. Fomin proved identity relating SRW connecting boundary points to LERW. Conjectured that it held for continuous processes as well: "discretize the model, compute the probability, and then pass to the limit. Technical details are omitted."

Simple Random Walk

Let X_i be i.i.d. with $\mathbb{P}\{X_i = e\} = 1/4$ for $|e| = 1$.

Set $S_n = x + X_1 + \cdots + X_n$.

The process $\{S_n : n \in \mathbb{N}\}$ is a **simple random walk** on $\mathbb{Z}^2$ starting at $x \in \mathbb{Z}^2$.

Write $S[0, j] = [S(0), S(1), \ldots, S(j)]$ for set of points visited by SRW (in order).

Suppose that $A \subset \mathbb{Z}^2$.

Let $\tau_A = \min\{n : S_n \notin A\} = \min\{n : S_n \in \partial A\}$.

We call τ_A the **exit time** of random walk from A (or the hitting time of A^c).

Background from Probability

A **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space with total measure $\mathbb{P}(\Omega) = 1$.

A **random variable** is a measurable mapping $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{C}, \mathcal{B})$.

X induces a probability measure on $(\mathbb{C}, \mathcal{B})$ called the **law** of Y, $\mathcal{L}_Y := \mathbb{P} \circ Y^{-1}$, defined as $\mathcal{L}_Y(A) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) \in A\})$ for each $A \in \mathcal{B}$.

A **stochastic process** is a collection of random variables $\{Y_i : i \in I\}$ for some indexing set I.

Background and Notation from Complex Analysis

Everything is exclusively two-dimensional ($\mathbb{C} \cong \mathbb{R}^2$).
Write w, x, y, z for points in $\mathbb{C}$, and t, n for time ($\in \mathbb{R}$).

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes open **unit disk**.

Define $\operatorname{inrad}(D) = \inf\{|z| : z \in \mathbb{C} \setminus D\}$ and $\operatorname{rad}(D) := \sup\{|z| : z \in \partial D\}$.

$\mathcal{D}^* := \{$domains $D \subset \mathbb{C} : 0 \in D;\ D$ simply connected, bounded; ∂D Jordan$\}$
$\mathcal{D}\ \ := \{D \in \mathcal{D}^* : \partial D$ piecewise analytic$\}$

For D, $D' \in \mathcal{D}^*$, let $\mathcal{T}(D, D') := \{$conformal transformations $f : D \to D'$ onto$\}$.

By Riemann mapping theorem, $\mathcal{T}(D, D') \neq \emptyset$.

"Classical" Convergence Result

Donsker's invariance principle. Simple random walk *converges* in *scaling limit* to Brownian motion which is *conformally invariant*.

- *scaling limit*: let lattice spacing tend to 0
- *converges*: weak convergence of probability measures

$\mu_n \Rightarrow \mu$ weakly if for every bounded, continuous function $h : \Xi \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_\Xi h(x)\,\mu_n(\mathrm{d}x) = \int_\Xi h(x)\,\mu(\mathrm{d}x).$$

If S_j is SRW define continuous time process Y_t by linear interpolation.

Let $X_t^{(n)} := n^{-1/2}\,Y_{nt}$ for each n (Brownian scaling).

$\mathbb{P}_n$: measure induced by $X^{(n)}$ on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$.

$\mathbb{W}$: Wiener measure on $C[0,\infty)$ under which $B_t(\omega) := \omega(t)$ is BM.

Theorem. $\mathbb{P}_n \Rightarrow \mathbb{W}$ weakly

Important Results from Complex Analysis

Riemann Mapping Theorem. Suppose that D, $D' \in \mathcal{D}^*$. Then $\exists!\ f \in \mathcal{T}(D, D')$ with $f(0) = 0$ and $f'(0) > 0$.

Carathéodory Extension Theorem. Suppose that D, $D' \in \mathcal{D}^*$. If $f \in \mathcal{T}(D, D')$, then f can be extended to a homeomorphism of $\overline{D} = D \cup \partial D$ onto $\overline{D'}$.

Koebe One-Quarter Theorem. If f is conformal mapping of $\mathbb{D}$ with $f(0) = 0$, then image of f contains open disk of radius $|f'(0)|/4$ about the origin.

Beurling Projection Theorem. There is a constant $c < \infty$ such that if $\gamma : [0, 1] \to \mathbb{C}$ is a curve with $\gamma(0) = 0$, $|\gamma(1)| = 1$, $\gamma(0, 1) \subset \mathbb{D}$, and $x \in \mathbb{D}$, then

$$\mathbb{P}^x\{B[0, T_{\mathbb{D}}] \cap \gamma[0, 1] = \emptyset\} \leq c\, |x|^{1/2}.$$

Supplementary Notes and Background Material

Conformal Invariance of Brownian Motion

Brownian motion until its exit from a domain is conformally invariant.

Theorem. Let $f \in \mathcal{T}(D, D')$. If B_t is a 2-D BM started at $x \in D$, stopped at τ_D, then $f(B_t)$ is a (time-changed) BM started at $f(x) \in D'$, stopped at $\tau_{D'}$.

If $A_s := \int_0^s |f'(B_r)|^2 \, \mathrm{d}r$ and $\sigma_t := \inf\{s : A_s \geq t\}$, then $B'_t := f(B_{\sigma_t})$ is BM.

Wiener measure is conformally invariant: $f \circ \mu_D(z) = \mu_{D'}(f(z))$

Overview of Main Result

$D \subset \mathbb{C}$: simply connected, bounded domain with Jordan boundary containing 0

excursion from z to w in D:
$\gamma : [0,T] \to \mathbb{C}$; $0 < T < \infty$; $\gamma(0) = z$; $\gamma(T) = w$; $\gamma(0,T) \subset D$

$\mu_{\partial D,z,w} := \mu_{\partial D}(z,w)$: excursion measure

$D_N = \frac{1}{N}\mathbb{Z}^2 \cap D$ with x, $y \in \partial D_N$.

discrete excursion from x to y in D_N:
S_j SRW on $\frac{1}{N}\mathbb{Z}^2$; $S_0 = x$; $S_n = y$; $[S_1, \ldots, S_{n-1}] \subset D_N$

$\nu_{\partial D_N,x,y} := \nu_{\partial D}(x,y)$: simple random walk excursion measure

Let Γ, $\Upsilon \subset \partial D$ be disjoint open arcs; associate Γ, Υ with Γ_N, $\Upsilon_N \subset \partial D_N$

"Theorem." $\displaystyle\sum_{x \in \Gamma_N} \sum_{y \in \Upsilon_N} \nu_{\partial D_N,x,y}(\cdot) \Rightarrow \int_\Gamma \int_\Upsilon \mu_{\partial D_N,x,y}(\cdot)\, |\mathrm{d}y|\, |\mathrm{d}x|$ weakly.

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Fomin's Conjecture and the Scaling Limit

Theorem. Suppose D, $D' \in \mathcal{D}$, and Γ, $\Upsilon \subset \partial D$. Let $f \in \mathcal{T}(D, D')$ with $f(0) = 0$, $f'(0) > 0$. If

$$\mathcal{H}_{\partial D}(\Gamma, \Upsilon; k) := \int_{V(\Upsilon;k)} \int_{V(\Gamma;k)} \det[H_{\partial D}(x^i, y^j)]_{1 \le i,j \le k} \, |\mathrm{d}x| \, |\mathrm{d}y|$$

where $V(\Gamma; k) := \{(x^1, \ldots, x^k) : x^i \in \Gamma, x^1 < \cdots < x^k\}$ and $V(\Upsilon; k) := \{(y^1, \ldots, y^k) : y^i \in \Upsilon, y^1 < \cdots < y^k\}$, then

$$\mathcal{H}_{\partial D}(\Gamma, \Upsilon; k) = \mathcal{H}_{\partial D'}(\Gamma', \Upsilon'; k).$$

For $D \in \mathcal{D}^*$ *define* $\mathcal{H}_{\partial D}(\Gamma, \Upsilon; k) := \mathcal{H}_{\partial \mathbb{D}}(\Gamma_{\mathbb{D}}, \Upsilon_{\mathbb{D}}; k)$.

Let D_N be $1/N$-scale discrete approximation to D with associated boundary arcs Γ_N, $\Upsilon_N \subset \partial D_N$. If, for each N, the labelling of $x^1, \ldots, x^k \in \Gamma_N$, $y^1, \ldots, y^k \in \Upsilon_N$ is such that the trajectory constraint is satisfied, then

$$\lim_{N \to \infty} \sum_{V(\Upsilon_N;k)} \sum_{V(\Gamma_N;k)} \det \mathbf{h}_{\partial D_N} = (1/4)^k \, \mathcal{H}_{\partial D}(\Gamma, \Upsilon; k).$$

In particular, the limit exists and is conformally invariant.

The Space of Curves $(\mathcal{K}, \mathbb{d})$

$\mathcal{K} :=$ { curves $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $0 < t_\gamma < \infty$}

$$\mathbb{d}(\gamma, \gamma') := \sup_{0 \le s \le 1} |\, \gamma(t_\gamma s) - \gamma'(t_{\gamma'} s)\,| + |t_\gamma - t_{\gamma'}|$$

Problem. $(\mathcal{K}, \mathbb{d})$ is an *incomplete* metric space.

Banach space: $\mathcal{X} := C[0,1] \times \mathbb{R}$ with $d_{\mathcal{X}} := d_\infty + \mathrm{abs}$
Embed $\mathcal{K} \hookrightarrow \mathcal{X}$ so that $(\mathcal{K}, \mathbb{d}) \cong (\mathcal{X}^+, d_{\mathcal{X}})$.

Theorem. If $\gamma_n \in \mathcal{X}^+$ and $d_{\mathcal{X}}(\gamma_n, \gamma) \to 0$ with $t_\gamma > 0$, then $\mathbb{d}(\gamma_n, \gamma) \to 0$.

Defn. Measure μ on $(\mathcal{K}, \mathbb{d})$ is σ-finite measure on $(\mathcal{X}, d_{\mathcal{X}})$ concentrated on $\mathcal{X}^+$.

conformal image of curve: use "Brownian scaling" so $(f \circ \gamma)(t) := f(\gamma_{\sigma_t})$

conformal image of measure on curves, $f \circ \mu$: scaling curves implicit in notation

The Prohorov Topology

(Ξ, ρ): complete, separable metric space (Polish space)

$\mathcal{M}(\Xi)$: collection of all finite measures on $(\Xi, \mathcal{B}_\rho)$

$(\mathcal{M}(\Xi), \wp)$ is also a Polish space where $\wp$ is the Prohorov metric.

Fact. If μ_n, $\mu \in \mathcal{M}$, then $\mu_n \Rightarrow \mu$ weakly iff $\wp(\mu_n, \mu) \to 0$.

Prop. Let γ_n, γ be $\mathcal{K}$-valued random variables with $\mathcal{L}(\gamma_n) = \mu_n$ and $\mathcal{L}(\gamma) = \mu$, respectively. If $\mathbb{P}(\mathbb{d}(\gamma_n, \gamma) \geq \varepsilon) \leq \varepsilon$, then $\wp(\mu_n, \mu) \leq \varepsilon$.

Fomin's Identity

Suppose $A \in \mathcal{A}^n$ and Γ, $\Upsilon \subset \partial A$ are disjoint.

Let $x^1, x^2, \ldots, x^k \in \Gamma$.

Let $S^1, S^2, \ldots, S^k$ be independent SRW starting at $x^1, x^2, \ldots, x^k$, respectively.

Set $\tau_A^i := \min\{j > 0 : S_j^i \notin A\}$.

Let $L^i = \Lambda(S^i)$ be loop erasure of the path $[S^i(0) = x^i, S^i(1), \ldots, S^i(\tau_A^i)]$.

Let $y^1, y^2, \ldots, y^k \in \Upsilon$ be such that any SRW excursion from x^i to y^j intersects any excursion from $x^{i'}$, $i' > i$, to $y^{j'}$, $j' < j$.

Let $\mathcal{E} = \mathcal{E}(x^1, \ldots, x^k, y^1, \ldots, y^k; A)$ be the event

- $S^i_{\tau_A^i} = y^i, \quad i = 1, \ldots, k$, and
- $S^i[0, \tau_A^i] \cap \{L^1 \cup \cdots \cup L^{i-1}\} = \emptyset, \quad i = 2, \ldots, k$.

Theorem (Fomin). $\mathbb{P}(\mathcal{E}) = \det[\mathbf{h}_{\partial A}(x^i, y^j)]_{1 \le i,j \le k}$

The Principal Theorem

Let $D \in \mathcal{D}^*$ with $\mathrm{inrad}(D) = 1$. Let Γ, $\Upsilon \subset \partial D$ by disjoint open boundary arcs.

Theorem. For every $\varepsilon > 0$, there exists an N such that

(a) $\left| h_{\partial D_N}(\Gamma_N, \Upsilon_N) - \frac{1}{4} H_{\partial D}(\Gamma, \Upsilon) \right| \le \varepsilon$,

(b) $\wp(\, \mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N),\ \mu^{\#}_{\partial D}(\Gamma, \Upsilon)\,) \le \varepsilon$, and

(c) $\wp(\, \mu^{\mathsf{rw},\#}_{\partial D_N}(\Gamma_N, \Upsilon_N),\ \mu^{\#}_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N)\,) \le \varepsilon$.

In particular, **(a)**, **(b)**, and **(c)** imply that

$$\lim_{N \to \infty} \wp(\, 4\, \mu^{\mathsf{rw}}_{\partial D_N}(\Gamma_N, \Upsilon_N),\ \mu_{\partial D}(\Gamma, \Upsilon)\,) = 0.$$

Defn and Theorem. $\wp(\, 4\, \mu^{\mathsf{rw}}_{\partial D_N},\ \mu_{\partial D}\,) \to 0$

Classical Potential Theory

Harmonic measure is absolutely continuous wrt Lebesgue measure:

$$\mathbb{P}^z\{B_{T_D} \in \mathrm{d}y\} = H_D(z,y)\,|\mathrm{d}y|$$

Poisson kernel is R-N derivative of HM and is related to Green's function:

$$H_D(z,y) = \frac{\partial}{\partial \mathbf{n}_y} g_D(z,y)$$

Green's function in the unit disk:

$$g_{\mathbb{D}}(w,z) := \log \left| \frac{\overline{z}w - 1}{z - w} \right|$$

Green's function is related to BM:

$$g_D(x) := g_D(0,x) = g_D(x,0) = \mathbb{E}^x[\log|B_{T_D}|] - \log|x| \text{ for } x \in D \quad (*)$$

Green's function is related to Riemann mapping theorem: If $f_D \in \mathcal{T}(D,\mathbb{D})$ with $f_D(0) = 0$, $f'_D(0) > 0$, then

$$f_D(x) = \exp\{-g_D(x) + i\theta_D(x)\}.$$

Behaviour under Conformal Transformation

Suppose D, $D' \in \mathcal{D}^*$ and $f \in \mathcal{T}(D, D')$.
Let w, $z \in D$, $y \in \partial D$ with ∂D locally analytic at y, $\partial D'$ locally analytic at $f(y)$.

Conformal Invariance of Green's Function

$g_D(w, z) = g_{D'}(f(w), f(z))$

Conformal Invariance of Harmonic Measure

$\mathbb{P}^z\{B_{T_D} \in \mathrm{d}y\} = \mathbb{P}^{f(z)}\{B'_{T_{D'}} \in f(\mathrm{d}y)\}$ (B' is time-change of BM)

Conformal Covariance of Poisson Kernel

$H_D(z, y) = |f'(y)|\, H_{D'}(f(z), f(y))$

Conformal Invariance of Wiener Measure

$f \circ \mu_D(z) = \mu_{D'}(f(z))$

Green's Function Estimates, II

For $A \in \mathcal{A}^n$, let $A^{*,n} := \{x \in A : g_A(x) \geq n^{-1/16}\}$.

The choice of $1/16$ for exponent is arbitrary, and slightly better estimates can be obtained with different choice.

There exist sequences α_N, β_N, $\varepsilon_N \downarrow 0$ such that:

Prop. if $x \in D_N^*$, $y \in D_N$, and $|x - y| \geq \beta_N$, then

$$G_{D_N}(x,y) = \frac{2}{\pi} g_{D_N}(x,y) + O(\alpha_N).$$

Theorem. if $D \in \mathcal{D}^*$ with $\mathrm{inrad}(D) = 1$, and x, $y \in \partial D_N$ with $|\theta_{D_N}(x) - \theta_{D_N}(y)| \geq \varepsilon_N$, then

$$h_{\partial D_N}(x,y) = \frac{(\pi/2)\; h_{D_N}(0,x)\; h_{D_N}(0,y)}{1 - \cos(\theta_{D_N}(x) - \theta_{D_N}(y))}[1 + O(\frac{\varepsilon_N^3}{|\theta_{D_N}(x) - \theta_{D_N}(y)|})]. \quad (***)$$

Proof. Let $A_N := 2N\, D_N \in \mathcal{A}^N$ and use last-exit decomposition for $h_{\partial A}(x,y)$.

Green's Function Estimates, I

There exists $(\Omega, \mathcal{F}, \mathbb{P})$ on which can be defined a 2-D BM B and a 2-D SRW S with $B_0 = S_0$ and a constant C such that for each $n \in \mathbb{N}$:

Theorem (Komlós-Major-Tusnády).

$$\mathbb{P}\{\max_{0 \le k \le n} \sup_{|t-k| \le \frac{1}{2}} |B_t - S_k| > C \log n\} \le C n^{-10}.$$

Prop. if $A \in \mathcal{A}^n$, $1 < r \le n^{20}$, and $x \in A$ with $|x| \le n^3$, then

$$\mathbb{P}^x\{|B_{T_A} - S_{\tau_A}| \ge Cr \log n\} \le C r^{-1/2}.$$

Excursion Poisson Kernel

Let D, $D' \in \mathcal{D}$ and $f \in \mathcal{T}(D, D')$.
Let $x \neq y \in \partial D$ s.t. ∂D locally analytic at x, y; $\partial D'$ locally analytic at $f(x)$, $f(y)$.

Defn. If $\mathbf{n}_x$ be the inward unit normal vector to D at x, then

$$H_{\partial D}(x, y) := \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} H_D(x + \varepsilon \mathbf{n}_x, y) = \frac{\partial}{\partial \mathbf{n}_x} \frac{\partial}{\partial \mathbf{n}_y} g_D(x, y) \quad (**).$$

Prop. $H_{\partial \mathbb{D}}(x, y) = \dfrac{1}{\pi} \dfrac{1}{|y - x|^2} = \dfrac{1}{2\pi} \dfrac{1}{1 - \cos(\theta_{\mathbb{D}}(x) - \theta_{\mathbb{D}}(y))}$

Prop. $H_{\partial D}(x, y) = |f'(x)|\,|f'(y)|\, H_{\partial D'}(f(x), f(y))$

Prop. $H_{\partial D}(x, y) = \dfrac{2\pi\, H_D(0, x)\, H_D(0, y)}{1 - \cos(\theta_D(x) - \theta_D(y))} \quad (***)$

Boundary-to-Boundary Excursion Measure

$\mu^{\#}_{\partial\mathbb{D}}(x,y) := \lim_{\varepsilon\to 0+} \mu^{\#}_{\mathbb{D}}((1-\varepsilon)x, y)$ (convergence in $\wp$ using metric $\mathbb{d}$)

$\mu_{\partial\mathbb{D}}(x,y) := H_{\partial\mathbb{D}}(x,y) \cdot \mu^{\#}_{\partial\mathbb{D}}(x,y)$

Prop. $f \circ \mu_{\partial D}(x,y) = |f'(x)|\,|f'(y)|\,\mu_{\partial D'}(f(x), f(y))$

Prop. $\mu_{\partial D}(x,y) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\mu_D(x+\varepsilon\mathbf{n}_x, y)$

$\mu_{\partial D}(\Gamma,\Upsilon) = \int_\Upsilon\int_\Gamma \mu_{\partial D}(x,y)\,|\mathrm{d}x|\,|\mathrm{d}y|$ and $f \circ \mu_{\partial D}(\Gamma,\Upsilon) = \mu_{\partial D'}(\Gamma',\Upsilon')$

$\mu_{\partial D} = \int_{\partial D}\int_{\partial D} \mu_{\partial D}(x,y)\,|\mathrm{d}x|\,|\mathrm{d}y|$ (σ-finite)

Theorem. If D, $D' \in \mathcal{D}$, $f \in \mathcal{T}(D,D')$, then $f \circ \mu_{\partial D} = \mu_{\partial D'}$.

Defn. For $D \in \mathcal{D}^*$, let $\mu_{\partial D} := f \circ \mu_{\partial\mathbb{D}}$.

Let $\mu_{\partial D}(\Gamma,\Upsilon)$ be $\mu_{\partial D}$ restricted to $\mathcal{K}^{\Upsilon}_{\Gamma}(D)$ with $H_{\partial D}(\Gamma,\Upsilon) := |\mu_{\partial D}(\Gamma,\Upsilon)|$

Approximate Boundary Arcs

Fix Γ, $\Upsilon \subset \partial D$ disjoint, open boundary arcs.

Let $\tilde{\Gamma}_N$ be image under Γ of $f_N \circ f^{-1}$. Similar for $\tilde{\Upsilon}_N$.

Associate to $\tilde{\Gamma}_N$ a subset $\Gamma_N \subset \partial D_N$. Similar for Υ_N.

Note. $H_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N) = H_{\partial D}(\Gamma, \Upsilon)$, but

$$\mu_{\partial \tilde{D}_N}(\tilde{\Gamma}_N, \tilde{\Upsilon}_N) = (f_N \circ f^{-1}) \circ \mu_{\partial D}(\Gamma, \Upsilon)$$

Prop. Suppose $f_n \to f$ uniformly on compacta of $\mathbb{D}$.

Let $\gamma \in \mathcal{K}(\mathbb{D})$ with $|\gamma(0)| < 1$ and $|\gamma(t_\gamma)| < 1$ (so that γ is NOT an excursion).

Then $\mathbb{d}(f_n \circ \gamma, f \circ \gamma) \to 0$.

Carathéodory Convergence of Approximate Domains

Defn and Theorem. $f_n \in \mathcal{T}(\mathbb{D}, E_n)$ with $f_n(0) = 0$, $f_n'(0) > 0$.
$f \in \mathcal{T}(\mathbb{D}, E)$ and $f(0) = 0$, $f'(0) > 0$.
$E_n \overset{\text{Cara}}{\to} E$ iff $f_n \to f$ uniformly on every compact subset of $\mathbb{D}$.

Let $D \in \mathcal{D}^*$ with $\operatorname{inrad}(D) = 1$.

D_N: connected component containing 0 of $D_N' = \{x \in \frac{1}{N}\mathbb{Z}^2 \cap D : \frac{1}{N}\mathcal{S}_x \subseteq D\}$.

$\tilde{D}_N \in \mathcal{D}$: union of scaled squares so that $\tilde{D}_N \cup \partial\tilde{D}_N = \bigcup_{x \in D_N} \frac{1}{N}\mathcal{S}_x$.

If $x \in \partial_i D_N$ or $\partial\tilde{D}_N$ or ∂D_N, then $\operatorname{dist}(x, \partial D) \leq CN^{-1}$.

Theorem. $\tilde{D}_N \overset{\text{Cara}}{\to} D$

Proof. Need to estimate $|I(z) - I_N(z)|$ where $I : \mathbb{D} \to \mathbb{D}$ is the identity and $I_N : \mathbb{D} \to \mathbb{D}_N$ is "almost the identity."

Discrete Potential Theory

Let $A \in \mathcal{A}^n$ and $\tau_A := \min\{j > 0 : S_j \in \partial A\}$.

Discrete Poisson kernel *is* probability SRW starting at x exits A at y:

$$h_A(x,y) := \mathbb{P}^x\{S_{\tau_A} = y\}$$

Discrete Green's function *is* expected # of visits to y starting at x before exiting A:

$$G_A(x,y) := \mathbb{E}^x[\sum_{j=0}^{\tau_A - 1} \mathbb{1}_{\{S_j = y\}}]$$

Write $G_A(x) := G_A(x,0) = G_A(0,x)$.

Fact. For $x \in A$, $\quad G_A(x) = \mathbb{E}^x[a(S_{\tau_A})] - a(x) \quad (*)$ where

$$a(x) = \frac{2}{\pi} \log|x| + k_0 + o(|x|^{-3/2})$$

is potential kernel for SRW and k_0 is a constant.

Discrete Excursion Poisson Kernel

Defn. For x, $y \in \partial A$, the **discrete excursion Poisson kernel** is

$$h_{\partial A}(x, y) := \mathbb{P}^x\{S_{\tau_A} = y, S_1 \in A\}.$$

Prop. If $A \in \mathcal{A}^n$, $x \in \partial A$, $y \in \partial A$, then

$$h_{\partial A}(x, y) = \left(\frac{1}{4}\right)^2 \sum_{(z,x)\in\partial_e A} \sum_{(w,y)\in\partial_e A} G_A(z, w) \quad (**).$$

Goal. Relate Green's function for BM to Green's function for SRW.

Why? To derive relationship between discrete excursion Poisson kernel and excursion Poisson kernel.

Discrete Excursion Measure

S_j: SRW on $\mathbb{Z}^2$

ω: discrete excursion (set of lattice points)

$\tilde{\omega}(t) \in \mathcal{K}$: lattice points joined together ($|\omega|$ steps, time $t_{\tilde{\omega}} := 2|\omega|$)

$$\Phi_N \tilde{\omega}(t) = \frac{1}{2N}\, \tilde{\omega}(4N^2 t) \quad \text{for} \quad 0 \le t \le t_{\Phi_N \tilde{\omega}} = \frac{t_{\tilde{\omega}}}{4N^2} = \frac{|\omega|}{2N^2}$$

Lemma. $\dfrac{1}{4N^2}\, \mathrm{d}(\gamma, \gamma') \le \mathrm{d}(\Phi_N \gamma, \Phi_N \gamma') \le \dfrac{1}{2N}\, \mathrm{d}(\gamma, \gamma')$

$\mu^{\mathrm{rw}}_{\partial D_N}(x, y)$ gives weight $4^{-|\omega|}$ to every "$1/N$-scale discrete excursion."

$$\mu^{\mathrm{rw}}_{\partial D_N}(\Gamma_N, \Upsilon_N) := \sum_{x \in \Gamma_N} \sum_{y \in \Upsilon_N} \mu^{\mathrm{rw}}_{\partial D_N}(x, y)$$