Classifying Elements of Table Algebra and C-algebra bases

Allen Herman*, Mikhael Muzychuk, Thomas Stanley†, and Bangteng Xu

July 5, 2017

Abstract

We will discuss the recognition problem for commutative reality-based algebras, C-algebras, and table algebras; that is, given a finite-dimensional commutative semisimple algebra with involution A over $\mathbb{C}$, and a fixed element b of A, we want to find a way to construct a C-algebra or table algebra basis that contains b. We show how to classify the entries of regular matrices of C-algebra bases of a given rank and involution type in terms of a variety. The polynomial generators of that variety yield an enumeration algorithm for integral table algebra bases and a decision algorithm for the C-algebra bases containing a fixed element. We use this to enumerate integral table algebras of rank up to 5 and order up to 100.

1 Introduction

Let A be a $(d+1)$-dimensional algebra with involution over $\mathbb{C}$. We assume A is a C^*-algebra with respect to the involution $*$, which means $*$ acts as complex conjugation on scalars, respects addition, and reverses multiplication in A. We say that the pair $(A, \mathbf{B})$ is a *reality-based algebra* (or RBA) if $\mathbf{B} = \{b_0, b_1, \dots, b_d\}$ is a basis of A for which

(i) the multiplicative identity of A is an element of $\mathbf{B}$ (we index the elements of $\mathbf{B}$ so that b_0 is the multiplicative identity of A);

(ii) $\mathbf{B}^2 \subseteq \mathbb{R}\mathbf{B}$ (in particular the structure constants λ_{ijk} generated by the basis $\mathbf{B}$ in the expressions $b_i b_j = \sum\limits_{k=0}^{d} \lambda_{ijk} b_k$ are all real numbers);

(iii) $\mathbf{B}^* = \mathbf{B}$, (so $*$ induces a transposition on the set $\{0, 1, \dots, d\}$ for which $0^* = 0$ and $(b_i)^* = b_{i^*}$ for $i \in \{1, \dots, d\}$;

(iv) $\lambda_{ij0} \neq 0 \iff j = i^*$; and

(v) $\lambda_{ii^*0} = \lambda_{i^*i0} > 0$.

Slightly stronger are the definitions of a table algebra basis or C-algebra basis. A *C-algebra* is an RBA that possesses a unique algebra homomorphism that is positive on elements of $\mathbf{B}$. This algebra homomorphism is called the *degree map* of A. It is always possible to positively rescale the defining basis of a C-algebra A to arrange that its degree map satisfies $\delta(b_i) = \lambda_{ii^*0}$ for all $b_i \in \mathbf{B}$. A C-algebra basis satisfying this condition is called a *standardized* C-algebra basis. A *table algebra* is a C-algebra for which all of the structure constants λ_{ijk} in the C-algebra basis $\mathbf{B}$ are nonnegative real numbers.

If $(A, \mathbf{B})$ be a C-algebra with standardized basis $\mathbf{B}$ and degree map δ, then its *order* is the positive real number $\delta(\mathbf{B}^+) = \delta(\sum_{i=0}^{d} b_i) = \sum_i \delta(b_i)$. The standard feasible trace of $(A, \mathbf{B})$ is the

*The first authour acknowledges the support of an NSERC Discovery Grant.
†The work of the third author was supported by an NSERC USRA grant.

$\mathbb{C}$-linear map defined by $\rho(b_i) = \delta(\mathbf{B}^+)\delta_{i0}$. It is easy to see that $\rho(xy) = \rho(yx)$ for all $x, y \in A$. Therefore, $(A, \mathbf{B})$ is a Frobenius algebra, and the standard feasible trace is a linear combination of irreducible characters of A. The constants m_χ for which $\rho = \sum_\chi m_\chi \chi$ where χ runs over the irreducible characters of A are called the *multiplicities* of the C-algebra $(A, \mathbf{B})$. These multiplicities are always positive real numbers [3].

We will sometimes refer to a C-algebra or table algebra as being *integral*. For C-algebras, this requires that both its structure constants in the standardized basis and its multiplicities are rational integers.

C-algebras and table algebras have significant structural advantages that allow them to behave more like groups than rings. To get an impression of this phenomenon, we direct the reader's attention to [1], [2], [3], and [9]. It is of fundamental importance, therefore, to be able to determine whether or not a semisimple algebra over $\mathbb{C}$ is a C-algebra, and if so to construct a C-algebra or table basis of A from its defining basis.

Motivated by phenomenon that arise for P-polynomial association schemes and distance regular graphs, we also consider the more specific question of constructing a rational C-algebra or integral table algebra basis of a semisimple algebra A that contains a given element b of A.

2 The easy cases: Rank 2 and Rank 3

Let A be a 2-dimensional algebra over $\mathbb{C}$, and let $b \in A$ be such that $\{1, b\}$ is a basis of A. Using the left regular representation of A gives us a matrix embedding of A into $M_2(\mathbb{C})$, so we will identify b with a 2×2 matrix. If $\mu_b(x) = x^2 - ux - k$ is the minimal polynomial of b, then the rational canonical form of b is

$$b = \begin{bmatrix} 0 & k \\ 1 & u \end{bmatrix},$$

and this agrees with the image of b in the left regular representation of A written in terms of our basis $\{1, b\}$.

Theorem 1. *Let A be a 2-dimensional C-algebra with involution $*$ having dimension 2. Suppose b is an element of A for which $\{1, b\}$ is a basis of A, and suppose $b^2 = k1 + ub$ for some $k, u \in \mathbb{C}$.*

(i) $\{1, b\}$ is an RBA-basis of A if and only if the involution is trivial on A, $k, u \in \mathbb{R}$, and $k > 0$.

(ii) $\{1, b\}$ is a standardized C-algebra basis of A if and only if $u = k - 1$.

(iii) $\{1, b\}$ is a C-algebra basis of A if and only if there exists $t > 0$ such that $tu = t^2k - 1$. In this case $\{1, tb\}$ is a standardized C-algebra basis of A.

(iv) If $\{1, b\}$ is a standardized C-algebra basis of A, then it is a table algebra basis if and only if $k \geq 1$.

Proof. (i). If $\{1, b\}$ is to be a $*$-invariant basis of A, for an algebra involution $*$, then the fact that 1^* has to be 1 forces $b^* = b$. That $k, u \in \mathbb{R}$ and $k = \lambda_{11^*0} > 0$ follow from the definition of an RBA-basis.

(ii). Assume $\{1, b\}$ is an RBA-basis of A with $b^2 = k1 + ub$. If δ is the degree map of A then $\delta(b) = k$, $\delta(1) = 1$, and $k^2 = \delta(b)^2 = \delta(b^2) = k + ku$. Therefore, $k = u + 1$.

(iii) When we rescale by multiplying b by some $t > 0$, we will have the identity $(tb)^2 = t^2k1 + (tu)(tb)$. If the basis $\{1, tb\}$ is standardized for a positive degree map δ, then we must have $\delta(tb) = t^2k > 0$. Since the degree map must be an algebra homomorphism,

$$\delta(tb)^2 = t^2k + (tu)\delta(tb) \implies t^4k^2 = t^2k + t^3uk,$$

so $t^2k = 1 + tu$.

(iii). Assume $\{1, b\}$ is a standardized C-algebra basis of A. By (ii), we have $b^2 = k1 + ub$ with $u = k - 1$. To be a table algebra basis, the structure constant u must be a nonnegative real number, so we must have $k \geq 1$. ∎

Note that when $\{1, b\}$ is a standardized C-algebra basis of A, the minimal polynomial of b is forced to be $\mu_b(x) = (x - k)(x + 1)$. Therefore, the character table of $(A, \{1, b\})$ is

	1	b	m_χ
δ	1	k	1
ϕ	1	-1	k

where m_χ is the multiplicity of the irreducible character χ in the standard feasible trace.

Corollary 2. *A 2-dimensional algebra A over $\mathbb{C}$ with trivial involution has a table algebra basis iff A has an element b whose left regular matrix has distinct eigenvalues.*

Proof. Suppose $\{1, x\}$ is a basis of A. If the left regular matrix representing x has a multiple eigenvalue, then since x cannot be a scalar multiple of the identity, the Jordan canonical form of x would be $\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$, and thus $A \simeq \mathbb{C}[x]/(x - \alpha)^2$ is not semisimple. Therefore, we can assume x has two distinct eigenvalues $\theta_1, \theta_2 \in \mathbb{C}$. We need only to replace x in our basis by some $\alpha x + \beta$ where the α and β satisfy $\alpha\theta_1 + \beta = -1$ and $\alpha\theta_2 + \beta = k$ for some fixed positive integer $k > 1$. Since θ_1 and θ_2 are distinct, this system of equations has a unique solution for α and β in $\mathbb{C}$, so such a substitution will be possible. □

Note that any 2-dimensional semisimple algebra A over $\mathbb{C}$ has such an element, because in this case $A = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ for two orthogonal idempotents e_1 and e_2 that sum to 1, and the element $b = 2e_1 - e_2$ has the required property.

Corollary 3. *Let A be a finite-dimensional commutative C-algebra over $\mathbb{C}$ with standardized C-algebra basis $\mathbf{B} = \{b_0, b_1, \dots, b_d\}$. Let x_0 be the eigenvector of A (in its left-regular representation) corresponding to the degree map $\delta : A \to \mathbb{C}$, and let $\{x_1, \dots, x_d\}$ be the distinct eigenvectors for the other eigenspaces of elements of A. Suppose $b_i x_j = k_{ij} x_j$ for $1 \le i, j \le d$.*

Then $\sum_{i=1}^{d} \sum_{j=1}^{d} k_{ij} = -1$.

Proof. This follows from the observation that $\{1 = b_0, b = \sum_i b_i\}$ is a C-algebra basis for a 2-dimensional C-algebra fusion of $(A, \mathbf{B})$. The eigenspaces of b are $\mathbb{C}x_0$ and $\mathbb{C}y$ where $y = \sum_{j=1}^{d} x_j$. The eigenvalue of b corresponding to the eigenvector y is $\sum_{i=1}^{d} \sum_{j=1}^{d} k_{ij}$. When $\{1, b\}$ is a C-algebra basis, it follows from Theorem 1 that this eigenvalue of b has to be -1. □

For algebras of dimension 3, we treat the symmetric and asymmetric cases separately. First, let us consider the symmetric case. Assume A a 3-dimensional algebra over $\mathbb{C}$ with trivial involution. Let $b \in A$, $b \notin \mathbb{C}1$. First, suppose the minimal polynomial of b has degree 2, so we have $b^2 = k1 + sb$ for some $k, s \in \mathbb{R}$. We require $k > 0$. Let $d \in A$ for which $\{1, b, d\}$ is a basis of A. Suppose $(bd)_1 = \alpha$. We need to replace d by c_0 to arrange for $(bc_0)_1 = 0$. Replacing d by $c_0 = d - \frac{\alpha}{k} b$ makes $(bc_0)_1 = 0$ in the new basis $\{1, b, c_0\}$. The regular matrix of b in this new basis has the form

$$\tilde{b} = \begin{bmatrix} 0 & k & 0 \\ 1 & s & u \\ 0 & 0 & v \end{bmatrix},$$

and $\tilde{b}$ will satisfy the same condition $\tilde{b}^2 = kI + s\tilde{b}$. This implies $u = 0$ and $v^2 = sv + k$. If $(c_0)^2 = \ell 1 + tb + wc_0$, then the regular matrix of c_0 has the form

$$\tilde{c}_0 = \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & t \\ 1 & v & w \end{bmatrix},$$

and the fact that $\tilde{c_0}^2 = \ell I + t\tilde{b} + w\tilde{c_0}$ imposes the conditions $\ell v = tk$ and $tv = \ell + ts$. The result of these conditions is that $k = v(v-s)$ and $\ell = t(v-s)$. So the remaining structure constants in these regular matrices are determined by s, v, t, and w. When we replace c_0 by $c = c_0 + n1 + mb$ with n, m satisfying $mk = nv$, then in the new basis $\{1, b, c\}$ we will have $(bc)_1 = 0$ and $(c^2)_1 = \ell + n^2 + m^2k$. So as long as $k > 0$ we can always find an RBA-basis of A containing b.

Now we attempt to construct a C-algebra basis of A that contains b. We will construct a standard C-algebra basis, so we assume b has been rescaled by a positive constant so that $(b^2)_1 = k = \theta_0$ is a positive eigenvalue of b. Since the minimal polynomial of b has degree 2, $\{1, b\}$ is a closed subset of rank 2, hence a standard C-algebra basis with respect to δ, where $\delta(1) = 1$ and $\delta(b) = k$. By Theorem 1(ii) the minimal polynomial of b must factor as $(x-k)(x+1)$. If $\{1, b, c\}$ is an RBA-basis of A containing b, then we will be in the case above where $s = k-1$, which implies $v = k$ or $v = -1$. We need to modify c so that we can extend δ to a degree homomorphism on A. We can adjust the basis of the ambient space $\mathbb{C}^3$ so that the eigenvector corresponding the eigenvalue k of $\tilde{b}$ is the all 1's vector. This arrangement means that evaluation of δ agrees with row sum on $\tilde{c}$ as well when $\{1, b, c\}$ is a standard C-algebra basis. When $v = k$, $t = \ell$ will be implied, so the condition $w = \ell - k - 1$ is the only requirement. If it is not satisfied, then we can simply replace c by rc, with r chosen so that $r^2\ell = rw + k + 1$. Since $\ell, k > 0$ this quadratic equation in r always will have 2 solutions. When $v = -1$ then $bc = -c$, so there is no way to define a positive degree map δ on A with $\delta(bc) = \delta(b)\delta(c)$.

Proposition 4. *Suppose A is a semisimple algebra of dimension 3 with trivial involution. Let $b \in A \setminus \mathbb{C}1$. Suppose the minimal polynomial of b is $x^2 - sx - k$.*

(i) If $k > 0$ then there exists an RBA-basis $\{1, b, c\}$ of A containing b.

(ii) Suppose $\{1, b, c\}$ is an RBA-basis of A containing b, with structure constants given by $b^2 = k1 + sb$, $bc = cb = ub + vc$, and $c^2 = \ell 1 + tb + wc$, with $k, \ell > 0$. Then $\{1, b, c\}$ is a standardized C-algebra basis of A iff $s = k-1$, $u = 0$, $v = k$, $t = \ell$, and $w = \ell - k - 1$.

(iii) Suppose $\{1, b, c\}$ is an RBA-basis of A with structure constants given by $b^2 = k1 + sb$, $bc = cb = ub + vc$, and $c^2 = \ell 1 + tb + wc$, with $k, \ell > 0$. In order for b to belong to a C-algebra basis of A, it must be possible to rescale b so that $s = k-1$, $v = k$, and $u = 0$. In this case we can rescale c to rc for some $r \neq 0$ to arrange that $t = \ell$ and $w = \ell - k - 1$.

In the second case, the minimal polynomial of b has degree 3, so $A = \mathbb{C}[b]$ and the involution acts trivially on b. Let $\mu_b(x) = x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$ be the minimum polynomial of b. We need to find a $c \in A$ for which $\{1, b, c\}$ is an RBA-basis. Pick any $k > 0$, and let $s_k = \sigma_1 + \frac{\sigma_3}{k}$. If c_k is the symmetric element of A determined by $b^2 = k1 + s_k b + c_k$, then the fact that the minimal polynomial of b has degree 3 implies that $c_k \neq 0$ for all $k > 0$. The choice of s_k ensures that $(bc_k)_1 = 0$, since

$$\begin{aligned} bc_k &= (k\sigma_1 - ks_k + \sigma_3)1 + (\sigma_1 s_k - \sigma_2 - s_k)b - \sigma_1 c_k \\ &= (\sigma_1^2 - \sigma_1 - \sigma_2 + (\sigma_1 + 1)\tfrac{\sigma_3}{k})b - \sigma_1 c_k. \end{aligned}$$

The value of $(c_k^2)_1$ in the basis $\{1, b, c_k\}$ is $\ell_k = (-\sigma_1\sigma_3 - \sigma_2 k - k^2 - \frac{\sigma_3^2}{k})$. For $\{1, b, c_k\}$ to be an RBA-basis of A it is necessary and sufficient that $\ell_k > 0$. This time, when we replace c_k by $c' = c_k + n1 + mb$, then $(b^2)_1$ becomes $k - n$, so $c' = c_{k-n}$ and nothing new results.

Proposition 5. *Suppose A is a w-dimensional semisimple algebra with trivial involution, and let $b \in A \setminus \mathbb{C}1$. Suppose the minimal polynomial of b is $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$. Then b is an element of an RBA-basis of A if and only if there is a positive $k > 0$ for which*

$$\ell := -(k^2 + \sigma_2 k + \sigma_1\sigma_3 + \frac{\sigma_3^2}{k}) > 0. \qquad (1)$$

In this situation, $b^2 = k1 + (\sigma_1 + \frac{\sigma_3}{k})b + c$, for some $c \in A$ such that $\{1, b, c\}$ is an RBA-basis of A, and $(c^2)_1 = \ell$.

Now suppose $\{1, b, c\}$ is an RBA-basis of a 3-dimensional algebra A with trivial involution. We first rescale b so that $(b^2)_1 = k$ is a positive eigenvalue of A, and set $\delta(b) = k$. As before, our regular matrices for b and c are

$$b \mapsto \begin{bmatrix} 0 & k & 0 \\ 1 & q & u \\ 0 & r & v \end{bmatrix}, \text{ and } c \mapsto \begin{bmatrix} 0 & 0 & \ell \\ 0 & u & s \\ 1 & v & t \end{bmatrix}.$$

The structure constant equations $b^2 = k1 + qb + rc$, $bc = cb = ub + vc$, and $c^2 = \ell 1 + sb + tc$ apply directly to these regular matrices as well. Equating matrices leads to five conditions on our variables:

$$ks = v\ell, \quad ku = r\ell, \quad sr = uv, \quad \ell + qs + ut = u^2 + sv, \text{ and } k + qv + rt = ru + v^2.$$

Since $k, \ell > 0$ the third condition is a consequence of the first two, and the fifth can be obtained from the fourth using the first two conditions for appropriate substitutions. When $\{1, b, c\}$ is a standardized C-algebra basis, we can arrange as before that the regular matrices have constant row sum condition. So this gives the further would require $q = k - u - 1$, $r = k - v$, $s = \ell - u$, and $t = \ell - v - 1$. This reduces the fourth equation to $\ell k = uk + v\ell$. This implies the degree homomorphism condition $\delta(b)\delta(c) = \delta(bc)$. The conditions $\delta(b^2) = \delta(b)^2$ and $\delta(c^2) = \delta(c)^2$ are satisfied since k and ℓ are eigenvalues of the respective regular matrices.

Theorem 6. *Let A be a 3-dimensional algebra over $\mathbb{C}$ with trivial involution. Suppose $b \in A \setminus \mathbb{C}1$.*

(i) b is an element of an RBA basis $\{1, b, c\}$ of A with structure constants given by $b^2 = k1+qb+rc$, $bc = cb = ub + vc$, and $c^2 = \ell 1 + sb + tc$ iff $k, \ell, q, r, s, t, u, v \in \mathbb{R}$, $k, \ell > 0$, $ku = rl$, $ks = v\ell$, and $k + qv + rt = ru + v^2$.

(ii) Suppose b is an element of an RBA-basis $\{1, b, c\}$ of A with structure constants as in (i). Then $\{1, b, c\}$ is a standardized C-algebra basis iff $q = k - u - 1$, $r = k - v$, $s = \ell - u$, $t = \ell - v - 1$, and $\ell = \frac{ku}{k-v}$. In this case the basis is determined uniquely by the spectrum of b.

(iii) $\{1, b, c\}$ is a C-algebra basis with structure constants as in (i) iff there exists $\alpha, \beta > 0$ such that $\{1, \alpha b, \beta c\}$ is a standardized C-algebra basis. This requires precisely the additional conditions $\alpha^2 k = 1+\alpha q+\beta u$, $\alpha\beta k = \alpha r+\beta v$, $\beta^2\ell = 1+\alpha v+\beta t$, $\alpha\beta\ell = \alpha u+\beta s$, and $\alpha\beta k\ell = \alpha uk+\beta v\ell$.

(iv) Suppose $\{1, b, c\}$ is a C-algebra basis of A with structure constants as in (i). Then $\{1, b, c\}$ is a table algebra basis of A iff $q, r, s, t, u, v \geq 0$ and the condition of (ii) is satisfied.

When $\{1, b, c\}$ is a standardized C-algebra basis of A, the above theorem says the left multiplication operators for b and c expressed in terms of the basis $\{1, b, c\}$ can be expressed in terms of three structure constants: k, ℓ, and u:

$$b \mapsto \begin{bmatrix} 0 & k & 0 \\ 1 & k-u-1 & u \\ 0 & k-v & v \end{bmatrix}, \text{ and } c \mapsto \begin{bmatrix} 0 & 0 & \ell \\ 0 & u & \ell - u \\ 1 & v & \ell - v - 1 \end{bmatrix},$$

with $v = \frac{k}{\ell}(\ell - u)$, $k, \ell > 0$, and $u \in \mathbb{R}$. In particular, there are infinitely many standard integral rank 3 symmetric C-algebra bases with negative stucture constants, and the structure constant u is not bounded in terms of k and ℓ. So to attempt an enumeration of integral bases only makes sense in the case of integral table algebras of a given involution type, and this will be discussed in the last section.

Suppose $\{1, b, c\}$ is a standardized C-algebra basis of A. Let $\mu_b(x) = x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$ be the minimum polynomial of b and suppose $\theta_0 = k, \theta_1, \theta_2$ are the eigenvalues of the regular matrix of b. The σ_i's are naturally symmetric functions in the θ_i's. They can also be directly calculated from the above matrix form of b, which produces the equations:

$\sigma_1 = \theta_0 + \theta_1 + \theta_2 = q + v$,
$\sigma_2 = \theta_0\theta_1 + \theta_0\theta_2 + \theta_1\theta_2 = qv - ru - k$, and
$\sigma_3 = \theta_0\theta_1\theta_2 = -kv$.

Since $\delta(b) = k = \theta_0$ is an eigenvalue of b, the above three equations then reduce to the conditions $\theta_1 + \theta_2 = v - u - 1$ and $\theta_1\theta_2 = -v$. These allow us to solve for u and v, and hence all the entries of c, in terms of the eigenvalues of b. We obtain the solutions

$$v = -\theta_1\theta_2, \text{ and } u = -\theta_1\theta_2 - \theta_1 - \theta_2 - 1.$$

We remark that if $\{1, b, c\}$ is a standardized C-algebra basis of A, then all of the structure constants are determined by the spectrum of b.

Suppose $Irr(\mathbb{C}\mathbf{B}) = \{\delta, \phi, \psi\}$. Then the eigenvalues of b are $\theta_0 = k, \phi_1 = \theta_1, \psi_1 = \theta_2$ and of c are ℓ, ϕ_2, ψ_2. The character table of the C-algebra is

	1	b	c	m_χ
δ	1	k	ℓ	1
ϕ	1	ϕ_1	ϕ_2	m_ϕ
ψ	1	ψ_1	ψ_2	m_ψ

Orthogonality relations tell us the second and third row sums of the character table are zero, so $\phi_2 = -1 - \phi_1$ and $\psi_2 = -1 - \psi_1$. Solving the equations $\phi_1 + \psi_1 = v - u - 1$ and $\phi_1\psi_1 = -v$ gives us (up to sign choice) $\phi_1 = \frac{1}{2}((v-u-1) + \sqrt{(v-u-1)^2 + 4v})$ and $\psi_1 = \frac{1}{2}((v-u-1) - \sqrt{(v-u-1)^2 + 4v})$. Note that whenever ϕ_1 is an irrational quadratic, the last two characters will be galois conjugate. The multiplicities can be calculated from the other orthogonality relations, which lead to the equation $\frac{1+k+\ell}{m_\phi} = 1 + \frac{\phi_1^2}{k} + \frac{\phi_2^2}{\ell}$ and a similar equation involving m_ψ. These reduce to (for example)

$$m_\phi = \frac{k\ell + k^2\ell + k\ell^2}{k\ell + \ell\phi_1^2 + k\phi_2^2},$$

and a similar equation with ψ replacing ϕ throughout. The character table P is the first eigenmatrix of the C-algebra, the multiplicities also give the first row of the the second eigenmatrix $Q = nP^{-1}$, which is the character table of the dual algebra. So it follows that the multiplicities, character values, and the dual character values all lie in the field $\mathbb{Q}(\phi_1)$.

The asymmetric case for dimension 3 can be handled in a similar fashion. Let A be a 3-dimensional algebra over $\mathbb{C}$ with $\mathbb{C}$-linear algebra involution $*$. Suppose $\{1, b, b^*\}$ is a basis of A. If the structure constants in this basis are real, then the involution condition on A tells us that we will have

$$b^2 = sb + tb^*, \quad (b^*)^2 = tb + sb^*, \text{ and } bb^* = (bb^*)^* = k1 + ub + ub^*,$$

with $k, s, t, u \in \mathbb{R}$. Applying these conditions to the matrices for b and b^* in the regular representation of A in the basis $\{1, b, b^*\}$ imposes the additional conditions $s = u$ and $k + u^2 = t^2$. These conditions are necessary requirements for $\{1, b, b^*\}$ to be a basis with real structure constants for an algebra with nontrivial involution.

Proposition 7. *Let A be a 3-dimensional $\mathbb{C}$-algebra with nontrivial involution $*$. Let $b \in A$ be such that $\{1, b, b^*\}$ is a basis of A. Suppose the structure constants in this basis are given by $b^2 = ub + tb^*$ and $bb^* = k1 + ub + ub^*$ for $k, u, t \in \mathbb{R}$ satisfying $k + u^2 = t^2$.*

(i) $\{1, b, b^*\}$ with the given structure constants is an RBA-basis of A iff $k > 0$.

(ii) An RBA-basis $\{1, b, b^*\}$ of A as in (i) is a standardized C-algebra basis of A iff $u = \frac{k-1}{2}$ and $t = \frac{k+1}{2}$.

(iii) An RBA-basis $\{1, b, b^*\}$ of A as in (i) can be rescaled to a standardized C-algebra basis $\{1, \alpha b, \alpha b^*\}$ of A iff there exists $\alpha > 0$ such that $u = \frac{\alpha^2 k - 1}{2\alpha}$ and $t = \frac{\alpha^2 k + 1}{2\alpha}$.

(iv) A standardized C-algebra basis as in (ii) is a table algebra basis of A iff $k \geq 1$.

Proof. (i). For $\{1, b, b^*\}$ to satisfy the definition of an RBA-basis it is only left to ensure that $\lambda_{bb^*0} = k > 0$. (ii). Suppose δ is a positive degree map for which $\{1, b, b^*\}$ is a standardized basis. Then $\delta(b) = \delta(b^*) = k$ is the constant row sum of the regular matrices for b and b^*. Therefore, $k = 1 + 2u$ and $k = u + t$. Now write u and t in terms of k. (iii). If $\alpha > 0$ and $\{1, \alpha b, \alpha b^*\}$ is a standardized C-algebra basis, then we will have $\delta(\alpha b) = \alpha^2 k$. Since δ is an algebra homomorphism, we have $\delta((\alpha b)^2) = \delta(\alpha b)^2$ and $\delta((\alpha b)(\alpha b^*)) = \delta(\alpha b)\delta(\alpha b^*)$. These identities reduce to $\alpha k = u + t$ and $\alpha^2 k = 1 + 2\alpha u$. (iv). Since (ii) implies $t > u$ and $k > 0$, for a table algebra basis we need only to ensure $u \geq 0$. □

Note that for a 3-dimensional asymmetric C-algebra, the entire algebra is determined by the valency k of a nonidentity basis element, since the regular matrices for b and b^* will satisfy $u = \frac{k-1}{2}$ and $t = \frac{k+1}{2}$. In particular, in an integral asymmetric table algebra of rank 3, this valency k must be an odd positive integer, and the order of an integral rank 3 asymmetric C-algebra will always be $\equiv 3 \mod 4$.

Suppose $k = 2u + 1$ for some nonnegative integer u. Then the regular matrices of b and b^* are:

$$b = \begin{bmatrix} 0 & 0 & 2u+1 \\ 1 & u & u \\ 0 & u+1 & u \end{bmatrix}, \text{ and } b^* = \begin{bmatrix} 0 & 2u+1 & 0 \\ 0 & u & u+1 \\ 1 & u & u \end{bmatrix}.$$

Suppose the eigenvalues of b are $k = 2u + 1, \phi_1$, and ψ_1. Using the trace and determinant of b we conclude $\phi_1 + \psi_1 = -1$ and $\phi_1\psi_1 = u + 1$. Solving for these gives $\phi_1 = \frac{1}{2}(-1 + \sqrt{-3 - 4u})$ and $\psi_1 = \frac{1}{2}(-1 - \sqrt{-3 - 4u})$. This is to be expected, as the character table can never be real for an asymmetric table algebra. The remaining character values are $\phi_2 = -1 - \phi_1$ and $\psi_2 = -1 - \phi_1$. Note that the second row sum being zero implies $\phi_2 = \psi_1$ in this case, so the columns of the character table will also interchanged by complex conjugation. It follows that the multiplicities m_ϕ and m_ψ of the last two irreducible characters must be the same, and since $1 + 2m_\phi = n$ we must have that $m_\phi = k$.

	1	b	b^*	m_χ
δ	1	$2u+1$	$2u+1$	1
ϕ	1	$\frac{1}{2}(-1+\sqrt{-3-4u})$	$\frac{1}{2}(-1-\sqrt{-3-4u})$	$2u+1$
$\bar{\phi}$	1	$\frac{1}{2}(-1-\sqrt{-3-4u})$	$\frac{1}{2}(-1+\sqrt{-3-4u})$	$2u+1$

3 Polynomial identities for standardized C-algebra bases

If the algebra A has dimension 4 or more, we have found that the C-algebra bases containing 1 and another fixed element b of A are no longer determined uniquely by the spectrum of b. In this section we investigate standard C-algebra bases of 4-dimensional C-algebras as an algebraic variety by determining polynomial identities that describe the bases in both the symmetric and asymmmetric cases. The resulting variety will only depend on the rank r, the permutation map τ resulting from the involution on the C-algebra basis, and a specific choice of the labelling of entries of the regular matrices of standard C-algebra basis elements. Therefore we denote it by $V_{r\tau}$ and its corresponding ideal by $I(V_{r\tau})$. For bases that contain a fixed element b, further polynomial identities corresponding to the coefficients of the characteristic polynomial of b can be appended to the list.

We will first illustrate this approach in the case of symmetric C-algebras of rank 4. Suppose A is a 4-dimensional algebra with trivial involution. Assume $\{1, b_1, b_2, b_3\}$ is a *standardized* C-algebra basis of A, and that the degrees of b_1, b_2, and b_3 are k, ℓ, and m. Since the algebra is commutative, the remaining structure constants relative this basis are determined by the products b^2, bb_2, and b_2b_3. In particular, the left regular representation of A in this basis has a precise form:

$$b_1 \mapsto \begin{bmatrix} 0 & k & 0 & 0 \\ 1 & k-x_1-x_4-1 & x_1 & x_4 \\ 0 & k-x_2-x_5 & x_2 & x_5 \\ 0 & k-x_3-x_6 & x_3 & x_6 \end{bmatrix}, b_2 \mapsto \begin{bmatrix} 0 & 0 & \ell & 0 \\ 0 & x_1 & \ell-x_1-x_7 & x_7 \\ 1 & x_2 & \ell-x_2-x_8-1 & x_8 \\ 0 & x_3 & \ell-x_3-x_9 & x_9 \end{bmatrix}, \text{ and}$$

$$b_3 \mapsto \begin{bmatrix} 0 & 0 & 0 & m \\ 0 & x_4 & x_7 & m-x_4-x_7 \\ 0 & x_5 & x_8 & m-x_5-x_8 \\ 1 & x_6 & x_9 & m-x_6-x_9-1 \end{bmatrix}.$$

The standardized property of the basis ensures the row sums of the left regular matrices of standardized basis elements agree with their value under the degree map. Applying the identities

$$\begin{aligned}
b_1^2 &= k1 + (k-x_1-x_4-1)b_1 + (k-x_2-x_5)b_2 + (k-x_3-x_6)b_3, \\
b_1b_2 &= x_1b_1 + x_2b_2 + x_3b_3, \\
b_1b_3 &= x_4b + x_5b_2 + x_4b_3, \\
b_2b_1 &= b_1b_2, \\
b_2^2 &= \ell 1 + (\ell-x_1-x_7)b_1 + (\ell-x_2-x_8-1)b_2 + (\ell-x_3-x_9)b_3 \\
b_2b_3 &= x_7b_1 + x_8b_2 + x_9b_3 \\
b_3b_1 &= b_1b_3 \\
b_3b_2 &= b_2b_3 \\
b_3^2 &= m1 + (m-x_4-x_7)b_1 + (m-x_5-x_8)b_2 + (m-x_6-x_9-1)b_3
\end{aligned}$$

to these regular matrices and equating entries produces a set of polynomials on which any standardized C-algebra basis corresponds to single point of $\mathbb{R}^{12}$ on which all of these polynomials vanish. We will denote this variety by V_{4s}, and its corresponding ideal in $\mathbb{R}[x_1, \dots, x_9, k, \ell, m]$ by $I(V_{4s})$. (We use GAP to collect and manipulate these polynomials.) In the case of symmetric C-algebras of rank 4 we get 84 nonzero polynomials arising from the above nine matrix equations, and reducing from this collection to a minimal generating set for $I(V_{4s})$ leaves us with just 10 polynomials:

$$\begin{aligned}
&x_3m - x_7k, \\
&x_3m + x_8\ell + x_9m - \ell m, \\
&x_3m + x_4k + x_6m - km, \\
&x_1k + x_2\ell + x_7k - k\ell, \\
&x_1k + x_2\ell + x_5\ell - k\ell, \\
&x_1x_6 - x_1k + x_2x_9 - x_2\ell - x_3x_4 + x_3x_5 - x_3x_6 - x_3x_9 + x_5x_9 - x_5\ell - x_6x_9 + k\ell - x_3, \\
&x_1x_5 - x_1k + x_2x_7 - x_2\ell - x_3x_4 - x_3x_7 + x_5x_7 - x_5\ell - x_6x_7 + k\ell, \\
&x_1x_5 - x_1k - x_2x_4 + x_2x_5 + x_2x_8 - x_2\ell - x_3x_5 - x_3x_8 + x_5x_8 - x_5\ell - x_6x_8 + k\ell + x_5, \\
&x_1x_5 - x_1k + x_2x_7 - x_2\ell - x_3x_5 - x_3x_8 + x_5x_7 - x_5x_9 - x_7k + k\ell, \\
&x_1x_4 + x_1x_7 - x_1x_8 - x_1k + x_2x_7 - x_2\ell - x_3x_4 - x_3x_7 + x_4x_7 - x_4x_9 - x_7k + k\ell + x_7.
\end{aligned}$$

Minimal generating set for $I(V_{4s})$

Conversely, any point of $\mathbb{R}^{12}$ where these polynomials are all zero and k, ℓ, and m are all positive will correspond to an ordered standard basis of a symmetric C-algebra of rank 4. This correspondence is not quite unique, different orderings of basis elements in the same standard basis will produce different points in $\mathbb{R}^{12}$.

Starting from this set of 10 polynomials, it is easy to write an algorithm that determines all integral table algebras of rank 4 and order n for reasonably small positive integers n. We have carried out this enumeration for orders up to 100 and obtained the following numbers of permutation inequivalent symmetric rank 4 integral table algebras of each order:

n	#	n	#	n	#	n	#	n	#
1	0	21	39	41	189	61	519	81	1126
2	0	22	55	42	209	62	489	82	1063
3	0	23	38	43	206	63	486	83	890
4	1	24	56	44	226	64	692	84	954
5	0	25	59	45	265	65	570	85	1221
6	1	26	75	46	320	66	720	86	1164
7	1	27	65	47	235	67	607	87	1010
8	3	28	96	48	276	68	597	88	1219
9	2	29	73	49	310	69	659	89	1179
10	7	30	94	50	342	70	745	90	1115
11	3	31	92	51	327	71	713	91	1487
12	11	32	108	52	383	72	692	92	1407
13	8	33	112	53	332	73	787	93	1304
14	15	34	142	54	354	74	732	94	1368
15	12	35	120	55	407	75	729	95	1223
16	29	36	195	56	496	76	992	96	1582
17	16	37	155	57	475	77	824	97	1475
18	28	38	163	58	487	78	1024	98	1352
19	21	39	154	59	407	79	888	99	1524
20	35	40	221	60	450	80	857	100	1899

Number of Integral TAs: Symmetric, Rank 4, Order n

One might ask about the Gröbner basis of the ideals corresponding to C-algebra bases. Unfortunately these generating sets are not close to being monomial, so the Gröbner bases one finds for these ideals tend to be considerably larger than our minimal generating sets. For example, the smallest Gröbner basis we found for $I(V_{4s})$ has 14 polynomials.

When we know the spectrum of b_1 in advance, we know the coefficients of the characteristic polynomial of the matrix representing b. If $spec(b_1) = \{\theta_0, \theta_1, \theta_2, \theta_3\}$, then the characteristic polynomial of b_1 is

$$\mu(x) = x^4 - \sigma_1 x^3 + \sigma_2 x^2 - \sigma_3 x + \sigma_4,$$

where the σ_i for $i = 1, 2, 3, 4$ are the usual usual symmetric functions of the θ's. We have that $\theta_0 = k > 0$ is a positive eigenvalue of b. Equating coefficients of the characteristic polynomial of b_1 gives us four more polynomials in the variables k, x_1, x_2, x_3, x_4, x_5, and x_6 that must vanish for a given b_1 to lie in a standardized C-algebra basis:

$$-x_2x_6k + x_3x_5k - \sigma_4,$$
$$-x_1x_5k + x_1x_6k + x_2x_4k - x_2x_6k - x_3x_4k + x_3x_5k + x_2x_6 + x_2k - x_3x_5 + x_6k + \sigma_3,$$
$$x_1x_5 - x_1x_6 - x_1k - x_2x_4 + x_2x_6 + x_2k + x_3x_4 - x_3x_5 - x_4k + x_6k - x_2 - x_6 - k - \sigma_2, \text{ and}$$
$$x_1 - x_2 + x_4 - x_6 - k + 1 + \sigma_1.$$

The entries of b_1 represent a point in the variety generated by these four polynomials. So b_1 lies in a standard C-algebra basis of a symmetric C-algebra of rank 4 iff $I(V_{4s})$ contains points whose coordinates representing the values of $x_1, \ldots, x_6$ and k agree with those coming from b_1, and the values of ℓ and m determined by these points are positive. There are 5 remaining variables ℓ, m, x_7, x_8, and x_9 whose values would be determined by the polynomials generating $I(V_{4s})$. In our algorithm this decision can always be made using only the values of ℓ and x_8, and in some cases x_8 is not required.

If we substitute the entries of the regular matrix b into our polynomials, then in small ranks a simple Gröbner basis calculation can determine the entries of the remaining matrices to produce the standard table algebra bases that contain b, or show that such a basis does not exist. We will illustrate this approach with an example to demonstrate that b may not lie in a standard integral table algebra basis of $\mathbb{Q}[b]$.

Example 8. Let A_4 be the distance 4 adjacency matrix for the Coxeter graph Γ with intersection array $[3,2,2,1;1,1,1,2]$. A_4 is the 5×5 matrix

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 2 & 2 \end{bmatrix}.$$

This example was noted by Brouwer and Fiol in [6] because A_4 has fewer eigenvalues, four, than the adjacency matrix for Γ, which has five. One can thus ask if A_4 is an element of a table algebra basis of the 4-dimensional algebra $\mathbb{Q}[A_4]$. The minimal polynomial for A_4 is $x^4-8x^3+6x^2+40x-24$, which becomes the characteristic polynomial of the matrix for A_4 resulting from its image in any regular representation of $\mathbb{Q}[A_4]$. If $\{I, A_4, A_{42}, A_{43}\}$ is a standardized C-algebra basis of $\mathbb{Q}[A_4]$, the regular representation of these basis matrices has the form determined by our polynomial identities above. The degree of A_4 is its largest eigenvalue, 6, so we must have $k = 6$, and from the coefficients of the characteristic polynomial we have $\sigma_1 = 8$, $\sigma_2 = 6$, $\sigma_3 = -40$, and $\sigma_4 = -24$. When we substitute these values into our polynomial generators of $I(V_{4s})$, we find that the polynomials still generate a proper ideal of $\mathbb{Q}[x_1,\dots,x_9,k,\ell,m]$. This is evidence that A_4 should be an element of some rational C-algebra basis of $\mathbb{Q}[A_4]$. A simple search beginning with the initial substitutions $(x_1,x_3) = (1,0)$ leads to two permutation-equivalent rational solutions corresponding to $x_2 = 4,1$. When $(x_1,x_2,x_3) = (1,4,0)$ the solution is:

$$A_4 \mapsto b := \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix},$$

with the regular matrices of b_2 and b_3 determined by $(x_7,x_8,x_9,\ell,m) = (0,\frac{4}{3},-\frac{1}{3},3,\frac{6}{5})$. In particular, the characteristic polynomial of b matches the minimal polynomial of A_4. More rational C-algebra bases occur when $(x_1,x_2,x_3) = (0,3,1),(3,0,4),(3,3,3),(4,2,\frac{11}{2})$, and $(4,3,\frac{17}{5})$, but none of these are integral. Since our search included all integral possibilities for $x_1 \in \{0,1,\dots,5\}$ and $x_3 \in \{0,\dots,6\}$, we can be certain that A_4 is not an element of a standard integral table algebra basis of $\mathbb{Q}[A_4]$. (In particular, the graph with adjacency matrix A_4 is not distance regular.)

For symmetric rank 4 C-algebras no uniform character table formula can be expected. The cases depend on the minimal polynomials $\mu_b(x)$ of the nontrivial basis elements. If one of these minimal polynomials has degree 4, then A is the ring of matrix polynomials $\mathbb{Q}[b]$, which is isomorphic to the quotient polynomial ring $\mathbb{Q}[x]/\mu_b(x)$ as an algebra. The field of character values of the C-algebra $(A,\mathbf{B})$ is the splitting field of $\mu_b(x)$. This can be a cyclotomic extension of degree 1, 2, 3, or a non-cyclotomic field of degree 6. The first non-cyclotomic integral table algebra example we found has order 13 and the minimal polynomial of one basis element is x^3+3x^2-x-4. If the degrees of minimal polynomials of basis elements are all 3 or less, then the C-algebra is a direct or circle (wedge) product of C-algebras of smaller rank.

We have carried out a similar process for asymmetric C-algebras of rank 4 and found a small set of generators for the corresponding ideal $I(V_{4a})$. For these a standard basis has the form $\{1,b,c,c^*\}$, and the general pattern for regular matrices of the non-identity standard basis elements will be:

$$b \mapsto \begin{bmatrix} 0 & k & 0 & 0 \\ 1 & k-2x_1-1 & x_1 & x_1 \\ 0 & k-x_2-x_3 & x_2 & x_3 \\ 0 & k-x_2-x_3 & x_3 & x_2 \end{bmatrix},$$

$$c \mapsto \begin{bmatrix} 0 & 0 & 0 & \ell \\ 0 & x_1 & x_4 & \ell - x_1 - x_4 \\ 1 & x_2 & x_5 & \ell - x_2 - x_5 - 1 \\ 0 & x_3 & x_6 & \ell - x_3 - x_6 \end{bmatrix}, \text{ and } c^* \mapsto \begin{bmatrix} 0 & 0 & \ell & 0 \\ 0 & x_1 & \ell - x_1 - x_4 & x_4 \\ 0 & x_3 & \ell - x_3 - x_6 & x_6 \\ 1 & x_2 & \ell - x_2 - x_5 - 1 & x_5 \end{bmatrix}.$$

In addition to the polynomials arising expressing products of these matrices using the structure constants, we again have polynomials arising from commutativity. This time there are 92 nonzero polynomials arising from the matrix equations, and from these we get a minimal generating set of size 7:

$$\begin{aligned}
&-x_2 + x_3 - x_5 + x_6 - 1, \\
&-x_3\ell - x_5\ell - x_6\ell + \ell^2, \\
&-x_3\ell + x_4k, \\
&-x_1k - x_2\ell - x_4k + k\ell, \\
&-x_1x_3 - x_2x_4 + x_3^2 - x_3x_4 + x_3x_5 + x_3x_6 - x_3\ell + x_4k, \\
&-x_1x_2 - x_2x_4 + x_3^2 - x_3x_4 + 2x_3x_6 - x_3\ell + x_4k, \\
&-x_1^2 + x_1x_3 - 2x_1x_4 + x_1x_5 + x_1x_6 - x_2x_4 + x_3x_4 - x_3\ell - x_5\ell - x_6\ell + \ell^2 - x_4.
\end{aligned}$$

Minimal generating set for $I(V_{4a})$

The corresponding Gröbner basis of $I(V_{4a})$ has size 10. We have determined all permutation inequivalent asymmetric rank 4 integral table algebra bases for orders up to 100. The number for each order are shown here:

n	#	n	#	n	#	n	#	n	#
1	0	21	5	41	8	61	12	81	18
2	0	22	6	42	9	62	13	82	18
3	0	23	4	43	8	63	13	83	16
4	1	24	6	44	9	64	24	84	18
5	0	25	5	45	13	65	13	85	21
6	2	26	5	46	11	66	16	86	18
7	1	27	5	47	9	67	13	87	18
8	2	28	10	48	11	68	14	88	25
9	2	29	5	49	10	69	15	89	18
10	3	30	7	50	11	70	18	90	19
11	1	31	6	51	11	71	14	91	22
12	3	32	7	52	15	72	16	92	21
13	8	33	7	53	10	73	15	93	20
14	3	34	8	54	12	74	15	94	21
15	3	35	6	55	11	75	15	95	19
16	8	36	12	56	17	76	22	96	30
17	3	37	7	57	13	77	17	97	20
18	4	38	8	58	13	78	20	98	20
19	3	39	8	59	11	79	16	99	21
20	4	40	15	60	13	80	17	100	32

# of Integral TAs: Asymmetric, Rank 4, Order n

A decision algorithm for deciding when b or c belongs to a standard basis of an asymmetric C-algebra of rank 4 can be also done using the coefficients of its characteristic polynomial. The procedure is exactly as described in the symmetric case.

For table algebras of rank 5 a similar enumeration procedure can be implemented, but the computational complexity grows quickly with the order. The variety $I(V_{5s})$ lies in $\mathbb{R}^{25}$. The matrix equations produce 220 nonzero polynomial generators, which reduce to a $\mathbb{Q}$-linearly independent set of just 43 polynomals. (We found that a $\mathbb{Q}$-linear independent can be found quickly, and it is almost always a minimal generating set or very close to one. It much more expensive to find a minimal generating set or a Gröbner basis. We eventually found a reduced Gröbner basis of $I(V_{5s})$ consisting of 564 polynomials. We are indebted to Alexander Konovalov for his assistance in these efforts, which were completed with the computer algebra software `Singular` after our attempts with GAP proved unsuccessful.)

The enumeration of symmetric rank 5 integral table algebras of order up to 100 was a challenging computational exercise. The number of these integral table algebras is quite high for each order because there are relatively few defining polynomials relative to the number of variables involved. A naive algorithm was successful up to order 50 and an improved selective case-by-case modification of this single-thread algorithm was successful up to order 90 in about two weeks. For orders 90 to 100 a parallel approach was needed to cover the large search space in under one month, and the data collected needed to be compiled and duplicates removed with a final check for permutation equivalence.

n	#	n	#	n	#	n	#	n	#
1	0	21	30	41	304	61	1318	81	4341
2	0	22	48	42	499	62	1725	82	4484
3	0	23	27	43	334	63	1570	83	3811
4	0	24	72	44	594	64	2131	84	4720
5	0	25	53	45	566	65	1904	85	4957
6	0	26	89	46	636	66	2437	86	5318
7	0	27	62	47	495	67	1848	87	4815
8	3	28	127	48	728	68	2441	88	5700
9	3	29	77	49	652	69	2182	89	4970
10	2	30	155	50	893	70	2736	90	6284
11	1	31	90	51	794	71	2268	91	6383
12	7	32	208	52	1041	72	3065	92	7254
13	2	33	154	53	776	73	2528	93	6477
14	10	34	215	54	1110	74	3148	94	7067
15	6	35	193	55	1013	75	2956	95	6477
16	18	36	335	56	1453	76	3673	96	8322
17	9	37	209	57	1244	77	3327	97	6860
18	21	38	333	58	1442	78	4184	98	7855
19	9	39	260	59	1134	79	3323	99	8101
20	44	40	424	60	1603	80	4263	100	9949

# of Integral TAs: Symmetric, Rank 5, Order n

The numbers of permutation inequivalent rank 5 integral table algebra bases containing one or two $*$-asymmetric pairs are much smaller for each order. The matrix equations for $I(V_{5a1})$ produce a set of 213 nonzero polynomials that contains a minimal generating set of size 54. With more generating polynomials we get a smaller variety and smaller numbers for each order: 15 (two), 19 (one), 23 (one), 27 (three), 28 (one), 30 (one), 31 (one), 35 (two), 39 (five), 43 (two), 47 (two), 48 (one), 51 (eight), 52 (two), 53 (one), 55 (three), 56 (two), 57 (two), 59 (three), 63 (five), 64 (one), 66 (one), 67 (five), 70 (one), 71 (four), 75 (nine), 78 (three), 79 (five), 82 (one), 83 (five), 84 (three), 86 (one), 87 (seven), 88 (two), 91 (four), 92 (one), 93 (one), 95 (five), 96 (one), 97 (one), 99 (eleven), and 100 (one).

The matrix equations for $I(V_{5a2})$ produce a set of 154 polynomial generators, that reduces to a minimal generating set of size 64. Using these our numbers of permutation inequivalent rank 5 table algebra bases containing two $*$-asymmetric pairs for each order up to 100 are: 5 (one), 9 (one), 13 (three), 17 (one), 21 (four), 25 (two), 29 (five), 33 (three) 37 (eight), 41 (three), 45 (nine), 49 (four), 53 (eight), 57 (five), 61 (twelve), 65 (five), 69 (twelve), 73 (six), 77 (twelve), 81 (seven), 85 (eighteen), 89 (seven), 93 (sixteen), 97 (eight). For these the order is always equal to $4k+1$, for some integer k. We did not know this before trying to enumerate them, but later we realized it follows from applying the integrality condition to our polynomial generators of $I(V_{5a2})$.

Remark. Those familiar with Brouwer's online lists of parameters for distance regular graphs [5] will appreciate that we have checked the table algebras corresponding to the intersection arrays with open feasibility problem, and found them to be compatible with the generators of our C-algebra varieties. In doing so, we noticed that in a few small order cases one can find association schemes appearing in Hanaki and Miyamoto's classification [7] for which at least one adjacency matrix in the scheme is that of a graph matching an intersection array in Brouwer's list. So we can conclude these intersection arrays are feasible.

int. array	scheme id.
$[4,2,2;1,1,2]$	`as21no09`
$[5,4,2;1,1,4]$	`as36cl4no55`
$[6,4,2;1,2,3]$	`as27no386`
$[4,3,2,1;1,2,3,4]$	`as16no30`
$[3,2,2,1;1,1,2,3]$	`as18no24`
$[3,2,2,2;1,1,1,3]$	`as30no51`
$[5,4,1,1;1,1,4,5]$	`as32no36`
$[4,3,3,1\ 1,1,3,4]$	`as32no2111`
$[6,5,4,1\ 1,2,5,6]$	`as36cl4no36`
$[8,7,4,1;1,4,7,8]$	`as32no53`

References

[1] Z. Arad, E. Fisman, and M. Muzychuk, Generalized table algebras, *Israel J. Math.*, **114** (1999), 29-60.

[2] H. Blau, Quotient structures in C-algebras, *J. Algebra*, **175** (1995), 24-64; Erratum: **177** (1995), 297-337.

[3] H. Blau, Table algebras, *European J. Combin.*, **30**, (2009), 1426-1455.

[4] H. Blau and G. Chen, Reality-based algebras, generalized Camina-Frobenius pairs, and the non-existence of degree maps, **40** (2012), (4), 1547-1562.

[5] A. Brouwer, *BCN tables*, `https://www.win.tue.nl/~aeb/drg/drgtables.html`

[6] A. Brouwer and M. Fiol, Distance-regular graphs where the distance-d graph has fewer distinct eigenvalues, (2014) `arXiv:1409.0389v1`

[7] A. Hanaki and I. Miyamoto, *Classification of Small Association Schemes*, `http://math.shinshu-u.ac.jp/~hanaki/as/`

[8] D.G. Higman, Coherent algebras, *Linear Algebra Appl.*, **93** (1987), 209-239.

[9] P.-H. Zieschang, *Theory of Association Schemes*, Springer-Verlag, Berlin, 2005.