The Basic Theory of Univalent Functions

A Guide to Some of the Complex Analysis Necessary for Understanding the Schramm-Loewner Evolution (SLE)

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Preface

When I started writing these notes, the original goal was to provide a comprehensive introduction to the Schramm-Loewner evolution (i.e., the stochastic Loewner evolution or SLE) suitable for advanced undergraduates or beginning graduate students. SLE is one of the most exciting areas of current mathematical research as evidenced by the International Mathematical Union's awarding of the Fields medal in 2006 to Wendelin Werner "for his contributions to the development of stochastic Loewner evolution, the geometry of two-dimensional Brownian motion, and conformal field theory." The envisioned prerequisites that one would need include a first undergraduate course in complex analysis and a first undergraduate course in probability.

However, in order to develop the Schramm-Loewner evolution (SLE), it is necessary to derive Loewner's partial differential equation (PDE). Charles Loewner introduced this PDE in 1923, and the non-elementary method he developed is still one of the most effective approaches to solving extremal problems for univalent functions. Before one can derive Loewner's equation, however, it is necessary to develop some of the theory of univalent functions. In addition to being prerequisites for establishing Loewner's equation, they are interesting in their own right. It is also worth noting that most of these results are quite elementary, and their proofs do not generally require techniques beyond the complex analysis usually presented in a typical undergraduate course.

Unfortunately, the original goal has not yet been met and all that exists at present is a set of notes on the basic theory of univalent functions. This chapter is meant to lead to a chapter on Loewner's equation, but such a chapter does not yet exist! The motivation for why someone should care about univalent functions as a prerequisite for Loewner's equation and SLE is also currently missing. Nonetheless, these notes are complete enough to be made available, and as a topic in a second complex analysis course, they could probably be covered in full detail in about nine lecture hours.

I still believe that the basics of SLE can be taught to a sufficiently interested advanced undergraduate or beginning graduate student, and that there is a need for such an introduction. As such, I will continue to expand these notes when time allows.

As noted elsewhere, these notes are heavily influenced by the excellent book of Peter Duren [4]. However, there are noticeable differences from that presentation, and we have also included a few pertinent results which appear in the recent monograph by Greg Lawler [9]. Special thanks are owed to Kevin Petrychyn who was supported by an NSERC undergraduate student research award (USRA) in 2007 for his help in the preparation of these notes.

I would appreciate hearing from anyone who stumbles upon these notes and who has comments, corrections, and/or suggestions.

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Brief Review of Complex Analysis

We include a very brief review of some of the basic facts from undergraduate complex analysis that we will need. In fact, essentially all of the results given in this introductory chapter may be found in [11], a book designed to introduce the methods of complex analysis to undergraduate students in mathematics, science, and engineering.

Let x and y be real numbers and let i denote the imaginary unit having the property that $i^2 = -1$. A *complex number* is an expression of the form $x + iy$. Write $\mathbb{C}$ to denote the set of *complex numbers*. That is,

$$\mathbb{C} = \{z = x + iy : x \in \mathbb{R}, \quad y \in \mathbb{R}\}.$$

There is a natural identification between $\mathbb{C}$ and $\mathbb{R}^2$. Let $z = x + iy$ be a complex number. We call x the *real part* of z, denoted $\mathrm{Re}\{z\}$, and call y the *imaginary part* of z, denoted $\mathrm{Im}\{z\}$. In order to perform complex arithmetic, we simply use the rules of ordinary (real) arithmetic remembering that $i^2 = -1$. In particular, suppose that $z = x+iy$ and $w = a+ib$ are complex numbers. The binary operations of addition, subtraction, multiplication, and division are as follows:

- addition

$$z + w = (x + iy) + (a + ib) = (x + a) + i(y + b)$$

- subtraction

$$z - w = (x + iy) - (a + ib) = (x + a) - i(y + b)$$

- multiplication

$$z \cdot w = (x + iy) \cdot (a + ib) = xa + iya + ixb + i^2yb = (xa - yb) + i(ya + xb)$$

- division

$$\frac{z}{w} = \frac{x + iy}{a + ib} = \frac{x + iy}{a + ib} \cdot \frac{a - ib}{a - ib} = \frac{(xa + yb) + i(ya - xb)}{a^2 + b^2} = \frac{xa + yb}{a^2 + b^2} + i\frac{ya - xb}{a^2 + b^2}$$

  (assuming that $w \neq 0$).

The *complex conjugate* of $z = x + iy$ is $\bar{z}$ defined by $\bar{z} = x - iy$ and the *modulus* of z is the positive real number $|z|$ with $|z|^2 = z\bar{z} = \bar{z}z = x^2 + y^2$.

A *domain* $D \subset \mathbb{C}$ is an open and connected non-empty subset of the complex plane. The domain D is *simply connected* if both D and $\mathbb{C} \setminus D$ are connected. A complex function $f : D \to \mathbb{C}$ defined for all $z \in D$ is said to be *complex differentiable* at $z_0 \in D$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The function $f : D \to \mathbb{C}$ is *analytic at* z_0 (or *holomorphic at* z_0) if it is complex differentiable at every point in some neighbourhood $\mathcal{N}(z_0;\epsilon)$ of $z_0 \in D$. We say that f is *analytic on* D if f is analytic at z_0 for every $z_0 \in D$. A function $f : D \to \mathbb{C}$ with the property that $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$ with $z_1 \neq z_2$ is said to be *one-to-one on* D (or *univalent* or *schlict*). Chapter 2 is devoted to the study of analytic, one-to-one functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Finally, a function $f : D \to \mathbb{C}$ which is both analytic on D and one-to-one on D is called *conformal on* D. We will often refer to such an f as a *conformal mapping of* D.

1.1 Cauchy-Riemann equations

Suppose that $f : D \to \mathbb{C}$ is analytic on D. If we write $f(z) = u(z) + iv(z)$, then u and v, the real and imaginary parts of f, respectively, satisfy the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Conversely, if $u : D \to \mathbb{R}$ and $v : D \to \mathbb{R}$ have continuous partials satisfying

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then $u + iv : D \to \mathbb{C}$ is analytic on D.

1.2 Cauchy integral theorem

Suppose that $D \subsetneq \mathbb{C}$ is a simply connected domain and that $f : D \to \mathbb{C}$ is analytic on D. Suppose further that D' is a simply connected domain with $\overline{D'} \subset D$ such that its boundary $\partial D'$ is a rectifiable Jordan curve. It necessary follows that f is analytic inside D' and on $\partial D'$. The *Cauchy integral theorem* states that

$$\int_{\partial D'} f(z)\,\mathrm{d}z = 0$$

and the *Cauchy integral formula* states that

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D'} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta$$

where the contour integrals are taken counterclockwise. If, in addition, f is continuous on $\overline{D}$ and ∂D is itself a rectifiable Jordan curve, then a straightforward limiting argument (see [12] for instance) proves that the Cauchy integral theorem

$$\int_{\partial D} f(z)\,\mathrm{d}z = 0$$

and the Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta$$

both hold where, again, the contour integrals are taken counterclockwise.

1.3 Taylor series expansion

If the complex function f is analytic at $z_0 \in \mathbb{C}$, then it is known that f must have derivatives of all orders at z_0. Furthermore, f has a *Taylor series expansion* given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{1.1}$$

which is convergent in some open disk centred at z_0. More precisely, if f is analytic in the disk $D_r = \{|z - z_0| < r\}$, then the Taylor series expansion (1.1) converges to $f(z)$ for all $z \in D_r$. Moreover, the convergence is uniform in any closed subdisk $\{|z - z_0| \le r'\}$ with $r' < r$.

Conversely, if a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1.2}$$

converges for every z in the disk D_r, $r > 0$, then (1.2) defines a function that is analytic on D_r.

1.4 Möbius transformations

As noted earlier, we will be particularly concerned with functions which are analytic and one-to-one on the unit disk $\mathbb{D}$. A particular class of such functions are the so-called *Möbius transformations* (or *fractional linear transformations*) which map $\mathbb{D}$ conformally onto $\mathbb{D}$.

Proposition 1.4.1 *If $z_0 \in \mathbb{D}$ is fixed, then*

$$g(z) = \frac{z - z_0}{1 - \bar{z}_0 z} \tag{1.3}$$

is a conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$ with $z_0 \mapsto 0$ and $|g'(z)| > 0$.

Proof Clearly, $g(z_0) = 0$. Furthermore,

$$g'(z) = \frac{(1 - \bar{z}_0 z) + \bar{z}_0 (z - z_0)}{(1 - \bar{z}_0 z)^2} = \frac{1 - |z_0|^2}{(1 - \bar{z}_0 z)^2}$$

and so we see that $|g'(z)| > 0$ for every $z \in \mathbb{D}$ since $|z_0| < 1$. In order to show that g maps $\mathbb{D}$ into $\mathbb{D}$, we must show that $|g(z)| < 1$ for every $|z| < 1$. This is accomplished via the following set of inqualities. Clearly, $|z| < 1$ and so $|z|^2 < 1$. Since $1 - |z_0|^2 \in \mathbb{R}$ we find $(1 - |z_0|^2)|z|^2 < 1 - |z_0|^2$ which upon rearranging gives

$$|z|^2 + |z_0|^2 < 1 + |z_0|^2 |z|^2.$$

We now subtract $\bar{z} z_0 + z \bar{z}_0$ from both sides of the inequality and note that this is a valid operation since $\bar{z} z_0 + z \bar{z}_0 \in \mathbb{R}$ as an easy calculation shows (Exercise 1.4.2). That is,

$$|z|^2 + |z_0|^2 - \bar{z} z_0 - z \bar{z}_0 < 1 + |z_0|^2 |z|^2 - \bar{z} z_0 - z \bar{z}_0$$

and so

$$z \bar{z} - \bar{z} z_0 - z \bar{z}_0 + z_0 \bar{z}_0 < 1 - \bar{z} z_0 - z \bar{z}_0 - \bar{z}_0 z_0 \bar{z} z$$

which implies

$$(z - z_0)(\bar{z} - \bar{z}_0) < (1 - \bar{z}_0 z)(1 - z_0 \bar{z}) \quad \text{or} \quad |z - z_0|^2 < |1 - \bar{z}_0 z|^2.$$

Thus,

$$|g(z)| = \frac{|z - z_0|}{|1 - \bar{z}_0 z|} < 1$$

as required. We complete the proof by showing that g is onto $\mathbb{D}$. Recall that in order to show a function g is onto we must show that for every $w \in \text{Range}(g)$ there is some $z \in \text{Domain}(g)$ with $w = g(z)$. Therefore, suppose that $w \in \mathbb{D}$ be arbitrary. Setting

$$w = \frac{z - z_0}{1 - \bar{z}_0 z}$$

and solving for z implies that

$$w - \bar{z}_0 z w = z - z_0 \quad \text{and so} \quad z = \frac{w + z_0}{1 + \bar{z}_0 w}.$$

Noting that $\frac{w+z_0}{1+\bar{z}_0 w} \in \mathbb{D}$ and

$$g\left(\frac{w + z_0}{1 + \bar{z}_0 w}\right) = w,$$

we conclude that $g : \mathbb{D} \to \mathbb{D}$ is onto, and the proof is complete. $\square$

Exercise 1.4.2 *Let z, $z_0 \in \mathbb{C}$ be arbitrary complex numbers. Prove that $\bar{z} z_0 + z \bar{z}_0 \in \mathbb{R}$.*

We will also have occasion to use the Taylor expansion of the Mobius transformation (1.3). That is, if $z_0 \in \mathbb{D}$ is fixed, then

$$\begin{aligned} g(z) = \frac{z - z_0}{1 - \bar{z}_0 z} &= (z - z_0)\left(1 + \bar{z}_0 z + \bar{z}_0^2 z^2 + \bar{z}_0^3 z^3 + \cdots\right) \\ &= z_0 + (1 - \bar{z}_0 z_0) z + (\bar{z}_0 - \bar{z}_0^2 z_0) z^2 + (\bar{z}_0^2 - \bar{z}_0^3 z_0) z^3 + \cdots \\ &= z_0 + (1 - |z_0|^2) z + \bar{z}_0 (1 - |z_0|^2) z^2 + \bar{z}_0^2 (1 - |z_0|^2) z^3 + \cdots \\ &= z_0 + (1 - |z_0|^2) \sum_{n=1}^{\infty} \bar{z}_0^{n-1} z^n \end{aligned} \tag{1.4}$$

Exercise 1.4.3 *Let $a \in \mathbb{C}$ with $|a| > 1$. Find the image of the unit disk under the analytic function*

$$f(z) = \frac{1}{z + a}.$$

1.5 Riemann mapping theorem

The Riemann mapping theorem, one of the most remarkable results from complex analysis, states that any simply connected proper subset of $\mathbb{C}$ can be mapped conformally onto the unit disk $\mathbb{D} = \{|z| < 1\}$.

Theorem 1.5.1 (Riemann Mapping Theorem) *Let $D \subsetneq \mathbb{C}$ be a simply connected domain, and let $z_0 \in D$ be any given point. Then there exists a unique analytic, one-to-one function $f : D \to \mathbb{D}$ which maps D onto $\mathbb{D}$ and has the properties that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Since the inverse image of a conformal map is also conformal, the Riemann mapping theorem implies that any two simply connected domains (neither of which is $\mathbb{C}$ itself) are conformally equivalent. That is, if $D \subsetneq \mathbb{C}$ and $D' \subsetneq \mathbb{C}$ are simply connected, $z \in D$, and $w \in D'$, then there exists a unique conformal transformation $f : D \to D'$ with $f(z) = w$ and $f'(z) > 0$.

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The Basic Theory of Univalent Functions

2.1 Introduction to univalent functions

Let $D \subset \mathbb{C}$ be a domain; that is, an open and connected non-empty subset of the complex plane. Recall that a function $f : D \to \mathbb{C}$ is *analytic at* z_0 if it is complex differentiable at every point in some neighbourhood of $z_0 \in D$. We say that f is *analytic on* D if f is analytic at z_0 for every $z_0 \in D$.

Definition 2.1.1 *A function $f : D \to \mathbb{C}$ is called* univalent on D *(or* schlict *or* one-to-one*) if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$ with $z_1 \neq z_2$.*

Fact It follows from Rouché's theorem that if f is analytic on D, then $f'(z_0) \neq 0$ if and only if f is *locally univalent at* z_0, i.e., if f is univalent in some neighbourhood of z_0. We leave this as an exercise for the reader familiar with Rouché's theorem; see [12, page 198] for details.

Definition 2.1.2 *A function $f : D \to \mathbb{C}$ which is both analytic on D and univalent on D is called* conformal on D*. We will often refer to such an f as a* conformal mapping of D*.*

A word of notation is needed here. If we are interested in both the domain and the range of a conformal mapping f, then we will write this explicitly as "let $f : D \to D'$ be a *conformal transformation*." That is, $f : D \to D'$ is a conformal mapping of D which is onto D'; i.e., $f(D) = D'$.

Example 2.1.3 *It is important to remember that the underlying domain is an integral part of the definition of a univalent function (or a conformal mapping). Suppose that*

$$D = \{z \in \mathbb{C} : 0 < |z| < 1, \mathrm{Im}\{z\} > 0, \mathrm{Re}\{z\} > 0\} = \{z \in \mathbb{C} : 0 < |z| < 1, 0 < \arg z < \pi/2\}$$

which is that part of the unit disk in the first quadrant. The function $f(z) = z^2$ then maps D conformally onto $\mathbb{D} \cap \mathbb{H} = \{z \in \mathbb{C} : 0 < |z| < 1, \mathrm{Im}\{z\} > 0\}$. That is, $f : D \to \mathbb{D} \cap \mathbb{H}$ is analytic and univalent on D, and onto $\mathbb{D} \cap \mathbb{H}$. However, the function $g(z) = z^2$ does NOT map $\mathbb{D}$ conformally onto the unit disk $\mathbb{D}$, although $g(\mathbb{D}) = \mathbb{D}$. While $g : \mathbb{D} \to \mathbb{D}$ is analytic, it is not univalent. For instance, $g(1/2) = g(-1/2) = 1/4$. In fact, $g'(0) = 0$ which means that there is no neighbourhood of 0 in which g is univalent.

Example 2.1.4 *It is also important to note that an analytic function may be locally univalent throughout a domain although it need not be univalent in that domain. Consider the domain*

$$D = \{z \in \mathbb{C} : 1 < |z| < 2,\ \ 0 < \arg z < 3\pi/2\}$$

and the function $f : D \to \mathbb{C}$ given by $f(z) = z^2$. It is clear that f is analytic on D and locally univalent at every $z_0 \in D$ since $f'(z_0) = 2z_0 \neq 0$ for all $z_0 \in D$. However, f is not univalent on D since

$$f\left(\frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}i\right) = f\left(-\frac{3}{2\sqrt{2}} - \frac{3}{2\sqrt{2}}i\right) = \frac{9}{4}i.$$

Exercise 2.1.5 *Suppose that*

$$D = \{z \in \mathbb{C} : (\mathrm{Re}\{z\} - 1)^2 + \mathrm{Im}\{z\}^2 < 1/4\} = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1/4\}.$$

Let $f : D \to \mathbb{C}$ be given by $f(z) = z^2$. Show that f is univalent on D.

Recall from the Riemann mapping theorem (Theorem 1.5.1) that any simply connected proper subset of the complex plane is conformally equivalent to the unit disk. That is, if $D \subsetneq \mathbb{C}$ is simply connected and $z_0 \in D$, then there exists a unique conformal transformation $f : \mathbb{D} \to D$ with $f(0) = z_0$ and $f'(0) > 0$. Therefore, statements about univalent functions on arbitrary simply connected domains can be translated to statements about univalent functions on the unit disk. For this reason, we will study in detail the following class of univalent functions.

Notation Let $\mathcal{S}$ denote the set of analytic, univalent functions on the unit disk $\mathbb{D}$ normalized by the conditions that $f(0) = 0$ and $f'(0) = 1$. That is,

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic and univalent on } \mathbb{D},\ f(0) = 0,\ f'(0) = 1\}.$$

It follows that every $f \in \mathcal{S}$ has a Taylor expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1,$$

where $a_n \in \mathbb{C}$, $n = 2, 3, \ldots$. In order to simplify certain formulæ, we will sometimes follow the convention of setting $a_1 = 1$ for $f \in \mathcal{S}$.

Example 2.1.6 *Perhaps the most important member of $\mathcal{S}$ is the Koebe function which is given by*

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$$

and maps the unit disk to the complement of the ray $(-\infty, -1/4]$. This can be verified by writing

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$

and noting that $\frac{1+z}{1-z}$ maps the unit disk conformally onto the right half-plane $\{\mathrm{Re}\{z\} > 0\}$; see Fig. 2.1. The

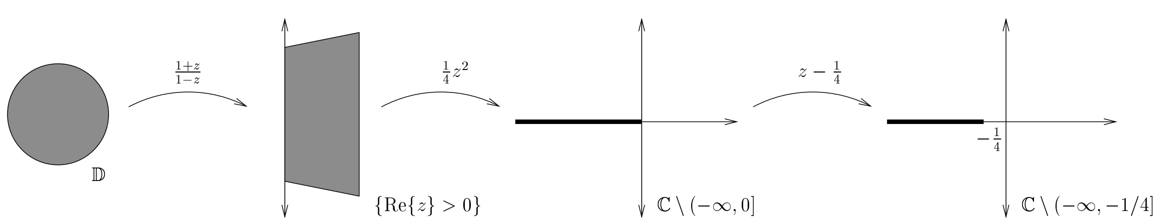

Fig. 2.1. The Koebe function maps $\mathbb{D}$ conformally onto $\mathbb{C} \setminus (-\infty, -1/4]$.

extremal nature of the Koebe function will be seen as we progress through this chapter. In fact, many of the results that we will discuss give bounds for functions in $\mathcal{S}$ and these bounds will be attained only by the Koebe function. See Exercise 2.4.2 for details.

Example 2.1.7 *Other examples of functions belonging to $\mathcal{S}$ include*

(i) *the identity map, $f(z) = z$;*

(ii) $f(z) = \dfrac{z}{1-z} = z + z^2 + z^3 + \cdots$ *which maps $\mathbb{D}$ onto the half-plane $\{\mathrm{Re}\{z\} > -1/2\}$;*

(iii) $f(z) = \dfrac{z}{1-z^2} = z + z^3 + z^5 + \cdots$ *which maps $\mathbb{D}$ onto the plane minus the two half-lines $[1/2, \infty)$ and $(-\infty, -1/2]$;*

(iv) $f(z) = \dfrac{1}{2}\log\dfrac{1+z}{1-z}$ *which maps* $\mathbb{D}$ *onto the horizontal strip* $\{-\pi/4 < \mathrm{Im}\{z\} < \pi/4\}$*; and*
(v) $f(z) = z - \dfrac{1}{2}z^2 = \dfrac{1}{2}[1-(1-z)^2]$ *which maps* $\mathbb{D}$ *onto the interior of a cardioid.*

It is natural to ask which transformations preserve the class $\mathcal{S}$. The following example shows that $f \in \mathcal{S}$ and $g \in \mathcal{S}$ need not imply that $f + g \in \mathcal{S}$ so that $\mathcal{S}$ is not closed under addition.

Example 2.1.8 *Let*

$$f(z) = \frac{z}{1-z} \quad \textit{and} \quad g(z) = \frac{z}{1+iz}$$

so that $f, g \in \mathcal{S}$. *However,*

$$f'(z) = \frac{1}{(1-z)^2} \quad \textit{and} \quad g'(z) = \frac{1}{(1+iz)^2}$$

so that

$$f'(z) + g'(z) = \frac{2-2(1-i)z}{(1-z)^2(1+iz)^2}$$

from which we conclude that $f'(z) + g'(z) = 0$ *if*

$$z = \frac{1}{1-i} = \frac{1+i}{2}.$$

This shows that $f + g \notin \mathcal{S}$.

We now present seven transformations which do preserve the class $\mathcal{S}$. They are naturally divided into three groups, and so we present the first three (rotation, dilation, conjugation) in Theorem 2.1.9, the second three (disk automorphism, range transformation, omitted-value transformation) in Theorem 2.1.11, and the final one (square root transformation) in Theorem 2.1.15.

Theorem 2.1.9 *The class* $\mathcal{S}$ *is preserved under the following transformations.*
(i) (rotation) *If* $f \in \mathcal{S}$, $\theta \in \mathbb{R}$, *and* $g(z) = e^{-i\theta}f(e^{i\theta}z)$, *then* $g \in \mathcal{S}$.
(ii) (dilation) *If* $f \in \mathcal{S}$, $0 < r < 1$, *and* $g(z) = \frac{1}{r}f(rz)$, *then* $g \in \mathcal{S}$.
(iii) (conjugation) *If* $f \in \mathcal{S}$ *and* $g(z) = \overline{f(\bar{z})}$, *then* $g \in \mathcal{S}$.

Proof In order to prove that $\mathcal{S}$ is preserved under rotation, dilation, and conjugation, we begin by noting that the composition of one-to-one mappings is again a one-to-one mapping.

(i) Suppose that $f \in \mathcal{S}$. Let $R(z) = e^{i\theta}z$ and $T(z) = e^{-i\theta}z$ so that $R : \mathbb{C} \to \mathbb{C}$ and $T : \mathbb{C} \to \mathbb{C}$ are clearly one-to-one. Since $g(z) = e^{-i\theta}f(e^{i\theta}z) = (T \circ f \circ R)(z)$ is a composition of one-to-one mappings, we conclude that g is univalent on $\mathbb{D}$. Since

$$g'(z) = e^{-i\theta} \cdot e^{i\theta} \cdot f'(e^{i\theta}z) = f'(e^{i\theta}z)$$

we see that g is analytic on $\mathbb{D}$. Furthermore, $g(0) = f(0) = 0$ and $g'(0) = f'(0) = 1$ so that $g \in \mathcal{S}$ as required. We also note that the Taylor expansion of g is given by

$$g(z) = e^{-i\theta}\left(e^{i\theta}z + a_2e^{2i\theta}z^2 + a_3e^{3i\theta}z^3 + \cdots\right) = z + a_2e^{i\theta}z^2 + a_3e^{2i\theta}z^3 + \cdots.$$

(ii) Suppose that $f \in \mathcal{S}$ and let $0 < r < 1$. Let $R(z) = rz$ and $T(z) = \frac{z}{r}$ so that $R : \mathbb{C} \to \mathbb{C}$ and $T : \mathbb{C} \to \mathbb{C}$ are clearly one-to-one. Since $g(z) = \frac{1}{r}f(rz) = (T \circ f \circ R)(z)$ is a composition of one-to-one mappings, we conclude that g is univalent on $\mathbb{D}$. Since

$$g'(z) = \frac{1}{r} \cdot r \cdot f'(rz) = f'(rz)$$

we see that g is analytic on $\mathbb{D}$. Furthermore, $g(0) = f(0) = 0$ and $g'(0) = f'(0) = 1$ so that $g \in \mathcal{S}$ as required. We also note that the Taylor expansion of g is given by

$$g(z) = \frac{1}{r}\left(rz + a_2 r^2 z^2 + a_3 r^3 z^3 + \cdots\right) = z + a_2 r z^2 + a_3 r^2 z^3 + \cdots .$$

(iii) Suppose that $f \in \mathcal{S}$. Let $w(z) = \bar{z}$ so that $w : \mathbb{C} \to \mathbb{C}$ is clearly one-to-one. Since $g(z) = \overline{f(\bar{z})} = (w \circ f \circ w)(z)$ is a composition of one-to-one mappings, we conclude that g is univalent on $\mathbb{D}$. Note that $w(z)$ is not analytic on $\mathbb{D}$ (see Exercise 2.1.10 below), and so we cannot simply use the fact that a composition of analytic functions is analytic as was done in (i) and (ii). Instead, we note that the Taylor series for f, namely

$$z + \sum_{n=2}^{\infty} a_n z^n \tag{2.1}$$

has radius of convergence 1. That is, the Taylor series (2.1) converges to $f(z)$ for all $|z| < 1$ with the convergence uniform on every closed disk $|z| \le r < 1$. It then follows that the Taylor series

$$z + \sum_{n=2}^{\infty} \bar{a}_n z^n \tag{2.2}$$

has radius of convergence 1, and so (2.2) defines an analytic function on $\mathbb{D}$. Hence, we conclude that

$$g(z) = \overline{f(\bar{z})} = \overline{\bar{z} + a_2 \bar{z}^2 + a_3 \bar{z}^3 + \cdots} = z + \bar{a}_2 z^2 + \bar{a}_3 z^3 + \cdots$$

is analytic on $\mathbb{D}$ with $g(0) = 0$ and $g'(0) = 1$. Thus, $g \in \mathcal{S}$ as required.

Taking (i), (ii), and (iii) together completes the proof. $\square$

Exercise 2.1.10 *Let $w : \mathbb{D} \to \mathbb{D}$ be given by $w(z) = \bar{z}$. Prove that w is not analytic at $z_0 = 0$, i.e., show that the limit*

$$\lim_{z \to z_0} \frac{w(z) - w(z_0)}{z - z_0}$$

does not exist.

Theorem 2.1.11 *The class $\mathcal{S}$ is preserved under the following transformations.*

(i) (disk automorphism) *If $f \in \mathcal{S}$ and*

$$g(z) = \frac{f\left(\frac{z+z_0}{1+\bar{z}_0 z}\right) - f(z_0)}{(1 - |z_0|^2) f'(z_0)}$$

for any $|z_0| < 1$, then $g \in \mathcal{S}$.

(ii) (range transformation) *If $f \in \mathcal{S}$, $\phi : f(\mathbb{D}) \to \mathbb{C}$ is analytic and univalent on $f(\mathbb{D})$, and*

$$g(z) = \frac{(\phi \circ f)(z) - \phi(0)}{\phi'(0)},$$

then $g \in \mathcal{S}$.

(iii) (omitted-value transformation) *If $f \in \mathcal{S}$ with $f(z) \neq w$ and*

$$g(z) = \frac{w f(z)}{w - f(z)},$$

then $g \in \mathcal{S}$.

Proof In order to prove that $\mathcal{S}$ is preserved under disk automorphism, range transformation, and omitted-value transformation, we again note as in the proof of Theorem 2.1.9 that the composition of one-to-one mappings is a one-to-one mapping.

(i) Suppose that $f \in \mathcal{S}$ and let $w(z) = \frac{z+z_0}{1+\bar{z}_0 z}$ be the Möbius transformation which maps the unit disk $\mathbb{D}$ conformally onto itself with $w(0) = z_0$. Since $z_0 \in \mathbb{D}$, we conclude that

$$g(z) = \frac{f(w(z)) - f(z_0)}{(1-|z_0|^2)f'(z_0)}$$

is univalent on $\mathbb{D}$ with $g(0) = 0$. Furthermore,

$$g'(z) = \frac{w'(z)f'(w(z))}{(1-|z_0|^2)f'(z_0)} = \frac{f'(w(z))}{(1-\bar{z}_0 z)^2 f'(z_0)}$$

so that g is analytic on $\mathbb{D}$ with $g'(0) = 1$. Thus, $g \in \mathcal{S}$ as required.

(ii) Suppose that $f \in \mathcal{S}$ and let $\phi : f(\mathbb{D}) \to \mathbb{C}$ be analytic and univalent on $f(\mathbb{D})$. If

$$g(z) = \frac{(\phi \circ f)(z) - \phi(0)}{\phi'(0)},$$

then g is clearly univalent on $\mathbb{D}$ with $g(0) = 0$. Furthermore,

$$g'(z) = \frac{f'(z)\phi'(f(z))}{\phi'(0)}$$

so that g is analytic on $\mathbb{D}$ with $g'(0) = 1$. Thus, $g \in \mathcal{S}$ as required.

(iii) Suppose that $f \in \mathcal{S}$ with $f(z) \neq w$ and let

$$g(z) = \frac{wf(z)}{w - f(z)}.$$

If $T(\zeta) = \frac{w\zeta}{w-\zeta}$ which is clearly one-to-one if $\zeta \neq w$, then it follows that $g(z) = (T \circ f)(z)$ is univalent on $\mathbb{D}$. Furthermore,

$$g'(z) = \frac{w^2 f'(z)}{(w - f(z))^2},$$

and since $w \neq f(z)$ it follows that g is analytic on $\mathbb{D}$ with $g'(0) = 1$. Thus, $g \in \mathcal{S}$ as required. $\square$

Finally, we will prove Theorem 2.1.15 which states that $\mathcal{S}$ is preserved under square root transformation. The proof, however, requires two lemmas about analytic functions.

Lemma 2.1.12 *If f is analytic on $\mathbb{D}$ with $0 \notin f(\mathbb{D})$, then there exists an analytic function h on $\mathbb{D}$ with $h^2 = f$.*

Proof Let $g(0)$ be any complex number with $\exp\{g(0)\} = f(0)$. For any other $w \in \mathbb{D}$, let

$$g(w) = g(0) + \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

where $\gamma : [0,1] \to \mathbb{D}$ is any C^1 curve from 0 to w. From the fundamental theorem of calculus, it follows that

$$g'(w) = \frac{f'(w)}{f(w)}. \tag{2.3}$$

Note that $f(z) \neq 0$ for $z \in \mathbb{D}$ so that $g'(z)$ is well-defined for all $z \in \mathbb{D}$ implying that g is analytic on $\mathbb{D}$. It now follows from (2.3) that

$$[fe^{-g}]'(w) = f'(w)e^{-g(w)} - g'(w)e^{-g(w)}f(w) = e^{-g(w)}[f'(w) - g'(w)f(w)] = 0.$$

The equation $[fe^{-g}]'(w) = 0$ implies that $f(w) = e^{g(w)}$. Hence, the proof is completed by taking

$$h(z) = \exp\left\{\frac{g(z)}{2}\right\}$$

so that h is analytic on $\mathbb{D}$ with $h^2(z) = f(z)$ for every $z \in \mathbb{D}$. $\square$

Remark The analytic function g in the previous lemma is unique up to translation by integral multiples of $2\pi i$. In fact,

$$g(z) = \log|f(z)| + i\arg f(z).$$

Exercise 2.1.13 *With minor modifications, the following generalization of Lemma 2.1.12 can be proved. Suppose that D is a simply connected domain and that f is analytic on D. If $0 \notin f(D)$, then there exists an analytic function h on D with $h^2 = f$.*

Remark In fact, it follows from the previous exercise that if $a \in \mathbb{C} \setminus \{0\}$ and we let $h(z) = \exp\left\{\frac{g(z)}{a}\right\}$, then h is analytic on D with $h^a(z) = f(z)$ for every $z \in D$. This says, of course, that it is possible to find roots of non-zero analytic functions on D.

Lemma 2.1.14 *If $f \in \mathcal{S}$, then there exists an odd function $h \in \mathcal{S}$ such that $h^2(z) = f(z^2)$ for every $z \in \mathbb{D}$.*

Proof If $f \in \mathcal{S}$, then $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ so that

$$\frac{f(z)}{z} = 1 + a_2z + a_3z^2 + \cdots$$

is a non-zero, analytic function on $\mathbb{D}$. By Lemma 2.1.12 there exists an analytic function F on $\mathbb{D}$ with

$$F^2(z) = \frac{f(z)}{z}.$$

If we define $h(z) = zF(z^2)$, then it is clear that h is odd with $h^2(z) = f(z^2)$, $h(0) = 0$, and $h'(0) = F(0) = 1$. Let $z_1, z_2 \in \mathbb{D}$ and suppose that $h(z_1) = h(z_2)$. The univalence of f implies that $z_1^2 = z_2^2$. Therefore, it must be the case that either $z_1 = z_2$ or $z_1 = -z_2$. If $z_1 = -z_2$, then this implies that $h(z_1) = -h(-z_1) = -h(z_2)$ since h is odd. However, this contradicts the assumption that $h(z_1) = h(z_2)$, and so we conclude that $z_1 = z_2$. This shows that $h \in \mathcal{S}$ and the proof is complete. $\square$

Theorem 2.1.15 *The class $\mathcal{S}$ is preserved under the square root transformation. That is, if $f \in \mathcal{S}$ and $g(z) = \sqrt{f(z^2)}$, then $g \in \mathcal{S}$.*

Proof Suppose that $f \in \mathcal{S}$ and $g(z) = \sqrt{f(z^2)}$. In order to define g some care must be shown. Since $f(z) = 0$ only when $z = 0$, it is possible to choose a single-valued branch of the square root by writing

$$g(z) = \sqrt{f(z^2)} = z\left(1 + a_2z^2 + a_3z^4 + a_4z^6 + \cdots\right)^{1/2} = z + b_3z^3 + b_5z^5 + \cdots$$

for $|z| < 1$ for some coefficients $b_n \in \mathbb{C}$. It now follows from Lemma 2.1.14 that g is univalent on $\mathbb{D}$ and that g is analytic on $\mathbb{D}$ with $g(0) = 0$ and $g'(0) = 1$. That is, $g \in \mathcal{S}$ as required. $\square$

Exercise 2.1.16 *As noted in the second remark following Lemma 2.1.12, it is possible to find roots of non-zero analytic functions. Modify the proof of Theorem 2.1.15 to prove that for any $f \in \mathcal{S}$ and for any $a = 2, 3, 4, \ldots$, if $g(z) = [f(z^a)]^{1/a}$ is the ath root transformation of f, then $g \in \mathcal{S}$.*

2.2 The area theorem and its consequences

In this section we establish our first geometric results for univalent functions. We present two versions of the area theorem, one of which expresses the area of the image of the unit disk under a univalent function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}$ in terms of the coefficients a_n in the Taylor expansion of f. From the area theorem follows another geometric result, the Koebe one-quarter theorem, as well as Bieberbach's theorem for the second coefficient a_2, namely $|a_2| \leq 2$. Bieberbach's theorem is the basis for the famous Bieberbach conjecture that $|a_n| \leq n$ for all $n = 2, 3, \ldots$, which was finally settled in 1985 by de Branges. In fact, Loewner's original motivation for introducing the partial differential equation now bearing his name was to prove the Bieberbach conjecture. Loewner was able to use his equation to prove that $|a_3| \leq 3$. In fact, this same differential equation of Loewner turned out to be a key tool for de Branges' 1985 proof of the entire Bieberbach conjecture.

Theorem 2.2.1 (Area Theorem) *If $f : \mathbb{D} \to f(\mathbb{D})$ is a conformal mapping of $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$ so that f has a Taylor expansion*

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1, \tag{2.4}$$

(with $a_1 \in \mathbb{R}$, $a_1 > 0$), then

$$\text{Area}(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

The proof we present using Green's theorem may seem unduly complicated. However, we have chosen this approach since it will also work with only minor modifications for the second version of the area theorem which follows this one. For an alternative proof, see the remark following Corollary 2.2.2 and Exercise 2.2.3.

Proof Write $D = f(\mathbb{D})$. Schematically, this is shown in Fig. 2.2 below.

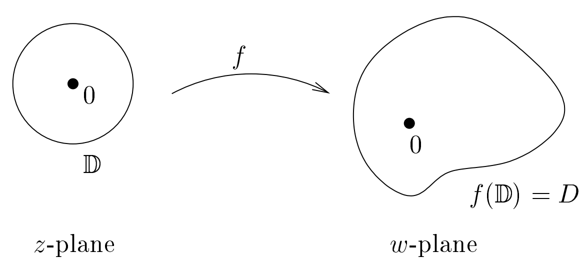

Fig. 2.2. The conformal transformation $f : \mathbb{D} \to D$.

Thus, Green's theorem implies

$$\text{Area}(D) = \frac{1}{2i} \int_{\partial D} \bar{w} \, \mathrm{d}w$$

which upon changing variables gives

$$\frac{1}{2i} \int_{\partial D} \bar{w} \, \mathrm{d}w = \frac{1}{2i} \int_{\partial \mathbb{D}} \overline{f(z)} f'(z) \, \mathrm{d}z.$$

Substituting in the expression (2.4) and changing to polar coordinates implies

$$\begin{aligned}
\operatorname{Area}(D) &= \frac{1}{2i}\int_{\partial\mathbb{D}} \left(\sum_{n=1}^{\infty} \bar{a}_n \bar{z}^n\right)\left(\sum_{m=1}^{\infty} m a_m z^{m-1}\right) \mathrm{d}z \\
&= \frac{1}{2i}\int_0^{2\pi} \left(\sum_{n=1}^{\infty} \bar{a}_n e^{-in\theta}\right)\left(\sum_{m=1}^{\infty} m a_m e^{i(m-1)\theta}\right) ie^{i\theta}\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(\sum_{n=1}^{\infty} \bar{a}_n e^{-in\theta}\right)\left(\sum_{m=1}^{\infty} m a_m e^{im\theta}\right) \mathrm{d}\theta
\end{aligned}$$

recalling that $\partial\mathbb{D} = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. Finally, we expand the product of sums to conclude

$$\begin{aligned}
\operatorname{Area}(D) = \frac{1}{2}\int_0^{2\pi} \left(\sum_{n=1}^{\infty} \bar{a}_n e^{-in\theta}\right)\left(\sum_{m=1}^{\infty} m a_m e^{im\theta}\right) \mathrm{d}\theta &= \frac{1}{2}\int_0^{2\pi} \left(\sum_{k=1}^{\infty} k\bar{a}_k a_k\right) \mathrm{d}\theta \\
&= \left(\sum_{k=1}^{\infty} k|a_k|^2\right) \frac{1}{2}\int_0^{2\pi} \mathrm{d}\theta \\
&= \pi \sum_{k=1}^{\infty} k|a_k|^2
\end{aligned}$$

having noted that the "off-diagonal terms" of the product are of the form $e^{i(m-n)\theta}$, $m \neq n$, all of which integrate to 0 over the range $0 \leq \theta \leq 2\pi$. □

Suppose that $f \in \mathcal{S}$ so that $f : \mathbb{D} \to f(\mathbb{D})$ is analytic and univalent on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. We say that f is *bounded* if $\operatorname{Area}(f(\mathbb{D})) < \infty$. Theorem 2.2.1 above gave one characterization of $\operatorname{Area}(f(\mathbb{D}))$, namely

$$\operatorname{Area}(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n|a_n|^2. \tag{2.5}$$

However, we can use the change-of-variables theorem to give another expression for $\operatorname{Area}(f(\mathbb{D}))$, namely

$$\operatorname{Area}(f(\mathbb{D})) = \int\int_{f(\mathbb{D})} \mathrm{d}x\,\mathrm{d}y = \int\int_{\mathbb{D}} |f'(z)|^2\,\mathrm{d}x\,\mathrm{d}y.$$

The expression

$$\int\int_{\mathbb{D}} |f'(z)|^2\,\mathrm{d}x\,\mathrm{d}y$$

is sometimes called the *Dirichlet integral* of f. In other words, the function $f \in \mathcal{S}$ is bounded if and only if it has a finite Dirichlet integral. Note that (2.5) implies that the coefficients a_n of a bounded function in $\mathcal{S}$ are $o(n^{-1/2})$; that is, if $f \in \mathcal{S}$ is bounded, then

$$\lim_{n\to\infty} \frac{|a_n|}{n^{-1/2}} = \lim_{n\to\infty} |a_n|\sqrt{n} = 0.$$

We summarize these results with the following corollary. Recall that in order to simplify formulæ we follow the convention that $a_1 = 1$ for $f \in \mathcal{S}$.

Corollary 2.2.2 *If $f \in \mathcal{S}$, then*

$$\int\int_{\mathbb{D}} |f'(z)|^2\,\mathrm{d}x\,\mathrm{d}y = \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

If $f \in \mathcal{S}$ is bounded so that

$$\int\int_{\mathbb{D}} |f'(z)|^2\,\mathrm{d}x\,\mathrm{d}y = \pi \sum_{n=1}^{\infty} n|a_n|^2 < \infty,$$

then $a_n = o(n^{-1/2})$.

Remark It is, of course, possible to prove Theorem 2.2.1 directly from

$$\mathrm{Area}(f(\mathbb{D})) = \int\int_{\mathbb{D}} |f'(z)|^2 \,\mathrm{d}x\,\mathrm{d}y$$

without recourse to Green's theorem. That is, if $f : \mathbb{D} \to f(\mathbb{D})$ is a conformal mapping of $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$ so that f has a Taylor expansion $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$, $|z| < 1$, then

$$\int\int_{\mathbb{D}} |f'(z)|^2 \,\mathrm{d}x\,\mathrm{d}y = \pi \sum_{n=1}^{\infty} n|a_n|^2. \tag{2.6}$$

Exercise 2.2.3 *Complete the steps in the proof indicated in the previous remark. That is, derive (2.6) without using Green's theorem as in the proof of Theorem 2.2.1 given above.*

Example 2.2.4 *Consider the function* $f(z) = z - \frac{1}{2}z^2$ *so that* $f \in \mathcal{S}$. *In fact,* f *maps the unit disk onto the interior of a cardioid. Using Corollary 2.2.2 we can determine the area of the cardioid and verify that* f *is bounded. That is,* $a_1 = 1$, $a_2 = -1/2$, *and so*

$$\mathrm{Area}(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n|a_n|^2 = \pi\left(1 \cdot |1|^2 + 2 \cdot |-1/2|^2\right) = \frac{3\pi}{2}.$$

Alternatively, we find $f'(z) = 1 - z$ *so that* $|f'(z)|^2 = 1 - z - \bar{z} + |z|^2$, *and so*

$$\begin{aligned}
\mathrm{Area}(f(\mathbb{D})) = \int\int_{\mathbb{D}} |f'(z)|^2 \,\mathrm{d}x\,\mathrm{d}y &= \int\int_{\mathbb{D}} 1 - z - \bar{z} + |z|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&= \int\int_{\mathbb{D}} 1 - (x+iy) - (x-iy) + x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y \\
&= \int\int_{\mathbb{D}} 1 - 2x + x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{2\pi}\int_0^1 (1 - 2r\cos\theta + r^2) r \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{3}{4} - \frac{2}{3}\cos\theta \,\mathrm{d}\theta \\
&= \frac{3\pi}{2}
\end{aligned}$$

as before.

It is interesting to note that until 1966 the estimate of Corollary 2.2.2 was believed to be the best one possible; that was, however, until Clunie and Pommerenke [3] found an improvement. We present the following theorem which is a weaker version of their result, but has the advantage that the proof uses only elementary results such as the (primary lemma for the) growth and distortion theorems of the next section. Consequently, we defer the proof until the end of Section 2.3.

Theorem 2.2.5 *There exists an absolute constant* $\delta > 0$ *such that* $a_n = O(n^{-1/2-\delta})$ *for every bounded function* $f \in \mathcal{S}$.

We now give a second area theorem which was first discovered by Gronwall [5] in 1914. This one is for functions which are analytic and univalent on the domain $\mathbb{U} = \{z : |z| > 1\}$ except for a simple pole at infinity with residue 1. Such functions have a Laurent expansion of the form

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \quad |z| > 1,$$

where $b_n \in \mathbb{C}$. We write $\mathcal{L}$ to denote the set of such functions. We also note that if $g \in \mathcal{L}$, then g maps $\mathbb{U}$ onto the complement of a compact, connected set E. Furthermore, if $f \in \mathcal{S}$ and g is defined by *inversion* as $g(z) = \frac{1}{f(1/z)}$, then

$$g(z) = z - a_2 + (a_2^2 - a_3)z^{-1} + \cdots, \quad |z| > 1. \tag{2.7}$$

In fact, inversion establishes a one-to-one correspondence between $\mathcal{S}$ and the subclass $\mathcal{L}'$ of $\mathcal{L}$ for which $0 \in E$ (i.e., for which $g(z) \neq 0$ in $\mathbb{U}$).

A strong restriction is placed on the size of the coefficients of $g \in \mathcal{L}$ by univalence, and this is captured by the folowing area theorem. As noted earlier, the proof is similar to that of Theorem 2.2.1.

Theorem 2.2.6 (Area Theorem) *If $g \in \mathcal{L}$ so that g has a Laurent expansion*

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}, \quad |z| > 1, \tag{2.8}$$

then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

Proof Let $g \in \mathcal{L}$ and denote by E the compact, connected set of points which are omitted by g. Suppose that $r > 1$, and let C_r be the image of the circle of radius r under g. Since g is a univalent function, it follows that C_r is a Jordan curve. Therefore, C_r encloses a domain E_r which contains E. See Fig. 2.3.

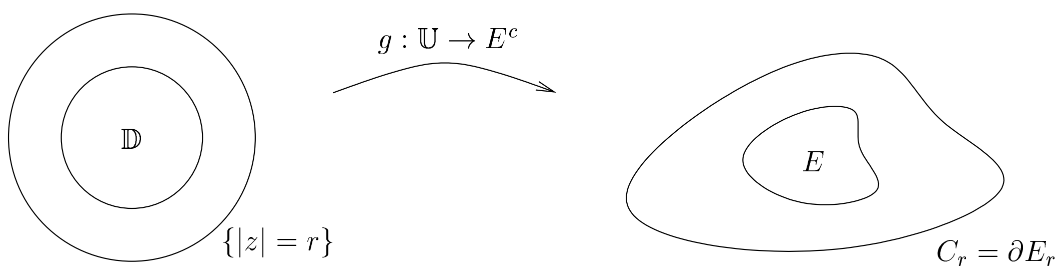

Fig. 2.3. The function $g : \mathbb{U} \to E^c$ and C_r, the image of $\{|z| = r\}$ under g.

It now follows from Green's theorem and a change of variables that

$$\text{Area}(E_r) = \frac{1}{2i} \int_{C_r} \bar{w}\, \mathrm{d}w = \frac{1}{2i} \int_{\{|z|=r\}} \overline{g(z)} g'(z)\, \mathrm{d}z.$$

Substituting in the expression (2.8) and changing to polar coordinates implies

$$\begin{aligned} \text{Area}(E_r) &= \frac{1}{2} \int_0^{2\pi} \left(re^{-i\theta} + \sum_{n=0}^{\infty} \bar{b}_n r^{-n} e^{in\theta} \right) \left(1 - \sum_{j=0}^{\infty} j b_j r^{-j-1} e^{-i(j+1)\theta} \right) re^{i\theta}\, \mathrm{d}\theta \\ &= \pi \left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \right). \end{aligned}$$

If we now let $r \to 1+$, then

$$\text{Area}(E) = \lim_{r\to 1+} \text{Area}(E_r) = \pi \left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right).$$

Since $\text{Area}(E) \geq 0$, it follows that

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$$

as required. □

Corollary 2.2.7 *If $g \in \mathcal{L}$, then*

$$|b_n| \leq n^{-1/2}, \quad n = 1, 2, 3, \ldots.$$

We remark that the upper bound given in the corollary is not sharp if $n \geq 2$ since

$$g(z) = z + n^{-1/2} z^{-n}$$

is not univalent on $\mathbb{U}$. In fact,

$$g'(z) = 1 - n^{1/2} z^{-n-1}$$

which equals zero at certain points in $\mathbb{U}$ if $n \geq 2$. However, it can be easily checked that the function

$$g(z) = z + b_0 + \frac{b_1}{z} \tag{2.9}$$

with $|b_1| = 1$ is univalent on $\mathbb{U}$ and so the upper bound is sharp for $n = 1$. In fact, the function g of (2.9) maps $\mathbb{U}$ conformally onto the complement of a line segment of length 4.

Bieberbach was the first to establish bounds on any of the coefficients a_n of $f \in \mathcal{S}$ when he proved [1] in 1916 that $|a_2| \leq 2$.

Theorem 2.2.8 (Bieberbach) *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}$, then $|a_2| \leq 2$.*

Proof Suppose that $f \in \mathcal{S}$. Apply a square root transformation (Theorem 2.1.15) and invert f to give

$$g(z) = \left[f(z^{-2})\right]^{-1/2} = z - \frac{a_2}{2} z^{-1} + \cdots$$

so that $g \in \mathcal{L}$. It therefore follows from the previous corollary (Corollary 2.2.7) that

$$|b_1| = \left|\frac{a_2}{2}\right| \leq 1$$

and so $|a_2| \leq 2$ as required. □

Exercise 2.2.9 *Show that if $f \in \mathcal{S}$, then $|a_2^2 - a_3| \leq 1$. (*Hint*: Use (2.7) and Corollary 2.2.7.)*

Bieberbach conjectured that if $f \in \mathcal{S}$, then the coefficients a_n of f satisfied $|a_n| \leq n$. This difficult open problem, known as Bieberbach's conjecture, was finally proved by L. de Branges [2] in 1985.

Theorem 2.2.10 (Bieberbach Conjecture–de Branges' Theorem) *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}$, then $|a_n| \leq n$ for all $n \geq 2$.*

The final major geometric result for univalent functions $f \in \mathcal{S}$ of this section is a theorem due to Koebe. In 1907, Koebe [7] showed that the images $f(\mathbb{D})$ of all functions $f \in \mathcal{S}$ contained a common disk $\{|w| < r\}$ for some $r \leq 1/4$. It follows from Bieberbach's theorem that $r = 1/4$. For obvious reasons, Theorem 2.2.11 is known as the *Koebe one-quarter theorem*.

Theorem 2.2.11 (Koebe One-Quarter Theorem) *The range of every $f \in \mathcal{S}$ contains the disk $\{w \in \mathbb{C} : |w| < 1/4\}$. That is,* $\operatorname{dist}(0, \partial f(\mathbb{D})) \geq 1/4$.

Proof Let $f \in \mathcal{S}$ so that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1,$$

and note that $|a_2| \leq 2$ by Bieberbach's theorem (Theorem 2.2.8). Suppose that $z_0 \notin f(\mathbb{D})$ and consider the omitted-value transformation

$$g_{z_0}(z) = \frac{z_0 f(z)}{z_0 - f(z)} = z + \left(a_2 + \frac{1}{z_0}\right) z^2 + \cdots.$$

We know from Theorem 2.1.11 that $g_{z_0} \in \mathcal{S}$ and so from Bieberbach's theorem we conclude that

$$\left| a_2 + \frac{1}{z_0} \right| \leq 2.$$

Combined with the fact that $|a_2| \leq 2$, we conclude that

$$\left| \frac{1}{z_0} \right| \leq 4 \quad \text{or} \quad |z_0| \geq \frac{1}{4}.$$

In other words, every omitted value of $f \in \mathcal{S}$ lies outside the disk of radius $1/4$ centred at the origin. □

Remark We also note that univalence is crucial to the Koebe one-quarter theorem. If

$$f_n(z) = \frac{1}{n}(e^{nz} - 1), \quad n = 1, 2, \ldots,$$

then $f_n(0) = 0$ and $f_n'(0) = 1$ (so that f_n is locally univalent at 0), but f_n omits the value $-1/n$ (which may of course be chosen arbitrarily close to 0).

The following theorem gives bounds on the derivative of a conformal transformation and can be considered as a preview of the distortion theorem (Theorem 2.3.2) of the next section. Its proof requires a slight generalization of the Koebe one-quarter theorem.

Theorem 2.2.12 *If $F : D \to D'$ is a conformal transformation with $F(z) = z'$, then*

$$\frac{d'}{4d} \leq |F'(z)| \leq \frac{4d'}{d}$$

where $d = \operatorname{dist}(z, \partial D)$ and $d' = \operatorname{dist}(z', \partial D')$; see Fig. 2.4.

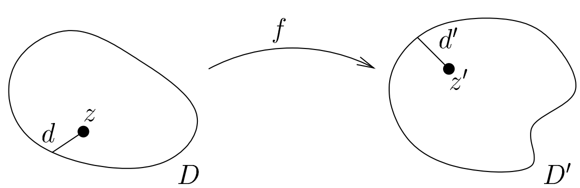

Fig. 2.4. The conformal transformation $F : D \to D'$ with $d = \operatorname{dist}(z, \partial D)$ and $d' = \operatorname{dist}(z', \partial D')$ indicated.

Proof Without loss of generality, suppose that $z = z' = 0$. In order to prove this theorem we will establish the following generalization of the Koebe one-quarter theorem. Let $F : D \to D'$ be a conformal transformation with $F(0) = 0$. It then follows that F has a Taylor expansion converging for all z in the disk of radius $d = \operatorname{dist}(0, \partial D)$ centred 0 and which is given by

$$F(z) = A_1 z + A_2 z^2 + A_3 z^3 + \cdots, \quad |z| < d.$$

The function f defined for $z \in \mathbb{D}$ by

$$f(z) = \frac{F(dz)}{dF'(0)} = \frac{F(dz)}{dA_1} = z + \frac{A_2 d}{A_1} z^2 + \frac{A_3 d^2}{A_1} z^3 + \cdots = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1,$$

is therefore analytic and univalent on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$, i.e., $f \in \mathcal{S}$. Suppose that w is an omitted-value of F restricted to the disk of radius d centred at 0. That is, suppose that $w \notin F(d\mathbb{D})$ or, in other words, suppose that $F(z) \neq w$ for any $|z| < d$. This is equivalent to supposing that

$$\frac{F(dz)}{dF'(0)} \neq \frac{w}{dF'(0)}$$

for any $|z| < 1$, i.e., that the function f omits the value $w_0 = w/(dF'(0))$. Consider the omitted-value transformation

$$g(z) = \frac{w_0 f(z)}{w_0 - f(z)} = \frac{w f(z)}{w - dF'(0) f(z)} = z + \left(a_2 + \frac{dF'(0)}{w} \right) z^2 + \cdots, \quad |z| < 1,$$

which belongs to $\mathcal{S}$ by Theorem 2.1.11. As in the proof of the Koebe one-quarter theorem (Theorem 2.2.11), we know that Bieberbach's theorem (Theorem 2.2.8) implies that both

$$|a_2| \leq 2 \quad \text{and} \quad \left| a_2 + \frac{dF'(0)}{w} \right| \leq 2.$$

Hence, we conclude that

$$\left| \frac{dF'(0)}{w} \right| \leq 4 \quad \text{and so} \quad |w| \geq \frac{d|F'(0)|}{4}.$$

Thus, if $F : D \to D'$ is a conformal transformation with $F(0) = 0$, then D' necessarily contains the disk of radius $d|F'(0)|/4$ centred at 0. If we write $d' = \operatorname{dist}(0, \partial D')$, then

$$\frac{d|F'(0)|}{4} \leq d' \tag{2.10}$$

and the upper bound follows. To derive the lower bound, we consider the inverse function $G(z) = F^{-1}(z)$ and note that $G : D' \to D$ is a conformal transformation of D' onto D. If we then define the function g for $z \in \mathbb{D}$ by

$$g(z) = \frac{G(d'z)}{d'G'(0)}, \quad |z| < 1,$$

so that $g \in \mathcal{S}$, we see that same argument as above implies

$$\frac{d'|G'(0)|}{4} \leq d. \tag{2.11}$$

Noting that $G'(0) = 1/F'(0)$ so that (2.11) implies $d' \leq 4d|F'(0)|$ gives the lower bound and together with (2.10) this proves the theorem. □

2.3 The growth and distortion theorems

In this section we establish two more fundamental theorems about univalent functions. The growth theorem and the distortion theorem provide bounds on $|f(z)|$ and $|f'(z)|$, respectively, over all $f \in \mathcal{S}$. The name "distortion" refers to the fact that geometrically $|f'(z)|$ represents the infinitesmal magnification of arclength (or, equivalently, that the Jacobian $|f'(z)|^2$ represents the infinitesmal magnification of area; see Corollary 2.2.2). We begin with the following lemma which provides the basic estimate leading to the growth and distortion theorems.

Lemma 2.3.1 *If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then*

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \le \frac{4|z|}{1-|z|^2}. \tag{2.12}$$

Proof Suppose that $f \in \mathcal{S}$. Let $\zeta \in \mathbb{D}$ be fixed, and let

$$T_\zeta(z) = \frac{z+\zeta}{1+\bar{\zeta}z}$$

which is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$ with $T_\zeta(0) = \zeta$ and $T'_\zeta(0) = 1 - |\zeta|^2$. Therefore, if we let

$$F_\zeta(z) = \frac{f(T_\zeta(z)) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}, \tag{2.13}$$

then $F_\zeta \in \mathcal{S}$ (since it is a disk automorphism; see Theorem 2.1.11). From (1.4) it follows that the Taylor expansion of $T_\zeta(z)$ is given by

$$T_\zeta(z) = \zeta + (1-|\zeta|^2)z - \bar{\zeta}(1-|\zeta|^2)z^2 + \cdots .$$

We therefore find that the Taylor expansion of $f(T_\zeta(z))$ is

$$f(T_\zeta(z)) = f(\zeta) + f'(\zeta)[T_\zeta(z) - \zeta] + \frac{f''(\zeta)}{2}[T_\zeta(z) - \zeta]^2 + \cdots$$

and so

$$f(T_\zeta(z)) - f(\zeta) = (1-|\zeta|^2)f'(\zeta)z + \left[\frac{f''(\zeta)}{2}(1-|\zeta|^2)^2 - \bar{\zeta}(1-|\zeta|^2)f'(\zeta)\right] z^2 + \cdots . \tag{2.14}$$

Substituting (2.14) into (2.13) yields

$$F_\zeta(z) = z + \left[\frac{f''(\zeta)(1-|\zeta|^2)}{2f'(\zeta)} - \bar{\zeta}\right] z^2 + \cdots .$$

By Bieberbach's Theorem (Theorem 2.2.8),

$$\left| \frac{f''(\zeta)(1-|\zeta|^2)}{2f'(\zeta)} - \bar{\zeta} \right| \le 2,$$

and so multiplying by $\frac{2|\zeta|}{1-|\zeta|^2}$ gives

$$\left| \frac{\zeta f''(\zeta)}{f'(\zeta)} - \frac{2\zeta\bar{\zeta}}{1-|\zeta|^2} \right| \le \frac{4|\zeta|}{1-|\zeta|^2}. \tag{2.15}$$

Finally, replace ζ with z in (2.15) above so that (2.12) follows proving the lemma. $\square$

Theorem 2.3.2 (Distortion Theorem) *If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then*

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}. \tag{2.16}$$

Proof Note that the inequality $|\zeta| \le C$ implies that $-C \le \operatorname{Re}\{\zeta\} \le C$. Hence, Lemma 2.3.1 implies that

$$\frac{2|z|^2}{1-|z|^2} - \frac{4|z|}{1-|z|^2} \le \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \le \frac{2|z|^2}{1-|z|^2} + \frac{4|z|}{1-|z|^2}. \tag{2.17}$$

Without loss of generality, it now suffices to consider $z = x \in (0,1)$, for if $z = re^{i\theta}$, then by Theorem 2.1.9,

$$F_\theta(z) = e^{-i\theta} f(e^{i\theta}z) \in \mathcal{S}.$$

Since $f'(z) \neq 0$ and $f'(0) = 1$, Lemma 2.1.12 implies that there exists an analytic function g on $\mathbb{D}$ with $g(0) = 0$ such that $f'(z) = e^{g(z)}$. Hence, $\operatorname{Re}\{g(z)\} = \log|f'(z)|$, and a direct calculation shows that

$$x\frac{\partial}{\partial x}\operatorname{Re}\{g(x)\} = \operatorname{Re}\left\{\frac{xf''(x)}{f'(x)}\right\}.$$

Using (2.17), we find

$$\frac{2x^2-4x}{1-x^2} \leq x\frac{\partial}{\partial x}\operatorname{Re}\{g(x)\} \leq \frac{2x^2+4x}{1-x^2} \quad \text{or} \quad \frac{2x-4}{1-x^2} \leq \frac{\partial}{\partial x}\operatorname{Re}\{g(x)\} \leq \frac{2x+4}{1-x^2}. \tag{2.18}$$

Integrating (2.18) gives

$$\int_0^x \frac{2u-4}{1-u^2}\,\mathrm{d}u \leq \operatorname{Re}\{g(x)\} \leq \int_0^x \frac{2u+4}{1-u^2}\,\mathrm{d}u.$$

Using partial fractions we calculate

$$\int_0^x \frac{2u-4}{1-u^2}\,\mathrm{d}u = \int_0^x \frac{1}{u-1} - \frac{3}{1+u}\,\mathrm{d}u = \log(1-x) - 3\log(1+x)$$

and

$$\int_0^x \frac{2u+4}{1-u^2}\,\mathrm{d}u = \int_0^x \frac{1}{1+u} + \frac{3}{1-u}\,\mathrm{d}u = \log(1+x) - 3\log(1-x),$$

and so we find that

$$\log\frac{1-x}{(1+x)^3} \leq \log|f'(x)| \leq \log\frac{1+x}{(1-x)^3}$$

since $\log|f'(0)| = \log 1 = 0$. Exponentiating both sides gives

$$\frac{1-x}{(1+x)^3} \leq |f'(x)| \leq \frac{1+x}{(1-x)^3}$$

and the theorem is proved. □

Remark Univalence is also crucial for the distortion theorem. Consider the function

$$f_n(z) = \frac{1}{n}(e^{nz}-1), \quad n = 1, 2, \ldots,$$

which was discussed in the remark on page 16. Note that $f_n'(1/n) = e$ and that f_n does not satisfy the conclusion (2.16) of the distortion theorem for n large enough; although f_n is locally univalent on $\mathbb{D}$, it is not univalent on $\mathbb{D}$.

Theorem 2.3.3 (Growth Theorem) *If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}. \tag{2.19}$$

Proof As in the proof of the distortion theorem (Theorem 2.3.2), it suffices to consider $z = x \in (0,1)$ in order to derive the upper bound of (2.19); we simply integrate and use the upper bound of the distortion theorem. That is,

$$f(x) = \int_0^x f'(u)\,\mathrm{d}u \quad \text{and so} \quad |f(x)| \leq \int_0^x |f'(u)|\,\mathrm{d}u \leq \int_0^x \frac{1+u}{(1-u)^3}\,\mathrm{d}u = \frac{x}{(1-x)^2}$$

since $f(0) = 0$. The lower bound of (2.19) is more subtle, however. Note that if $|f(z)| \geq 1/4$, then the lower bound holds immediately since

$$\frac{|z|}{(1+|z|)^2} < \frac{1}{4} \quad \text{for all } |z| < 1.$$

Suppose, therefore, that $z \in \mathbb{D}$ with $|f(z)| < 1/4$. By rotating if necessary, it suffices to consider $f(z) = w \in [0, 1/4)$. Since the Koebe one-quarter theorem (Theorem 2.2.11) tells us that $\text{dist}(0, \partial f(\mathbb{D})) \geq 1/4$, it must be the case that the line segment $[0, w]$ lies entirely in $f(\mathbb{D})$. Let γ be the preimage under f of this line segment so that γ is a C^1 curve in $\mathbb{D}$ from 0 to z. Formally, we define $\gamma : [0, w] \to \mathbb{D}$ by setting $\gamma(t) = f^{-1}(t)$, $0 \leq t \leq w$. Therefore, by the fundamental theorem of calculus,

$$f(z) = w = \int_0^w f'(\gamma(t))\,\gamma'(t)\,\mathrm{d}t = \int_0^w |f'(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t = \int_\gamma |f'(\zeta)|\,|\mathrm{d}\zeta|$$

where the third equality uses the fact that $f'(\gamma(t))\,\gamma'(t)$ is non-negative (i.e., by construction, $f'(\zeta)\,\mathrm{d}\zeta$ has constant signum along γ), and so from the lower bound of the distortion theorem,

$$|f(z)| = \int_\gamma |f'(\zeta)|\,|\mathrm{d}\zeta| \geq \int_0^{|z|} \frac{1-u}{(1+u)^3}\,\mathrm{d}u = \frac{|z|}{(1+|z|)^2}.$$

Taken together, these bounds give (2.19) proving the theorem. □

Although the following corollary looks easy enough, it should be noted, however, that this result is not just an immediate application of the growth and distortion estimates, but rather requires the "trick" of disk automorphism as used in the proof of Lemma 2.3.1. In fact, if $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then the growth and distortion theorems only immediately imply the weaker result

$$\left(\frac{1-|z|}{1+|z|}\right)^3 \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \left(\frac{1+|z|}{1-|z|}\right)^3. \tag{2.20}$$

Corollary 2.3.4 *If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then*

$$\frac{1-|z|}{1+|z|} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|}. \tag{2.21}$$

Proof Suppose that $f \in \mathcal{S}$. As in the proof of Lemma 2.3.1, let $\zeta \in \mathbb{D}$ be fixed, let

$$T_\zeta(z) = \frac{z+\zeta}{1+\bar{\zeta}z},$$

and consider the disk automorphism

$$F_\zeta(z) = \frac{f(T_\zeta(z)) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)},$$

so that $F_\zeta \in \mathcal{S}$ by Theorem 2.1.11. By the growth theorem (Theorem 2.3.3), it follows that

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |F_\zeta(-\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|)^2}. \tag{2.22}$$

However,

$$F_\zeta(-\zeta) = \frac{f(T_\zeta(-\zeta)) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = \frac{f(0) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = \frac{-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}$$

since $T_\zeta(-\zeta) = f(0) = 0$, and so (2.22) implies

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq \frac{|f(\zeta)|}{(1-|\zeta|^2)|f'(\zeta)|} \leq \frac{|\zeta|}{(1-|\zeta|)^2}. \tag{2.23}$$

Finally, replace ζ with z in (2.23) above and simplify so that (2.21) follows proving the lemma. We again remark that (2.21) is not implied by (2.20). □

The following proposition can be proven by using Lemma 2.3.1 and the distortion theorem (Theorem 2.3.2) to obtain uniform bounds on $|f''(z)|$ for $|z| \leq r$. However, a much simpler proof can be given using the fact proved by de Branges that $|a_n| \leq n$ (Theorem 2.2.10).

Proposition 2.3.5 *For every $0 < r < 1$, there exists a constant $C_r < \infty$ such that if $f \in \mathcal{S}$ and $|z| \le r$, then*

$$|f(z) - z| \le C_r |z|^2. \tag{2.24}$$

In fact, the optimal choice of C_r is

$$C_r = \frac{2-r}{(1-r)^2}.$$

Proof If $f \in \mathcal{S}$, then $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ so that

$$|f(z) - z| = \left| \sum_{n=2}^{\infty} a_n z^n \right| \le \sum_{n=2}^{\infty} |a_n||z|^n = |z|^2 \sum_{n=2}^{\infty} |a_n||z|^{n-2} \le |z|^2 \sum_{n=2}^{\infty} n r^{n-2} = \frac{2-r}{(1-r)^2}|z|^2$$

where the second inequality used the estimates $|a_n| \le n$ and $|z| \le r$. □

Exercise 2.3.6 *Prove (2.24) of Proposition 2.3.5 using Taylor's theorem and the estimates of Lemma 2.3.1 and Theorem 2.3.2. That is, show that for every $0 < r < 1$, there exists a constant $C_r < \infty$ such that if $f \in \mathcal{S}$ and $|z| \le r$, then $|f(z) - z| \le C_r |z|^2$. (Note that Theorem 2.2.10, de Branges' proof of the Bieberbach conjecture, is necessary in order to obtain the optimal value of C_r.)*

The next exercise may be viewed as an extension of the upper bound of the distortion theorem to higher order derivatives. The solution, however, requires Theorem 2.2.10, de Branges' proof of the Bieberbach conjecture.

Exercise 2.3.7 *Show that if $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then*

$$|f^{(n)}(z)| \le \frac{n!(n+|z|)}{(1-|z|)^{n+2}}, \quad n = 1, 2, 3, \ldots.$$

Proposition 2.3.8 *If $0 < r < 1$ and $F : D \to D'$ is a conformal transformation with $F(z) = z'$, then*

$$|F(w) - z'| \le \frac{4|w-z|d'}{(1-r)^2 d}$$

for all w with $|w - z| \le rd$ where $d = \operatorname{dist}(z, \partial D)$ and $d' = \operatorname{dist}(z', \partial D')$.

Proof Without loss of generality, suppose that $F \in \mathcal{S}$; that is, $z = z' = 0$, $d = 1$, and $F'(0) = 1$. For if not, it suffices to consider $f \in \mathcal{S}$ given by

$$f(z) = \frac{F(dz)}{dF'(0)}, \quad |z| < 1,$$

as in the proof of Theorem 2.2.12. It follows from the growth theorem (Theorem 2.3.3) that

$$|F(w)| \le \frac{|w|}{(1-|w|)^2} \le \frac{|w|}{(1-r)^2}$$

and from the Koebe one-quarter theorem (Theorem 2.2.11) that $\operatorname{dist}(0, \partial D') \ge 1/4$. That is, $1 \le 4d'$ and so

$$|F(w)| \le \frac{|w|}{(1-r)^2} \le \frac{4|w|d'}{(1-r)^2}$$

which completes the proof. □

As noted in Section 2.2, we are now able to prove Theorem 2.2.5 whose proof requires the primary lemma for the growth and distortion theorems. For the benefit of the reader, we repeat the statement below. Recall that $f \in \mathcal{S}$ is bounded if

$$\text{Area}(f(\mathbb{D})) = \int\int_{\mathbb{D}} |f'(z)|^2 \, \mathrm{d}x \, \mathrm{d}y = \pi \sum_{n=1}^{\infty} n|a_n|^2 < \infty.$$

Our proof of this theorem follows [4].

Theorem 2.2.5 There exists an absolute constant $\delta > 0$ such that $a_n = O(n^{-1/2-\delta})$ for every bounded function $f \in \mathcal{S}$.

Proof Suppose that $f \in \mathcal{S}$ is bounded. If $\epsilon > 0$, then the Cauchy-Schwarz inequality implies that

$$\int_0^{2\pi} |f'(re^{i\theta})|^{1+\epsilon} \, \mathrm{d}\theta \le \left(\int_0^{2\pi} |f'(re^{i\theta})|^{2\epsilon} \, \mathrm{d}\theta \right)^{1/2} \left(\int_0^{2\pi} |f'(re^{i\theta})|^2 \, \mathrm{d}\theta \right)^{1/2}. \tag{2.25}$$

If we define

$$J(r) = \int_0^{2\pi} |f'(re^{i\theta})|^2 \, \mathrm{d}\theta, \quad 0 \le r < 1,$$

then it is clear that $r \mapsto J(r)$ is increasing in r and so

$$r(1-r)J(r) \le \int_r^1 tJ(t) \, \mathrm{d}t \le \int_0^1 tJ(t) \, \mathrm{d}t = \int_0^{2\pi} \int_0^1 |f'(re^{i\theta})|^2 t \, \mathrm{d}t \, \mathrm{d}\theta = \int\int_{\mathbb{D}} |f'(z)|^2 \, \mathrm{d}x \, \mathrm{d}y = \pi \sum_{n=1}^{\infty} n|a_n|^2 < \infty$$

since $f \in \mathcal{S}$ is bounded. It therefore follows that there exists a positive constant $C_1 < \infty$ (whose value depends on the function f) such that

$$J(r) \le \frac{C_1}{1-r}$$

provided that $0 \le r < 1$ is sufficiently bounded away from 0, say for all $r \ge 1/2$. In other words,

$$J(r) = O((1-r)^{-1}) \quad \text{as} \quad r \to 1-. \tag{2.26}$$

We also define

$$I(r) = \int_0^{2\pi} |f'(re^{i\theta})|^{2\epsilon} \, \mathrm{d}\theta \quad \text{and} \quad F(z) = [f'(z)]^{\epsilon},$$

and note that

$$F(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_0 = 1,$$

is analytic on the unit disk $\mathbb{D}$ since f is univalent on $\mathbb{D}$. Hence,

$$I(r) = \int_0^{2\pi} |F(re^{i\theta})|^2 \, \mathrm{d}\theta = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$

(which follows from a calculation similar to the one done in the proof of Theorem 2.2.1) and so it follows that

$$I''(r) \le 8\pi \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-2}. \tag{2.27}$$

By Lemma 2.3.1, we know that for every $f \in \mathcal{S}$,

$$\left| \frac{f''(z)}{f'(z)} \right| \le \frac{6}{1-r}, \quad |z| = r.$$

This implies that

$$2\pi \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-2} = \int_0^{2\pi} |F'(re^{i\theta})|^2 \, \mathrm{d}\theta = \epsilon^2 \int_0^{2\pi} \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right|^2 |f'(re^{i\theta})|^{2\epsilon} \, \mathrm{d}\theta \le \frac{36\epsilon^2}{(1-r)^2} I(r). \tag{2.28}$$

If we combine the inequalities (2.27) and (2.28), we find that

$$(\log I(r))'' = \frac{I''(r)}{I(r)} - \left(\frac{I'(r)}{I(r)}\right)^2 \le \frac{I''(r)}{I(r)} \le \frac{144\epsilon^2}{(1-r)^2}.$$

Integrating twice (and using the fact that $I'(0) = 0$) now yields

$$\log I(r) - \log I(0) = \int_0^r \int_0^{r_2} (\log I(r_1))''\, \mathrm{d}r_1\, \mathrm{d}r_2 \le \int_0^r \int_0^{r_2} \frac{144\epsilon^2}{(1-r_1)^2}\, \mathrm{d}r_1\, \mathrm{d}r_2 \le -144\epsilon^2 \log(1-r).$$

Since $I(0) = 2\pi$ we conclude that $\log I(r) \le \log 2\pi - 144\epsilon^2 \log(1-r)$ and so

$$I(r) \le 2\pi(1-r)^{-144\epsilon^2}. \tag{2.29}$$

Therefore, we conclude from (2.25), (2.26), and (2.29) that there exists a positive constant $C_2 < \infty$ (with C_2 depending on f) such that

$$\int_0^{2\pi} |f'(re^{i\theta})|^{1+\epsilon}\, \mathrm{d}\theta \le I(r)^{1/2} J(r)^{1/2} \le C_2(1-r)^{-1/2}(1-r)^{-72\epsilon^2},$$

or, equivalently,

$$\int_0^{2\pi} |f'(re^{i\theta})|^{1+\epsilon}\, \mathrm{d}\theta = O((1-r)^{-1/2-72\epsilon^2}) \quad \text{as} \quad r \to 1-. \tag{2.30}$$

Suppose that $0 < \alpha < 1/2$ is fixed, and define the sets

$$E_1 = \{\theta : |f'(re^{i\theta})| \le (1-r)^{-\alpha}\} \quad \text{and} \quad E_2 = \{\theta : |f'(re^{i\theta})| > (1-r)^{-\alpha}\}.$$

It then follows from (2.30) that

$$\begin{aligned}\int_0^{2\pi} |f'(re^{i\theta})|\, \mathrm{d}\theta &= \int_{E_1} |f'(re^{i\theta})|\, \mathrm{d}\theta + \int_{E_2} |f'(re^{i\theta})|\, \mathrm{d}\theta \\ &\le 2\pi(1-r)^{-\alpha} + (1-r)^{\alpha\epsilon} \int_0^{2\pi} |f'(re^{i\theta})|^{1+\epsilon}\, \mathrm{d}\theta \\ &\le 2\pi(1-r)^{-\alpha} + C_2(1-r)^{\alpha\epsilon - 1/2 - 72\epsilon^2}.\end{aligned}$$

If we now set $\epsilon = \alpha/144$, then the exponent $\alpha\epsilon - 1/2 - 72\epsilon^2$ is equal to $\alpha^2/288 - 1/2$. Furthermore, $\alpha^2/288 - 1/2$ is greater than $-\alpha$ for α sufficiently close to $1/2$. In fact, $\alpha^2/288 - 1/2 \ge -\alpha$ provided $\alpha \ge 12\sqrt{145} - 144$. Note that $12\sqrt{145} - 144$ is the positive root of the quadratic equation $\alpha^2 + 288\alpha - 144 = 0$. Thus, we can conclude that there exists some universal constant $\alpha < 1/2$ such that

$$\int_0^{2\pi} |f'(re^{i\theta})|\, \mathrm{d}\theta \le (2\pi + C_2)(1-r)^{-\alpha} = C(1-r)^{-\alpha} \tag{2.31}$$

where $C = 2\pi + C_2$ is a positive constant depending on f. The Cauchy integral formula now implies that

$$f^{(n)}(0) = (f')^{(n-1)}(0) = \frac{(n-1)!}{2\pi i} \int_{\partial\mathbb{D}} \frac{f'(z)}{z^n}\, \mathrm{d}z$$

and so

$$n!|a_n| = |f^{(n)}(0)| \le \frac{(n-1)!}{2\pi} \int_{\partial\mathbb{D}} \frac{|f'(z)|}{|z|^n}\, \mathrm{d}z = \frac{(n-1)!}{2\pi} r^{-n+1} \int_0^{2\pi} |f'(re^{i\theta})|\, \mathrm{d}\theta. \tag{2.32}$$

It now follows from (2.31) and (2.32) that

$$n|a_n| \le \frac{1}{2\pi} r^{-n+1} \int_0^{2\pi} |f'(re^{i\theta})|\, \mathrm{d}\theta \le Cr^{-n}(1-r)^{-\alpha}.$$

Choosing $r = 1 - 1/n$ for $n = 2, 3, \ldots$ immediately implies that $|a_n| \le 4Cn^{\alpha-1}$ and so $a_n = O(n^{\alpha-1})$. The proof is then completed by taking $\delta = 1/2 - \alpha$. □

Although this theorem established the relatively poor bound $\delta \geq 145 - 1/2 - 12\sqrt{145} > 0.000865$, the advantage of this proof is that it relied almost exclusively on direct means. The more sophisticated argument of Clunie and Pommerenke [3] proves that $\delta > 1/300$. The best value of δ is still unknown.

2.4 Additional exercises

Exercise 2.4.1 *There have been a number of instances in this chapter in which the fact that a composition of conformal mappings is again conformal has been used. The purpose of this exercise is to provide a complete proof of this fact. That is, suppose that $f : D \to D'$ is a conformal transformation. Prove that if g is a conformal mapping of D', then $g \circ f$ is a conformal mapping of D.*

Exercise 2.4.2 *As noted in Example 2.1.6, the Koebe function*

$$k(z) = \frac{z}{(1-z)^2}$$

is "extremal for $\mathcal{S}$" in the sense that many of the inequalities for functions $f \in \mathcal{S}$ are equalities if and only if f is (a rotation of) the Koebe function.

(a) *Show directly that if $f \in \mathcal{S}$ is the Koebe function, then equalities hold in each of the following results: Theorem 2.2.8, Exercise 2.2.9, and Theorem 2.2.11. Note that the proof of Theorem 2.2.11 shows that only the Koebe function and its rotations omit a value of modulus $1/4$. Every $f \in \mathcal{S}$ which is not (a rotation of) the Koebe function has a range covering a disk of radius strictly greater than $1/4$.*

(b) *Show that for each $z \in \mathbb{D}$, $z \neq 0$, equality holds in each of the following results if and only if $f \in \mathcal{S}$ is (a suitable rotation of) the Koebe function: Theorem 2.2.8, Exercise 2.2.9 Theorem 2.3.2, Theorem 2.3.3, and Corollary 2.3.4.*

Exercise 2.4.3 *Let $r > 0$. Show that the function*

$$f(z) = \frac{z}{(1-z)^3}$$

is analytic and univalent on the disk $\{|z| < r\}$ if and only if $r \leq 1/2$. (That is, show that f is conformal on the disk $\{|z| < 1/2\}$ but not on any larger disk centred at the origin.)

Exercise 2.4.4 *Suppose that $a_n \in \mathbb{C}$ for $n = 2, 3, \ldots$ with*

$$\sum_{n=2}^{\infty} n|a_n| \leq 1.$$

Prove that if f is defined for $|z| < 1$ by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$

then $f \in \mathcal{S}$.

The remaining exercises discuss the Schwarzian derivative, an important object in both the classical theory of univalent functions and in the modern stochastic theory.

Suppose that f is analytic and locally univalent on D. The *Schwarzian derivative* of f, denoted $\mathscr{S}f$, is defined for all $z \in D$ by

$$(\mathscr{S}f)(z) = \left[\frac{f''(z)}{f'(z)}\right]' - \frac{1}{2}\left[\frac{f''(z)}{f'(z)}\right]^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left[\frac{f''(z)}{f'(z)}\right]^2.$$

Exercise 2.4.5

(a) *Let $T : \mathbb{C} \to \mathbb{C}$ be a (non-degenerate) fractional linear transformation given by*

$$T(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0. \tag{2.33}$$

Show that $(\mathscr{S}T)(z) = 0$.

(b) *Let f be analytic and locally univalent on D, and let φ be an analytic function for which the composition $g = f \circ \varphi$ is well-defined. Show that*

$$(\mathscr{S}g)(z) = [\varphi'(z)]^2 (\mathscr{S}f)(\varphi(z)) + (\mathscr{S}\varphi)(z).$$

(c) *Using the results of **(a)** and **(b)**, conclude that if $g = f \circ T$ and $h = T \circ f$ where T is defined as in (2.33), then*

$$(\mathscr{S}g)(z) = [T'(z)]^2 (\mathscr{S}f)(T(z)) \quad \text{and} \quad (\mathscr{S}h)(z) = (\mathscr{S}f)(z).$$

In the language of functional analysis, this result shows that if $\mathbf{S}$ is the mapping from f to its Schwarzian derivative $\mathscr{S}f$, then $\mathbf{S}(T \circ f) = \mathbf{S}(f)$.

The following exercise, due first to Krauss [8] and then to Nehari [13], outlines a necessary condition for an analytic function f on $\mathbb{D}$ to be univalent.

Exercise 2.4.6 *Show that if $f \in \mathcal{S}$, then*

$$|(\mathscr{S}f)(z)| \leq \frac{6}{(1-|z|^2)^2}, \quad |z| < 1.$$

Hint*: Let $f \in \mathcal{S}$ have Taylor expansion $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ and let F be obtained from f by disk automorphism, so that the Taylor expansion of $F \in \mathcal{S}$ is $F(z) = z + A_2 z^2 + A_3 z^3 + \cdots$. Consider $G(z) = F(1/z)^{-1} = z + b_0 + b_1 z^{-1} + \cdots \in \mathcal{L}$. Conclude from Exercise 2.2.9 that $b_1 = A_2^2 - A_3$ satisfies $|b_1| \leq 1$. Verify that b_1 can be expressed explicitly as*

$$b_1 = -\frac{1}{6}(1-|z|^2)^2 (\mathscr{S}f)(z).$$

(Note that the Koebe function shows that the constant 6 is optimal.)

Replacing the constant 6 by the constant 2 gives a sufficient condition for an analytic function on $\mathbb{D}$ to belong to $\mathcal{S}$. This is Nehari's theorem whose proof is beyond our scope; see [4] for details.

Theorem 2.4.7 (Nehari) *If f is analytic and locally univalent on $\mathbb{D}$ and its Schwarzian derivative satisfies*

$$|(\mathscr{S}f)(z)| \leq \frac{2}{(1-|z|^2)^2}, \quad |z| < 1,$$

then f is univalent on $\mathbb{D}$.

Exercise 2.4.8 *Use Nehari's theorem to verify each of the following analytic functions on $\mathbb{D}$ do, in fact, belong to $\mathcal{S}$.*

(a) $f(z) = \dfrac{z}{1-z}$;

(b) $f(z) = \dfrac{1}{2} \log\left(\dfrac{1+z}{1-z}\right)$;

Exercise 2.4.9 *This exercise due to Hille [6] shows that the constant 2 in Nehari's theorem is the best possible and cannot be replaced with any larger number. For any $\alpha \in \mathbb{C}$, let*

$$f(z) = \left(\frac{1-z}{1+z}\right)^{\alpha}, \quad |z| < 1.$$

(a) *Verify that f is analytic and locally univalent on $\mathbb{D}$.* Hint*: $\frac{1-z}{1+z}$ maps $\mathbb{D}$ conformally onto the right half-plane.*

(b) *Show that*

$$(\mathscr{S}f)(z) = \frac{2(1-\alpha^2)}{(1-z^2)^2}.$$

(c) *Show that f is univalent on $\mathbb{D}$ if and only if $\alpha = a + bi$ satisfies $a^2 + b^2 \leq 2|a|$.*

(d) *Verify that $\alpha = ib$ (with $b \in \mathbb{R} \setminus \{0\}$) therefore gives a non-univalent function with*

$$|(\mathscr{S}f)(z)| \leq \frac{2(1+b^2)}{(1-|z|^2)^2}, \quad |z| < 1.$$

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