ON KOSZUL DUALITY BETWEEN POLYNOMIAL AND EXTERIOR ALGEBRAS

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Abstract

In this thesis we will study Ext-algebras over a polynomial and exterior algebra. We prove the classical fact that the Ext-algebra over a polynomial algebra is exterior and the Ext-algebra over an exterior algebra is polynomial, using the tautological Koszul complex. We also give a proof using Koszul duality for algebras.

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Chapter 1

Introduction

1.1 Motivation and background

Ext functors are a fundamental concept of homological algebra and are defined as the derived functors of the hom functors. The Koszul complex is an applicable general construction in homological algebra which was first introduced by Jean-Louis Koszul in 1950 for defining cohomology theory of Lie algebras [Kos50]. Moreover, this can be used to determine basic facts about a module or ideal. The behavior of Ext groups over a polynomial algebra and an exterior algebra is a classic topic. One exciting thing is that the Koszul complex can be used as a tool to find Ext groups over the above algebras.

Let k be a field, $P(n) = k[x_1, \ldots, x_n]$ the polynomial algebra over k on n generators, and $E(n) = \Lambda_k(x_1, \ldots, x_n)$ the exterior algebra over k on n generators. According to [Pri70, Example 2.2(2)], the following fact is "classical". The Ext-algebra over a polynomial algebra is exterior: $\operatorname{Ext}_{P(n)}(k,k) \cong E(n)$ and the Ext-algebra over an exterior algebra is polynomial: $\operatorname{Ext}_{E(n)}(k,k) \cong P(n)$.

In this thesis, we will give two proofs of the above theorem: one using the tautological Koszul complex $P(n) \otimes_k E(n)$, outlined in [Eis95, Exercise 17.21] (Proposition 4.3.7), and one using Koszul duality for algebras, outlined in [Tam18] and [PP05] (Proposition 5.4.3). There is a proof using Hopf algebras, outlined in [May13], but we are not covering it in this thesis.

1.2 Contribution

In the literature, the proofs that the Ext-algebra over a polynomial algebra is exterior and vice versa skip some details. We fill in some details and compute some explicit examples (Example 3.2.2, Example 4.1.4, Example 4.1.5, Example 4.2.8, Lemma 5.3.2 and Proposition 5.3.6). In particular, we work out the proof outlined in [Eis95, Exercise 17.21], which is structured as a list of exercises (Lemma 4.3.1, Lemma 4.3.2, Proposition 4.3.5 and Proposition 4.3.7).

Several references assume that the ground commutative ring k is a field, including [PP05] and [Pri70] when stating the fact that if A is a Koszul algebra, then the Ext-algebra $\operatorname{Ext}_A(k,k)$ is isomorphic to the Koszul dual algebra $A^!$. In this thesis, we check the special case of that result for polynomial and exterior algebras over any commutative ring k, using work of Positselski [Pos21].

1.3 Organization

Chapter 2 reviews some preliminaries from homological algebra and multilinear algebra, in particular symmetric algebras and exterior algebras.

Chapter 3 deals with additive structure of Ext-algebras. We calculate Ext groups over a polynomial algebra and exterior algebra using the Koszul complex.

Chapter 4 covers multiplicative structure of Ext-algebras. We discuss how endomorphism chain complexes directly provide the multiplicative structure. Then, we define the tautological Koszul complex and study its multiplicative structure. Finally, we provide a proof of the classical fact that the Ext-algebra over a polynomial algebra is exterior using the tautological Koszul complex (Proposition 4.3.7).

Chapter 5 introduces definitions and properties of quadratic algebras and Koszul algebras. Then, we observe that the symmetric algebra and exterior algebra are Koszul dual to each other. In Proposition 5.4.3, we give a second proof using Koszul duality for algebras.

Chapter 2

Preliminaries

In this thesis, all rings will be assumed unital but an R-algebra will not be assumed unital.

2.1 Hom modules

Facts about hom modules can be found in [DF04, §10.2].

Notation 2.1.1. *Let R be a commutative ring. For R-modules M and N, let* $\mathrm{Hom}_R(M, N)$ *denote the set of R-module maps $f : M \to N$.*

Proposition 2.1.2. *For a commutative ring R,* $\mathrm{Hom}_R(M, N)$ *is an R-module, functorial in both variables: contravariant in M and covariant in N.*

Proof. $\mathrm{Hom}_R(M, N)$ is endowed with pointwise addition and scalar multiplication.

Let $f,g \in \mathrm{Hom}_R(M,N)$ and $c,r \in R$. We have

$$(f+g)(cx) = f(cx) + g(cx) = c(f+g)(x) \text{ and}$$

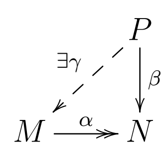
$$(rf)(cx) = r(f(cx)) = c(rf)(x), \text{ since } r \text{ commutes with } c.$$

Then $f+g$ and rf are R-linear maps. □

Note that for any ring R, $\mathrm{Hom}_R(M,N)$ is an abelian group.

Remark 2.1.3. *If S is any set, then the set of functions $\mathrm{Hom}_{Set}(S,N)$ into an R-module N becomes an R-module under pointwise addition and scalar multiplication. The subset $\mathrm{Hom}_R(M,N) \subseteq \mathrm{Hom}_{Set}(M,N)$ is an R-submodule.*

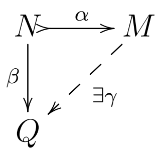
Definition 2.1.4. *An R-module P is called* ***projective*** *if for every epimorphism of modules $\alpha : M \twoheadrightarrow N$ and every map $\beta : P \to N$, there exists a map $\gamma : P \to M$ such that $\beta = \alpha\gamma$, as in the following diagram:*



[Eis95, §A3.2 Definition].

Lemma 2.1.5. *Free modules are projective. A module is projective if and only if it is a direct summand of a free module* [DF04, §10.5 Proposition 30].

Definition 2.1.6. *An R-module Q is called* ***injective*** *if for every monomorphism of R-modules $\alpha : N \rightarrowtail M$ and every homomorphism of R-modules $\beta : N \to Q$,* *there exists a homomorphism of R-modules $\gamma : M \to Q$ such that $\beta = \gamma\alpha$, as in the following diagram:*



[Eis95, §A3.4 Definition].

Characterizations of injective modules can be found in [DF04, §10.5 Proposition 34].

2.2 The hom complex between chain complexes

Throughout this subsection, let R be a commutative ring.

Definition 2.2.1. *1. A **chain complex** C of R-modules is a sequence*

$$\dots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \dots$$

*where $d_n : C_n \to C_{n-1}$ satisfies $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. A chain complex is also defined as **unbounded** or $\mathbb{Z}$-graded. A chain complex is said to be **non-negatively graded** if $C_n = 0$ for $n < 0$. The n^{th} **homology group** of the chain complex C is defined as $H_n(C) = \ker d_n / \operatorname{im} d_{n+1}$.*

*2. A **cochain complex** C of R-modules is a sequence*

$$\dots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \dots$$

where $d^n : C^n \to C^{n+1}$ *satisfies* $d^{n+1} \circ d^n = 0$ *for all* $n \in \mathbb{Z}$*. A cochain complex is also defined as* ***unbounded*** *or* $\mathbb{Z}$*-graded. A cochain complex is said to be* ***non-negatively graded*** *if* $C^n = 0$ *for* $n < 0$*. The* n^{th} ***cohomology group*** *of the cochain complex* C *is defined as* $H^n(C) = \ker d^n / \operatorname{im} d^{n-1}$ [DF04, §17.1].

Definition 2.2.2. *Let* C *and* D *be chain complexes of* R*-modules. The* ***tensor product*** *of* C *and* D *is the chain complex* $C \otimes_R D$ *having in degree* n *the* R*-module*

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes_R D_j$$

and with boundary operator given by

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^{|c|} c \otimes (\partial d)$$

where $|c|$ *denotes the degree of* $c \in C$*. To be more precise, for* $c \in C_i$ *and* $d \in D_j$*, the formula can be written as*

$$\partial_{i+j}^{C \otimes D}(c \otimes d) = (\partial_i^C c) \otimes d + (-1)^i c \otimes (\partial_j^D d)$$

where the subscript denotes the degree and the superscript denotes the chain complex to which the boundary operator belongs [HS97, §V.1 Example(a)].

Lemma 2.2.3. *The tensor product* $C \otimes_R D$ *is indeed a chain complex, i.e.,* $\partial^2 = 0$ *holds.*

Definition 2.2.4. *A* ***differential graded algebra*** *or* ***DG-algebra*** *is a graded* R*-algebra* A_* *equipped with a map* $d : A_i \to A_{i-1}$ *satisfying* $d^2 = 0$ *and the Leibniz* *rule:*

$$d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$$

where $|a| = i$ *for* $a \in A_i$ [Wei94, 4.5.2].

The Leibniz rule is saying that the multiplication map $A \otimes_R A \to A$ is a chain map.

Lemma 2.2.5. *Let A be a DG-algebra. The multiplication on A induces a multiplication on the homology $H(A)$:*

$$H_p(A) \otimes H_q(A) \to H_{p+q}(A)$$

defined for $[a] \in H_p(A)$ *and* $[b] \in H_q(A)$ *by*

$$[a][b] := [ab].$$

If A is unital with unit $1_A \in A_0$, then the homology algebra $H(A)$ is unital with unit $1_{H(A)} = [1_A] \in H_0(A)$.

Definition 2.2.6. *Let C and D be chain complexes of R-modules. The **hom complex** $\underline{\mathrm{Hom}}(C, D)$ has in degree n the maps of graded R-modules $f : C_* \to D_{*+n}$ that raise degree by n, that is:*

$$\underline{\mathrm{Hom}}(C, D)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(C_i, D_{i+n}) = \prod_{j-i=n} \mathrm{Hom}_R(C_i, D_j)$$

and with boundary operator given by

$$(\partial f)(c) = \partial(f(c)) + (-1)^{|f|+1} f(\partial c).$$

In alternate notation:

$$\partial^{\underline{\mathrm{Hom}}(C,D)} f = \partial^D \circ f + (-1)^{|f|+1} f \circ \partial^C.$$

The sign convention is chosen so that the formula

$$\partial(f(c)) = (\partial f)(c) + (-1)^{|f|} f(\partial c)$$

holds, consistent with the usual "Koszul sign rule" [Bre97, §VI.2].

Lemma 2.2.7. $\underline{\mathrm{Hom}}(C, D)$ *is indeed a chain complex, i.e.,* $\partial^2 = 0$ *holds.*

Definition 2.2.8. *Let C and D be chain complexes of R-modules. A **chain map of degree n** $f : C_* \to D_{*+n}$ is an n-cycle in the hom complex* $\underline{\mathrm{Hom}}(C, D)$*. More explicitly, for even degree n the chain map f commutes with the boundary maps:*

$$f(\partial c) = \partial f(c),$$

while for odd degree n the chain map f anticommutes with the boundary maps:

$$f(\partial c) = -\partial f(c).$$

Lemma 2.2.9. *Let C be a chain complex of R-modules. The endomorphism complex of C* $\underline{\mathrm{Hom}}(C, C)$ *is made into a (unital) differential graded algebra under function composition.*

Example 2.2.10. *Let C be the chain complex of abelian groups $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ concentrated in degrees 0 and 1. The endomorphism complex* $\underline{\mathrm{Hom}}(C, C)$ *is*

$$\mathbb{Z} \xrightarrow{\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]} \mathbb{Z} \times \mathbb{Z} \xrightarrow{[n\ -n]} \mathbb{Z}$$

concentrated in degrees -1, 0, and 1. Its homology algebra is

$$H_*\underline{\mathrm{Hom}}(C, C) = \begin{cases} \mathbb{Z}/n, & * = 0, -1 \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 2.2.11. *A morphism $C \to D$ of chain complexes is called a* ***quasi-isomorphism*** *if the induced maps of homology groups $H_n(C) \to H_n(D)$ are isomorphisms for all $n \in \mathbb{Z}$* [Wei94, 1.1.2].

Theorem 2.2.12. *Let P be a bounded below chain complex of projective R-modules. Let $f : C \xrightarrow{\sim} D$ be a quasi-isomorphism of chain complexes of R-modules. The induced map of hom complexes*

$$f_* : \underline{\mathrm{Hom}}(P, C) \to \underline{\mathrm{Hom}}(P, D)$$

is a quasi-isomorphism [Bro82, Theorem I.8.5].

2.3 Ext functors

The Ext functors are the derived functors of the hom functors. Ext can be defined by using an injective or a projective resolution.

Lemma 2.3.1. *Every R-module N has an injective resolution* [Wei94, Exercise 2.3.5].

Take any injective resolution

$$0 \to N \to I^0 \to I^1 \to \dots$$

Remove the term N, then we obtain a cochain complex by applying $\mathrm{Hom}_R(M,-)$

$$0 \to \mathrm{Hom}_R(M, I^0) \to \mathrm{Hom}_R(M, I^1) \to \dots$$

For each n, $\mathrm{Ext}_R^n(M,N)$ is the cohomology of the complex at position n

$$H^n(\mathrm{Hom}(M, I^\bullet)) = \mathrm{Ext}^n(M,N).$$

Take any projective resolution of M

$$\dots \to P_1 \to P_0 \to M \to 0.$$

Remove the term M, then we obtain a cochain complex by applying $\mathrm{Hom}_R(-,N)$

$$0 \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to \dots$$

For each n, $\mathrm{Ext}_R^n(M,N)$ is the cohomology of the complex at position n

$$H^n(\mathrm{Hom}(P_\bullet, N)) = \mathrm{Ext}^n(M,N).$$

Remark 2.3.2. *Both constructions using injective and projective resolutions yield the same* Ext *groups* [Wei94, Theorem 2.7.6].

We can construct Ext groups in terms of long extensions, also called Yoneda Ext groups. An element of the Yoneda $\mathrm{Ext}^n(M,N)$ is an equivalence class of exact sequence of the form

$$0 \to N \to X_n \to \dots \to X_1 \to M \to 0.$$

The equivalence relation is generated by the relation that identifies two extensions:

$$\xi : 0 \to N \to X_n \to \ldots \to X_1 \to M \to 0$$

$$\xi' : 0 \to N \to X'_n \to \ldots \to X'_1 \to M \to 0$$

if there are maps $X_k \to X'_k$ for all $k \in \{1, 2, \ldots, n\}$ so that every resulting square commutes, i.e., there is a chain map $\xi \to \xi'$ which is the identity on M and N [Wei94, Vista 3.4.6]. The Yoneda Ext groups agree with the Ext groups defined as right derived functors [HS97, §IV.9].

For a commutative ring R and R-modules M and N, $\text{Ext}^n_R(M, N)$ is an R-module and for a non-commutative ring R, $\text{Ext}^n_R(M, N)$ is only an abelian group. In general, if R is an algebra over a commutative ring S, then $\text{Ext}^n_R(M, N)$ is at least an S-module.

2.4 Properties of Ext groups

Here are some of the basic properties and computations of Ext groups [Wei94, §3.3].

Proposition 2.4.1. *For any R-modules M and N:*

1. $\text{Ext}^0_R(M, N) \cong \text{Hom}_R(M, N)$.

2. $\text{Ext}^n_R(M, N) = 0$ *for $n > 0$ if either M is projective or N is injective.*

3. *The converses also hold:*

   if $\mathrm{Ext}^1_R(M,N) = 0$ *for all* N*, then* M *is projective;*

   if $\mathrm{Ext}^1_R(M,N) = 0$ *for all* M*, then* N *is injective.*

4. $\mathrm{Ext}^n_R(\bigoplus_{i\in I} M_i, N) \cong \prod_{i\in I} \mathrm{Ext}^n_R(M_i, N)$.

5. $\mathrm{Ext}^n_R(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} \mathrm{Ext}^n_R(M, N_i)$.

Lemma 2.4.2. $\mathrm{Ext}^n_{\mathbb{Z}}(M,N) = 0$ *for all* $n \geq 2$ *and all abelian groups* M *and* N.

Proof. Let $P_\bullet$ be a length 1 projective (free) resolution of M. Therefore, $\mathrm{Ext}^*(M,N)$ is the cohomology of

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to 0.$$

□

Example 2.4.3. *Since* $\bigoplus_{i\in I} \mathbb{Z}$ *is a free abelian group,* $\mathrm{Ext}^1_{\mathbb{Z}}(\bigoplus_{i\in I} \mathbb{Z}, N) = 0$.

Example 2.4.4. $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, A) \cong A/nA$.

2.5 Yoneda product

Definition 2.5.1. *For* R*-modules* M*,* N*, and* P*, the* ***Yoneda product*** *is the pairing of Ext groups*

$$\mathrm{Ext}^n_R(N,P) \otimes \mathrm{Ext}^m_R(M,N) \to \mathrm{Ext}^{m+n}_R(M,P)$$

induced by the composition pairing

$$\mathrm{Hom}_R(N,P) \otimes \mathrm{Hom}_R(M,N) \xrightarrow{\circ} \mathrm{Hom}_R(M,P).$$

The Yoneda product can be expressed in terms of injective resolutions and projective resolutions. Let X be a projective resolution of M and Y be an injective resolution of P. The composition pairing

$$\mathrm{Hom}_R(N, P) \otimes \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, P)$$

gives a map of cochain complexes

$$\mathrm{Hom}_R(N, Y) \otimes \mathrm{Hom}_R(X, N) \to \mathrm{Tot}(\mathrm{Hom}_R(X, Y)).$$

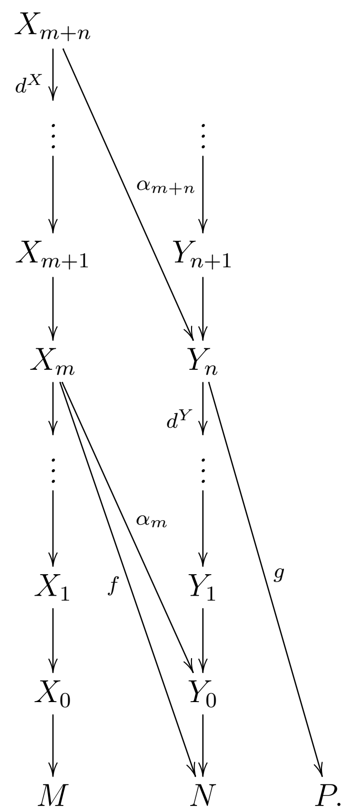
Passing to cohomology, we obtain a pairing

$$\mathrm{Ext}^n_R(N, P) \otimes \mathrm{Ext}^m_R(M, N) \to \mathrm{Ext}^{m+n}_R(M, P)$$

[May08, §4.8].

Remark 2.5.2. *The Yoneda product can also be constructed by using projective resolutions of M and N or injective resolutions of N and P. Let X and Y be projective resolutions of M and N. Let $\alpha \in \mathrm{Ext}^m_R(M, N)$ which is represented by a cocycle $f : X_m \to N$ and $\beta \in \mathrm{Ext}^n_R(N, P)$ which is represented by a cocycle $g : Y_n \to P$.*

Consider the following diagram:



Since X_m *is projective, there is a lift* $\alpha_m : X_m \to Y_0$ *of* f *through the map* $Y_0 \to N$*. We will construct R-module maps* $\alpha_{m+i} : X_{m+i} \to Y_i$ *inductively which satisfy* $d^Y \circ \alpha_{m+i} = \alpha_{m+i-1} \circ d^X$*. When* $i = 1$*, since* X_{m+1} *is projective and the image of* $\alpha_m \circ d^X$ *is contained in the image of* d^Y*, there exists such a map* α_{m+1}*. We assume that there exists the map* α_{m+i-1}*,* $i > 1$*. Since* X_{m+i} *is projective and the image of* $\alpha_{m+i-1} \circ d^X$ *is contained in the image of* d^Y*, there exists such a map* α_{m+i}*. The* n^{th} *step of the* n *lifting steps is* $\alpha_{m+n} : X_{m+n} \to Y_n$*. The given R-module maps* α_i *make the parallelograms commute. Since for even degree a chain map commutes with the* *boundary maps while for odd degree a chain map anticommutes with the boundary maps (see Definition 2.2.8), we change the sign convention to make it match the signs in the homology algebra $H_*\underline{\mathrm{Hom}}(P,P)$. If m is odd, then we choose $\alpha'_i = (-1)^{i-m}\alpha_i$ which make the parallelograms anticommute. The composition $g \circ \alpha'_{m+n} : X_{m+n} \to P$ represents an element $\beta \cdot \alpha \in \mathrm{Ext}_R^{m+n}(M,P)$. This Yoneda product construction agrees with the Yoneda product construction in Definition 2.5.1 up to a sign.*

The Yoneda product can also be constructed in terms of splicing long extensions as follows; see [May08, §4.6] and [ML95, §III.5]. Suppose given extensions

$$0 \to N \to E_m \to \dots \to E_1 \to M \to 0$$

and

$$0 \to P \to F_n \to \dots \to F_1 \to N \to 0$$

representing classes in $\mathrm{Ext}_R^m(M,N)$ and $\mathrm{Ext}_R^n(N,P)$. Rename F_i as E_{m+i} and splice the sequences using the composite map $F_1 \to N \to E_m$. This gives an extension

$$0 \to P \to E_{m+n} \to \dots \to E_1 \to M \to 0,$$

which is the Yoneda product of the given extensions. This Yoneda product construction agrees with the Yoneda product constructions in Definition 2.5.1 and Remark 2.5.2 up to a sign.

Definition 2.5.3. *For an R-module M, the* **Ext-*algebra*** *of M is defined to be*

$$\mathrm{Ext}_R(M,M) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}_R^i(M,M)$$

equipped with the Yoneda product.

2.6 Dual modules

Definition 2.6.1. *Let M be an R-module. The (R-linear)* ***dual*** *of M is the R-module*

$$M^* := \mathrm{Hom}_R(M, R).$$

Assuming that M is a free R-module and given a basis $\{e_i\}_{i\in I}$ of the R-module M, the ***dual elements*** *$e_i^* \in M^*$ are the linear functionals defined by the formula*

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Proposition 2.6.2. *Let $\{e_i\}_{i\in I}$ be a basis of a free R-module M. The dual elements $e_i^* \in M^*$ are linearly independent* [DF04, §11.3].

Lemma 2.6.3. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, $\alpha : F \Rightarrow G$ a natural transformation and X an object in $\mathcal{C}$ such that the component*

$$\alpha_X : F(X) \to G(X)$$

is an isomorphism. Assuming that A is a retract of X, the component

$$\alpha_A : F(A) \to G(A)$$

is also an isomorphism.

Definition 2.6.4. *An R-module M is called **finitely generated** if there is some finite subset A of M that generates M as an R-module* [DF04, §10.3].

Proposition 2.6.5. *Let R be a commutative ring. There is a natural map of R-modules*

$$\eta : M^* \otimes_R N \to \mathrm{Hom}_R(M, N)$$

defined by

$$\eta(f \otimes x) = f(-)x.$$

The natural map is an isomorphism if either M or N is finitely generated projective.

Proof. First, consider the case where N is finite free, so that there is an isomorphism of R-modules $\phi : N \xrightarrow{\cong} R^n$. We obtain a commutative diagram:

$$\begin{array}{ccc}
M^* \otimes_R N & \xrightarrow{\ \eta\ } & \mathrm{Hom}_R(M, N) \\
{\scriptstyle M^*\otimes\phi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{Hom}(M,\phi)} \\
M^* \otimes_R (\bigoplus_{i=1}^{n} R) & \xrightarrow{\ \eta\ } & \mathrm{Hom}_R(M, \bigoplus_{i=1}^{n} R) \\
\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow \\
\bigoplus_{i=1}^{n} M^* \otimes_R R & \xrightarrow[\cong]{\ \bigoplus \eta\ } & \bigoplus_{i=1}^{n} \mathrm{Hom}_R(M, R).
\end{array}$$

Using the fact that a finitely generated projective module is a retract of a finite free module (see Lemma 2.1.5), the natural map η is an isomorphism if N is finitely generated projective, by Lemma 2.6.3. The case where M is finitely generated projective can be proved similarly. □

Remark 2.6.6. *Consider the case $N = M$ where M is a finite free R-module with basis $\{e_1, \ldots, e_r\}$. The tensor*

$$\sum_{i=1}^{r} e_i^* \otimes e_i \in M^* \otimes_R M$$

corresponds to the identity map $\mathrm{id}_M : M \to M$. In particular, this tensor does not depend on the choice of basis.

Example 2.6.7. *We consider the case where the natural map $\eta : M^* \otimes N \to \mathrm{Hom}_R(M, N)$ in Proposition 2.6.5 fails to be an isomorphism. Take $R = \mathbb{Z}$ and $M = N = \mathbb{Z}/n$. Since $M^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = 0$, the left side of the map is equal to 0. The right side of the map is $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n) = \mathbb{Z}/n$. Thus, in this case, the map η is not an isomorphism.*

Lemma 2.6.8. *Let M be an R-module. The natural map*

$$\sigma : M \to M^{**}$$

defined for $x \in M$ by

$$\sigma(x)(f) = f(x)$$

for $f \in \mathrm{Hom}_R(M, R)$, is an isomorphism if M is finitely generated projective.

Example 2.6.9. *Consider the case $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Since $M^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, the map σ in Lemma 2.6.8 fails to be an isomorphism.*

Lemma 2.6.10. *Let R be a commutative ring. There is a natural map of R-modules*

$$\alpha : M^* \otimes N^* \to (M \otimes N)^*$$

defined for $f \in M^*$ and $g \in N^*$ by

$$\alpha(f \otimes g)(m \otimes n) = f(m)g(n)$$

for $m \in M$ and $n \in N$. The natural map is an isomorphism if either M or N is finitely generated projective.

Lemma 2.6.11. *Let C and D be chain complexes of R-modules. There is a natural map of cochain complexes*

$$\alpha : C^* \otimes D^* \to (C \otimes D)^*$$

defined for $f \in C^*$ and $g \in D^*$ by

$$\alpha(f \otimes g)(x \otimes y) = f(x)g(y)$$

for $x \in C$ and $y \in D$. The natural map is an isomorphism if either chain complex C or D is degreewise finitely generated projective.

Proof. We denote the boundary maps in chain complexes by ∂ and coboundary maps in cochain complexes by d. Consider the following diagram:

$$\begin{array}{ccc} (C^* \otimes D^*)^n & \xrightarrow{d} & (C^* \otimes D^*)^{n+1} \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n+1}} \\ (C \otimes D)^n & \xrightarrow{d} & (C \otimes D)^{n+1}. \end{array}$$

We show that the above diagram commutes, i.e., the map α is a cochain map. The tensor product $C^* \otimes D^*$ is the cochain complex with terms

$$(C^* \otimes D^*)^n := \bigoplus_{p+q=n} C^p \otimes D^q.$$

Let $f \in C^p$ and $g \in D^q$. We consider the composition $\alpha_{n+1} \circ d$ and obtain

$$\begin{aligned}\alpha_{n+1} \circ d(f \otimes g) &= \alpha_{n+1}(d(f) \otimes g + (-1)^p f \otimes d(g)) \\ &= \alpha_{n+1}(\partial^*(f) \otimes g + (-1)^p f \otimes \partial^*(g)),\end{aligned}$$

which is a map $(C \otimes D)_{n+1} \to R$ defined by

$$\begin{aligned}(\partial^*(f) \otimes g + (-1)^p f \otimes \partial^*(g))(x \otimes y) &= (\partial^* f)(x)g(y) + (-1)^p f(x)(\partial^* g)(y) \\ &= f(\partial x)g(y) + (-1)^p f(x)g(\partial y) \in R\end{aligned}$$

for $x \otimes y \in (C \otimes D)_{n+1}$. Now we consider the composition $d \circ \alpha_n$ and obtain the map $(d \circ \alpha_n)(f \otimes g) : (C \otimes D)_{n+1} \to R$ defined by

$$\begin{aligned}\left((d \circ \alpha_n)(f \otimes g)\right)(x \otimes y) &= \alpha_n(f \otimes g)(\partial(x \otimes y)) \\ &= \alpha_n(f \otimes g)(\partial(x) \otimes y + (-1)^{|x|} x \otimes \partial(y)) \\ &= f(\partial x)g(y) + (-1)^{|x|} f(x)g(\partial y) \in R.\end{aligned}$$

Given the cochain $f \in C^p$ and the chain $x \in C_i$, if $i = p$, then we obtain $(-1)^{|x|} f(x)g(\partial y) = (-1)^p f(x)g(\partial y)$. In the case $i \neq p$, we obtain $f(x) = 0$ and thus

$$(-1)^p f(x)g(\partial y) = (-1)^{|x|} f(x)g(\partial y) = 0.$$

In both cases we have $\alpha_{n+1} \circ d = d \circ \alpha_n$. $\square$

2.7 Multilinear algebra

Facts about multilinear algebra are from [DF04, §11.5] and [Con18]. In this section, we discuss tensor algebras, symmetric algebras and exterior algebras.

Definition 2.7.1. *Let M be an R-module and $n \geq 0$ an integer.*

1. *The n^{th} **symmetric power** of M is the quotient of the n^{th} tensor power*

$$S^n(M) = \mathrm{Sym}^n(M) := M^{\otimes n}/S$$

where S is the submodule generated by elements

$$m_1 \otimes \ldots \otimes m_n - m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(n)}$$

for any permutation $\sigma \in \Sigma_n$.

The equivalence class of $m_1 \otimes m_2 \otimes \ldots \otimes m_n$ in $\mathrm{Sym}^n(M)$ is denoted as a monomial $m_1 m_2 \ldots m_n$.

2. *The n^{th} **exterior power** of M is the quotient of the n^{th} tensor power*

$$\Lambda^n(M) := M^{\otimes n}/A$$

where A is the submodule generated by elements

$$m_1 \otimes \ldots \otimes m_n$$

with $m_i = m_j$ for some indices $1 \leq i < j \leq n$.

The equivalence class of $m_1 \otimes m_2 \otimes \ldots \otimes m_n$ in $\Lambda^n(M)$ is denoted as a wedge product $m_1 \wedge m_2 \wedge \ldots \wedge m_n$.

Remark 2.7.2. *If $2 \in R$ is invertible, then the submodule $A \subseteq M^{\otimes n}$ being quotiented out in the definition of $\Lambda^n(M)$ is the submodule generated by elements*

$$m_1 \otimes \ldots \otimes m_n - \operatorname{sgn}(\sigma) m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(n)}$$

for any permutation $\sigma \in \Sigma_n$.

Lemma 2.7.3. *Let R be a commutative ring and $\{e_1, \ldots, e_r\}$ the standard basis of R^r.*

1. *If S is the submodule being quotiented out from $T^2(R^r)$ to form the symmetric square $S^2(R^r)$, then S is a free R-module with basis*

$$\{e_i \otimes e_j - e_j \otimes e_i \mid 1 \leq i < j \leq r\}.$$

2. *If A is the submodule being quotiented out from $T^2(R^r)$ to form the exterior square $\Lambda^2(R^r)$, then A is a free R-module with basis*

$$\{e_i \otimes e_j + e_j \otimes e_i \mid 1 \leq i < j \leq r\} \cup \{e_i \otimes e_i \mid 1 \leq i \leq r\}.$$

Proposition 2.7.4. *Let R be a commutative ring and $\{e_1, \ldots, e_r\}$ the standard basis of R^r.*

1. *The tensor power $T^n(R^r)$ has canonical basis (as R-module)*

$$\{e_{i_1} \otimes \ldots \otimes e_{i_n} \mid 1 \leq i_j \leq r \,\text{for}\, 1 \leq j \leq n\}.$$

2. The monomials of length n

$$\{e_1^{n_1}e_2^{n_2}\dots e_r^{n_r} \mid n_1+\dots+n_r=n\} = \{e_{i_1}\dots e_{i_n} \mid 1\le i_1\le \dots \le i_n \le r\}$$

form a canonical basis (as R-module) of the symmetric power $S^n(R^r)$.

3. The exterior power $\Lambda^n(R^r)$ has canonical basis (as R-module)

$$\{e_{i_1}\wedge\dots\wedge e_{i_n} \mid 1\le i_1<i_2<\dots<i_n\le r\}.$$

The indices that appear form a subset $\{i_1,i_2,\dots,i_n\}\subseteq\{1,2,\dots,r\}$; there exist $\binom{r}{n}$ such subsets.

Definition 2.7.5. *Let M be an R-module.*

*1. The **tensor algebra** on M is*

$$T(M) := \bigoplus_{n\ge 0} T^n(M)$$

where $T^n(M) := M^{\otimes n}$ denotes the n^{th} tensor power of M. The multiplication map

$$T(M)\otimes T(M)\to T(M)$$

is given by the pairing

$$M^{\otimes p}\otimes M^{\otimes q}\to M^{\otimes p+q}$$

$$(x_1\otimes\dots\otimes x_p)\otimes(y_1\otimes\dots\otimes y_q)\mapsto x_1\otimes\dots\otimes x_p\otimes y_1\otimes\dots\otimes y_q.$$

2. *The* ***symmetric algebra*** *on* M *is*

$$S(M) := \bigoplus_{n\geq 0} S^n(M)$$

with multiplication

$$S(M) \otimes S(M) \to S(M)$$

given by the pairing

$$S^p(M) \otimes S^q(M) \to S^{p+q}(M)$$

given on monomials by the formula

$$(x_1 \dots x_p) \otimes (y_1 \dots y_q) \mapsto x_1 \dots x_p y_1 \dots y_q.$$

3. *The* ***exterior algebra*** *on* M *is*

$$\Lambda(M) := \bigoplus_{n\geq 0} \Lambda^n(M)$$

with multiplication

$$\Lambda(M) \otimes \Lambda(M) \to \Lambda(M)$$

given by the pairing

$$\Lambda^p(M) \otimes \Lambda^q(M) \to \Lambda^{p+q}(M)$$

given on elementary products by the formula

$$(x_1 \wedge \dots \wedge x_p) \otimes (y_1 \wedge \dots \wedge y_q) \mapsto x_1 \wedge \dots \wedge x_p \wedge y_1 \wedge \dots \wedge y_q.$$

Lemma 2.7.6. *Let M be an R-module and let $n \geq 0$. The following formulas define natural pairings:*

1.

$$T^n(M^*) \otimes T^n(M) \to R$$

$$(f_1 \otimes \ldots \otimes f_n) \otimes (m_1 \otimes \ldots \otimes m_n) \mapsto \prod_{i=1}^{n} f_i(m_i)$$

2.

$$S^n(M^*) \otimes S^n(M) \to R$$

$$(f_1 f_2 \ldots f_n) \otimes (m_1 m_2 \ldots m_n) \mapsto \sum_{\sigma \in \Sigma_n} \prod_{i=1}^{n} f_i(m_{\sigma(i)})$$

3.

$$\Lambda^n(M^*) \otimes \Lambda^n(M) \to R$$

$$(f_1 \wedge \ldots \wedge f_n) \otimes (m_1 \wedge \ldots \wedge m_n) \mapsto \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} f_i(m_{\sigma(i)}) = \det[f_i(m_j)].$$

Proposition 2.7.7. *Let M be a finite free R-module. Let $\{e_1, \ldots, e_r\}$ be a basis of M and $\{e_1^*, \ldots, e_r^*\}$ the dual basis of M^*.*

1. The pairing

$$T^n(M^*) \otimes T^n(M) \to R$$

is perfect, i.e., the corresponding map

$$T^n(M^*) \to (T^n(M))^*$$

is an isomorphism. Moreover, via the pairing, the collection in $T^n(M^*)$

$$\{e_{i_1}^* \otimes \ldots \otimes e_{i_n}^* \mid 1 \leq i_j \leq r \,\text{for}\, 1 \leq j \leq n\}$$

is dual to the basis of $T^n(M)$

$$\{e_{i_1} \otimes \ldots \otimes e_{i_n} \mid 1 \leq i_j \leq r \,\text{for}\, 1 \leq j \leq n\}.$$

2. *The pairing*

$$\Lambda^n(M^*) \otimes \Lambda^n(M) \to R$$

is perfect, i.e., the associated map

$$\Lambda^n(M^*) \to (\Lambda^n(M))^*$$

is an isomorphism. Moreover, via the pairing, the collection in $\Lambda^n(M^*)$

$$\{e_{i_1}^* \wedge \ldots \wedge e_{i_n}^* \mid 1 \leq i_1 < i_2 < \ldots < i_n \leq r\}$$

is dual to the basis of $\Lambda^n(M)$

$$\{e_{i_1} \wedge \ldots \wedge e_{i_n} \mid 1 \leq i_1 < i_2 < \ldots < i_n \leq r\}.$$

2.8 Extension of scalars

For a (not necessarily commutative) ring R, let R-Mod denote the category of left R-modules.

Definition 2.8.1. *Let $f : R \to S$ be a ring homomorphism.*

1. ***Restriction of scalars*** *along f is the functor*

$$f^* : S\text{-Mod} \to R\text{-Mod}$$

that sends an S-module M to the R-module f^*M which has the same underlying abelian group M, with R acting via f:

$$r \cdot m := f(r) \cdot m.$$

We sometimes write M instead of f^*M if the base ring is clear from the context.

2. ***Extension of scalars*** *along f is the functor*

$$f_! : R\text{-Mod} \to S\text{-Mod}$$

that sends an R-module M to the S-module $f_!M = S \otimes_R M$, with S acting via multiplication on the left:

$$s' \cdot (s \otimes m) := (s's) \otimes m.$$

The extension of scalars is also called change of rings [HS97, §IV.12].

Example 2.8.2. *Let R be a ring and $u : \mathbb{Z} \to R$ the unique ring homomorphism, given by $u(1) = 1_R$, the unit of R. Restriction of scalars along u*

$$u^* : R\text{-Mod} \to \mathbb{Z}\text{-Mod} = \text{Ab}$$

is the "underlying abelian group" functor, forgetting the scalar multiplication by R.

Proposition 2.8.3. *Let $f : R \to S$ be a ring homomorphism. There is a natural isomorphism of abelian groups*

$$\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M, N),$$

for all R-module M and S-module N. In other words, extension of scalars is left adjoint to restriction of scalars: $f_! \dashv f^*$ [DF04, §10.4].

Proposition 2.8.4. *Let $f : R \to S$ be a ring homomorphism between commutative rings, and let M be an R-module.*

1. *If A is an R-algebra, then $S \otimes_R A$ is canonically an S-algebra.*

2. *For any $n \geq 0$, there is a natural isomorphism of S-modules*

$$S \otimes_R T_R^n(M) \cong T_S^n(S \otimes_R M).$$

Here $T_R^n(M)$ denotes the n^{th} tensor power of M as an R-module:

$$T_R^n(M) = \overbrace{M \otimes_R \ldots \otimes_R M}^{n\,\text{times}}.$$

As $n \geq 0$ varies, the maps form an isomorphism of graded S-algebras

$$S \otimes_R T_R(M) \xrightarrow{\cong} T_S(S \otimes_R M)$$

[Bou70, §III.5.3].

3. *For any $n \geq 0$, there is a natural isomorphism of S-modules*

$$S \otimes_R \Lambda_R^n(M) \cong \Lambda_S^n(S \otimes_R M).$$

Here $\Lambda^n_R(M)$ denotes the n^{th} exterior power of M as an R-module:

$$\Lambda^n_R(M) = \overbrace{M \wedge_R \ldots \wedge_R M}^{n\,\text{times}}.$$

As $n \geq 0$ varies, the maps form an isomorphism of graded S-algebras

$$S \otimes_R \Lambda_R(M) \xrightarrow{\cong} \Lambda_S(S \otimes_R M)$$

[Bou70, §III.7.5].

Chapter 3

Additive structure of Ext-algebras

In this chapter, we discuss the computation of Ext groups over symmetric algebras and exterior algebras.

3.1 Ext over a polynomial algebra

Definition 3.1.1. *The* ***polynomial ring*** $k[x]$ *over a commutative ring* k *on one generator consists of the polynomials* $p = p_0 + p_1x + p_2x + \ldots + p_{m-1}x^{m-1} + p_mx^m$ *where* $p_0, p_1, \ldots, p_m \in k$ *are the coefficients of* p, $p_m \neq 0$ *if* $m > 0$ [DF04, §7.2].

The usual addition, multiplication, and scalar multiplication of polynomials make $k[x]$ into a commutative k-algebra.

We compute examples of Ext groups over polynomial algebras.

Example 3.1.2. *Let* $A = k[x]$ *be a polynomial algebra on one generator and consider* *the A-module $k \cong A/(x)$. A free resolution $P_\bullet$ of k is given by*

$$0 \to \underset{A}{k[x]} \xrightarrow{x} \underset{A}{k[x]} \to k \to 0.$$

Thus the cochain complex $\mathrm{Hom}_A(P_\bullet, k)$ *is*

$$0 \to \mathrm{Hom}_A(A, k) \xrightarrow{d} \mathrm{Hom}_A(A, k) \to 0.$$

Composition of A-module maps $A \xrightarrow{x} A \to k$ is zero. Hence the differential of the cochain complex is $d = 0$. The hom module is $\mathrm{Hom}_A(A, k) \cong k$. *The cochain complex has zero differential, that is:*

$$0 \to k \xrightarrow{0} k \to 0.$$

By definition of Ext, *the cohomology groups of the complex are the* Ext *groups:*

$$\mathrm{Ext}^*_A(k, k) = \begin{cases} k, & * = 0, 1 \\ 0, & \textit{otherwise.} \end{cases}$$

Since every class in $\mathrm{Ext}^1_A(k, k)$ *squares to zero,* $\mathrm{Ext}^*_A(k, k)$ *is an exterior algebra over k on a generator in cohomological degree 1.*

We express $\mathrm{Ext}^*_A(k, k)$ as a graded k-algebra:

$$\begin{aligned} \mathrm{Ext}^*_A(k, k) &= k[y]/(y^2), \quad |y| = 1 \\ &= \Lambda_k(y), \quad |y| = 1. \end{aligned}$$

In Example 4.2.8, we will explain that this generator in $\mathrm{Ext}^1_A(k, k)$ can be viewed as x^*, the dual of x.

Remark 3.1.3. *With more generators, an exterior algebra is not the same as a truncated polynomial algebra.*

Example 3.1.4. *Let $A = k[x, y]$ be the polynomial algebra on two generators. Consider the A-module $k \cong A/(x, y)$. A free resolution $P_\bullet$ of k is given by*

$$\ldots \to 0 \to A \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} A^2 \xrightarrow{[x\ y]} A \to k \to 0.$$

The cochain complex $\mathrm{Hom}_A(P_\bullet, k)$ *is*

$$\ldots \to 0 \to \mathrm{Hom}_A(A, k) \xrightarrow{d} \mathrm{Hom}_A(A^2, k) \xrightarrow{d} \mathrm{Hom}_A(A, k) \to 0.$$

Note that $d = 0$, $\mathrm{Hom}_A(A, k) \cong k$ *and* $\mathrm{Hom}_A(A^2, k) \cong k^2$*. Thus the cochain complex has the form*

$$0 \to k \xrightarrow{0} k^2 \xrightarrow{0} k \to 0.$$

By definition of Ext*, the cohomology groups of the complex are the* Ext *groups:*

$$\mathrm{Ext}_A^*(k, k) = \begin{cases} k, & * = 0, 2 \\ k^2, & * = 1 \\ 0, & \textit{otherwise.} \end{cases}$$

In Example 4.1.5 we will show that $\mathrm{Ext}_A^*(k, k)$ is an exterior algebra over k on two generators in cohomological degree 1, i.e., $\mathrm{Ext}_A(k, k) = \Lambda_k(x^*, y^*)$.

3.2 Ext over an exterior algebra

In this section, we compute examples of Ext groups over an exterior algebra.

Example 3.2.1. *Let $A = \Lambda_k(x)$ be the exterior algebra over k on one generator. Consider the A-module $k \cong A/(x)$. A free resolution $P_\bullet$ of k is given by*

$$\ldots \to A \xrightarrow{x} A \xrightarrow{x} A \to k \to 0.$$

The cochain complex $\mathrm{Hom}_A(P_\bullet, k)$ *is*

$$0 \to \mathrm{Hom}_A(A,k) \xrightarrow{d} \mathrm{Hom}_A(A,k) \xrightarrow{d} \mathrm{Hom}_A(A,k) \xrightarrow{d} \mathrm{Hom}_A(A,k) \to \ldots$$

Note that $d = 0$ and $\mathrm{Hom}_A(A,k) \cong k$. The cochain complex becomes

$$0 \to k \xrightarrow{0} k \xrightarrow{0} k \to \ldots$$

By definition of Ext*, the cohomology groups of the complex are the* Ext *groups:*

$$\mathrm{Ext}^*_A(k,k) = \begin{cases} k, & * \geq 0 \\ 0, & \textit{otherwise.} \end{cases}$$

In Example 4.1.4, we will show that $\mathrm{Ext}^*_A(k,k)$ is a polynomial algebra over k on a generator in cohomological degree 1, i.e., $\mathrm{Ext}_A(k,k) = k[x^*]$.

Example 3.2.2. *Let k be a commutative ring and $A = \Lambda_k(x,y)$ the exterior algebra over k on two generators. The exterior algebra A has a standard basis as a k-module:*

$$A =< 1, x, y, x \wedge y > .$$

Since A is a non-commutative k-algebra, the multiplication on the right by $x \in A$

$$\rho_x : A \xrightarrow{\cdot x} A$$

is a map of left A-modules. Consider the (left) A-module $k \cong A/(x,y)$ and we construct a free resolution of k starting with the quotient map $q : A \to k$. The kernel of the quotient map $q : A \to k$ is the A-submodule generated by x and y, i.e., $\ker(A \to k) =< x, y >$. *Now we find a free A-module which surjects onto the kernel of the morphism $A \to k$. We pick the map $A^2 \xrightarrow{[x\ y]} A$ as next step of the resolution. Now we compute* $\ker(A^2 \xrightarrow{[x\ y]} A)$. *For any $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \ker(A^2 \xrightarrow{[x\ y]} A)$, we obtain*

$$\left[\begin{smallmatrix} x & y \end{smallmatrix}\right]\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \rho_x(a) + \rho_y(b) = a \wedge x + b \wedge y = 0. \tag{3.1}$$

In Equation (3.1), *we get solutions for $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right]$. We express elements a and b in Equation* (3.1) *in terms of the basis of A as a k-module:*

$$(a_{00}1 + a_{10}x + a_{01}y + a_{11}x \wedge y) \wedge x + (b_{00}1 + b_{10}x + b_{01}y + b_{11}x \wedge y) \wedge y = 0$$

$$\Longleftrightarrow a_{00}x - a_{01}x \wedge y + b_{00}y + b_{10}x \wedge y = 0$$

$$\Longleftrightarrow (0)1 + (a_{00})x + (b_{00})y + (b_{10} - a_{01})x \wedge y = 0.$$

We have $a_{00} = b_{00} = 0$, $b_{10} = a_{01}$ and then

$$\begin{aligned} \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] &= \left[\begin{smallmatrix} a_{10}x + a_{01}y + a_{11}x\wedge y \\ a_{01}x + b_{01}y + b_{11}x \wedge y \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} a_{10}x + a_{11}(x\wedge y) \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ b_{01}y + b_{11}(x\wedge y) \end{smallmatrix}\right] + \left[\begin{smallmatrix} a_{01}y \\ a_{01}x \end{smallmatrix}\right] \\ &= (a_{10} - a_{11}y)\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] + (b_{01} + b_{11}x)\left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] + (a_{01})\left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right]. \end{aligned}$$

This implies that $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ lies in the A-submodule $< \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right] >$. Then, $\ker(A^2 \xrightarrow{[x\ y]} A) \subseteq < \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right] >$. *Thus,* $\ker(A^2 \xrightarrow{[x\ y]} A) =< \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right] >$. *We pick the map $A^3 \xrightarrow{\left[\begin{smallmatrix} x & 0 & y \\ 0 & y & x \end{smallmatrix}\right]} A^2$ as next step of the resolution. We compute* $\ker(A^3 \xrightarrow{\left[\begin{smallmatrix} x & 0 & y \\ 0 & y & x \end{smallmatrix}\right]} A^2)$. *For* *any* $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \ker(A^3 \xrightarrow{\begin{bmatrix} x & 0 & y \\ 0 & y & x \end{bmatrix}} A^2)$, *we obtain*

$$\begin{bmatrix} x & 0 & y \\ 0 & y & x \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \rho_x(a)+\rho_y(c) \\ \rho_y(b)+\rho_x(c) \end{bmatrix} = \begin{bmatrix} a\wedge x + c\wedge y \\ b\wedge y + c \wedge x \end{bmatrix} = 0. \tag{3.2}$$

We express elements a, b *and* c *in Equation* (3.2) *in terms of the basis of* A *as a* k*-module:*

$$(a_{00}1 + a_{10}x + a_{01}y + a_{11}x \wedge y) \wedge x + (c_{00}1 + c_{10}x + c_{01}y + c_{11}x \wedge y) \wedge y = 0$$

$$\Longleftrightarrow a_{00}x + c_{00}y + (c_{10} - a_{01})x \wedge y = 0$$

and

$$(b_{00}1 + b_{10}x + b_{01}y + b_{11}x \wedge y) \wedge y + (c_{00}1 + c_{10}x + c_{01}y + c_{11}x \wedge y) \wedge x = 0$$

$$\Longleftrightarrow c_{00}x + b_{00}y + (b_{10} - c_{01})x \wedge y = 0.$$

We have $a_{00} = b_{00} = c_{00} = 0$, $c_{10} = a_{01}$ *and* $b_{10} = c_{01}$ *and then*

$$\begin{aligned} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} a_{10}x+a_{01}y+a_{11}x\wedge y \\ b_{10}x+b_{01}y+b_{11}x\wedge y \\ a_{01}x+b_{10}y+c_{11}x\wedge y \end{bmatrix} \\ &= \begin{bmatrix} a_{10}x+a_{11}(x\wedge y) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_{01}y+b_{11}(x\wedge y) \\ 0 \end{bmatrix} + \begin{bmatrix} a_{01}y \\ 0 \\ a_{01}x \end{bmatrix} + \begin{bmatrix} 0 \\ b_{10}x \\ b_{10}y+c_{11}x\wedge y \end{bmatrix} \\ &= (a_{10} - a_{11}y)\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + (b_{01} + b_{11}x)\begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + (a_{01})\begin{bmatrix} y \\ 0 \\ x \end{bmatrix} + (b_{10} + c_{11}x)\begin{bmatrix} 0 \\ x \\ y \end{bmatrix}. \end{aligned}$$

Thus, $\ker(A^3 \xrightarrow{\begin{bmatrix} x & 0 & y \\ 0 & y & x \end{bmatrix}} A^2) =< \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}, \begin{bmatrix} y \\ 0 \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ x \\ y \end{bmatrix} >$. *We have the beginning of a free resolution* $P_\bullet$ *of* k*:*

$$\ldots \to A^4 \xrightarrow{\begin{bmatrix} x & 0 & y & 0 \\ 0 & y & 0 & x \\ 0 & 0 & x & y \end{bmatrix}} A^3 \xrightarrow{\begin{bmatrix} x & 0 & y \\ 0 & y & x \end{bmatrix}} A^2 \xrightarrow{[x \ y]} A \to k.$$

The cochain complex $\mathrm{Hom}_A(P_\bullet, k)$ *is*

$$0 \to \mathrm{Hom}_A(A, k) \xrightarrow{d} \mathrm{Hom}_A(A^2, k) \xrightarrow{d} \mathrm{Hom}_A(A^3, k) \xrightarrow{d} \mathrm{Hom}_A(A^4, k) \to \ldots$$

Then the above cochain complex becomes:

$$0 \to k \xrightarrow{0} k^2 \xrightarrow{0} k^3 \to \dots$$

We obtain the Ext *groups:*

$$\mathrm{Ext}^i_A(k,k) = k^{i+1}, i \leq 2.$$

In fact, this holds for all $i \geq 0$. *We will see in Proposition 5.4.3 that* $\mathrm{Ext}^*_A(k,k)$ *is a polynomial algebra over* k *on two generators in cohomological degree 1, i.e.,* $\mathrm{Ext}_A(k,k) = k[x^*, y^*]$.

3.3 The Koszul complex

The facts about the Koszul complex are from [Wei94, §4.5]. In this section we discuss the Koszul complex and how it is used to find Ext groups over algebras.

Definition 3.3.1. *Let* R *be a commutative ring. The* ***Koszul complex*** *associated to elements* $x_1, \dots, x_n \in R$ *is the exterior algebra on a free* R*-module of rank* n

$$K_*(\mathbf{x}) = K_*(x_1, \dots, x_n) := \Lambda^*(R^n)$$

equipped with the differential $d : K_p(\mathbf{x}) \to K_{p-1}(\mathbf{x})$ *given on the standard basis elements by*

$$d(e_{i_1} \wedge ... \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} e_{i_1} \wedge ... \wedge \widehat{e_{i_j}} \wedge ... \wedge e_{i_p},$$

where the 'hat' symbol $\widehat{\ }$ *indicates that the term is deleted.*

Example 3.3.2. *When* $p = 2, d(e_{i_1} \wedge e_{i_2}) = x_{i_1}e_{i_2} - x_{i_2}e_{i_1}$.

Lemma 3.3.3. *The Koszul complex* $K(\mathbf{x})$ *is indeed a chain complex, i.e., the differential* d *satisfies* $d^2 = 0$*, which moreover satisfies the Leibniz rule:*

$$d(a \wedge b) = d(a) \wedge b + (-1)^{|a|} a \wedge d(b)$$

where $a, b \in K(\mathbf{x})$*. In other words, the Koszul complex* $K(\mathbf{x})$ *is a DG-algebra.*

Example 3.3.4. *Let* R *be a commutative ring. We describe the Koszul complex* $K(x_1, \ldots, x_n)$ *explicitly for* $n = 1, 2, 3$.

- *For* $n = 1$*, let* $x \in R$*. The Koszul complex of the sequence* (x) *is*

$$K(x) = \ldots \to 0 \to \underset{e_1}{R} \xrightarrow{x} \underset{1}{R} \to 0$$

  where 1 and e_1 *are the basis elements of* $K_0(x)$ *and* $K_1(x)$.

- *For* $n = 2$*, let* $x, y \in R$*. The Koszul complex of the sequence* (x, y) *is*

$$K(x,y) = \ldots \to 0 \to \underset{e_1 \wedge e_2}{R} \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} \underset{e_1, e_2}{R^2} \xrightarrow{[x\ y]} \underset{1}{R} \to 0.$$

- *For* $n = 3$*, let* $x, y, z \in R$*. The Koszul complex of the sequence* (x, y, z) *is*

$$\ldots \to 0 \to \underset{e_1 \wedge e_2 \wedge e_3}{R} \xrightarrow{\left[\begin{smallmatrix} z \\ -y \\ x \end{smallmatrix}\right]} \underset{\substack{e_1 \wedge e_2 \\ e_1 \wedge e_3 \\ e_2 \wedge e_3}}{R^3} \xrightarrow{\left[\begin{smallmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{smallmatrix}\right]} \underset{\substack{e_1 \\ e_2 \\ e_3}}{R^3} \xrightarrow{[x\ y\ z]} \underset{1}{R} \to 0.$$

Lemma 3.3.5. *Let* R *be a commutative ring and* $x, y \in R$*. The Koszul complex* $K(x, y)$ *is isomorphic to the tensor product of complexes* $K(x) \otimes_R K(y)$*. More generally, the Koszul complex* $K(x_1, \ldots, x_d)$ *is isomorphic to the tensor product of complexes* $K(x_1) \otimes_R K(x_2) \otimes_R \ldots \otimes_R K(x_d)$ [Wei94, §4.5].

Proof. We will prove the case $d = 2$. The Koszul complex $K(x)$ is

$$K(x) = \ldots \to 0 \to \underset{<e_1>}{R} \xrightarrow{x} \underset{<1_x>}{R} \to 0.$$

The Koszul complex $K(y)$ is

$$K(y) = \ldots \to 0 \to \underset{<e_2>}{R} \xrightarrow{y} \underset{<1_y>}{R} \to 0.$$

We use $(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$ and the formula for the boundary in Definition 2.2.2 to find the chain complex $K(x) \otimes K(y)$:

$$\ldots \to 0 \to \underset{<e_1 \otimes e_2>}{R} \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} \underset{<e_1 \otimes 1_y, 1_x \otimes e_2>}{R^2} \xrightarrow{[x\ y]} \underset{<1_x \otimes 1_y>}{R} \to 0.$$

Therefore $K(x, y) \cong K(x) \otimes_R K(y)$. $\square$

Definition 3.3.6. *For a commutative ring R and an R-module M, $x \in R$ is called a* ***non-zero-divisor*** *on M if $xm = 0$ implies $m = 0$ for all $m \in M$.*

A sequence of elements $x_1, \ldots, x_d \in R$ in a commutative ring R is called ***regular*** *if x_1 is a non-zero-divisor in R and x_i is a non-zero-divisor in $R/(x_1, \ldots, x_{i-1})$ for all $2 \leq i \leq d$.*

Example 3.3.7. *Let $A = k[x_1, \ldots, x_n]$ be a polynomial algebra over k on n generators. The element x_1 is a non-zero-divisor in A by inspection. The quotient rings are also polynomial:*

$$A/(x_1, \ldots, x_{i-1}) = k[x_1, \ldots, x_n]/(x_1, \ldots, x_{i-1}) \cong k[x_i, \ldots, x_n].$$

By applying the previous argument, x_i is a non-zero-divisor on $A/(x_1, \dots, x_{i-1})$. Therefore $x_1, \dots, x_n$ is a regular sequence.

Proposition 3.3.8. *Let $I = (x_1, \dots, x_d)$ be an ideal of R generated by a regular sequence. Then the augmented complex $K(x_1, \dots, x_d) \to R/I$ is acyclic, that is, it has trivial homology.*

Equivalently, the homology of the unaugmented complex is

$$H_i K(x_1, \dots, x_d) = \begin{cases} R/I, & i = 0 \\ 0, & i \neq 0. \end{cases}$$

Consequently, the Koszul complex $K(x_1, \dots, x_d)$ is a free resolution of R/I as an R-module [Wei94, Corollary 4.5.4].

Example 3.3.9. *First, consider the case $d = 1$,*

$$K(x_1) = \ldots \to 0 \to R \xrightarrow{x_1} R \to 0.$$

We have $H_0 K(x_1) = R/I$.

Since x_1 is a non-zero-divisor in R, $H_1 K(x_1) = 0$ holds.

Thus, $H_i K(x_1) = 0$ for all $i > 0$.

Example 3.3.10. *Consider the case $d = 2$,*

$$K(x_1, x_2) = \ldots \to 0 \to R \xrightarrow{\left[\begin{smallmatrix} -x_2 \\ x_1 \end{smallmatrix}\right]} R^2 \xrightarrow{\left[\begin{smallmatrix} x_1 & x_2 \end{smallmatrix}\right]} R \to 0.$$

We have $H_0 K(x_1, x_2) = R/I$.

Since the sequence (x_1, x_2) is regular, $\ker\left[\begin{smallmatrix} x_1 & x_2 \end{smallmatrix}\right] =< \left[\begin{smallmatrix} -x_2 \\ x_1 \end{smallmatrix}\right] >$ and $\ker\left[\begin{smallmatrix} -x_2 \\ x_1 \end{smallmatrix}\right] = 0$.

We justify $\ker[\, x_1 \; x_2 \,] = < [\, {}^{-x_2}_{x_1} \,] >$ *as follows: Since* $[\, x_1 \; x_2 \,][\, {}^{-x_2}_{x_1} \,] = 0$, *then* $< [\, {}^{-x_2}_{x_1} \,] > \subseteq$ $\ker[\, x_1 \; x_2 \,]$. *Let* $(P, Q) \in \ker[\, x_1 \; x_2 \,]$, *that is:*

$$[\, x_1 \; x_2 \,]\left[\, {}^{P}_{Q} \,\right] = x_1 P + x_2 Q = 0. \tag{3.3}$$

Since x_2 *is not a zero divisor in* $R/(x_1)$, *the equation* $\overline{x_1} \cdot \overline{P} + \overline{x_2} \cdot \overline{Q} = 0$ *implies* $\overline{Q} = 0$ *in* $R/(x_1)$. *Since* $\overline{\mathrm{Q}} = 0$ *in* $R/(x_1)$, $Q = rx_1$ *for some* $r \in R$. *Using Equation* (4.3.2) *and the fact that* x_1 *is a non-zero-divisor, we get:*

$$x_1 P + x_2 Q = 0$$

$$\Longleftrightarrow x_1 P + x_2 r x_1 = 0$$

$$\Longleftrightarrow x_1 (P + r x_2) = 0$$

$$\Longleftrightarrow P = -r x_2.$$

Thus $(P, Q) = (-r x_2, r x_1)$, *where* $r \in R$. *Hence* $(P, Q) \in \langle [\, {}^{-x_2}_{x_1} \,] \rangle$, *proving the inclusion* $\ker[\, x_1 \; x_2 \,] \subseteq < [\, {}^{-x_2}_{x_1} \,] >$.

We get $\ker[\, {}^{-x_2}_{x_1} \,] = 0$ *as follows. For* $P \in R$, *assume that* $[\, {}^{-x_2}_{x_1} \,] P = 0$ *in* R^2, *equivalently,* $-x_2 P = 0$ *and* $x_1 P = 0$ *in* R. *Since* x_1 *is not a zero divisor in* R, *we have* $P = 0$. *Thus* $\ker[\, {}^{-x_2}_{x_1} \,] = 0$. *Since* $\ker[\, {}^{-x_2}_{x_1} \,] = 0$, $H_2 K(x_1, x_2) = 0$.

Hence

$$H_i K(x_1, x_2) = 0 \;\; \text{for all} \;\; i > 0.$$

We use the Koszul complex to compute Ext groups over a polynomial algebra on n generators.

Lemma 3.3.11. *Let $A = k[x_1, \ldots, x_n]$ be a polynomial algebra over k on n generators. The* Ext *groups are* $\mathrm{Ext}^i_A(k, k) \cong k^{\binom{n}{i}}$ *for all* $i \geq 0$.

Proof. Since $(x_1, \ldots, x_n)$ is a regular sequence in $A = k[x_1, \ldots, x_n]$, the Koszul complex

$$K(x_1, \ldots, x_n) \to A/(x_1, \ldots, x_n) = k$$

is a free resolution $P_\bullet$ of k as an A-module by Proposition 3.3.8. By Proposition 2.7.4, The Koszul complex is isomorphic to a chain complex of the form

$$\ldots \xrightarrow{d} A^{\binom{n}{i}} \xrightarrow{d} \ldots \xrightarrow{d} A^{\binom{n}{1}} \xrightarrow{d} A^{\binom{n}{0}} \to 0.$$

The cochain complex $\mathrm{Hom}_A(P_\bullet, k)$ is isomorphic to

$$0 \to k^{\binom{n}{0}} \xrightarrow{d^*} k^{\binom{n}{1}} \xrightarrow{d^*} \ldots \xrightarrow{d^*} k^{\binom{n}{i}} \xrightarrow{d^*} \ldots$$

with differential d^* the restriction along d, that is, $d^*(f) = f \circ d$. Consider the composition of A-module maps:

$$A^{\binom{n}{i+1}} \xrightarrow{d} A^{\binom{n}{i}} \xrightarrow{f} k.$$

We have

$$\begin{aligned} f(d(e_{k_1} \wedge \ldots \wedge e_{k_{i+1}})) &= f(\sum_{j=1}^{i+1} (-1)^{j-1} x_{k_j} e_{k_1} \wedge \ldots \wedge \widehat{e_{k_j}} \wedge \ldots \wedge e_{k_{i+1}}) \\ &= \sum_{j=1}^{i+1} (-1)^{j-1} f(x_{k_j} e_{k_1} \wedge \ldots \wedge \widehat{e_{k_j}} \wedge \ldots \wedge e_{k_{i+1}}) \\ &= \sum_{j=1}^{i+1} (-1)^{j-1} x_{k_j} f(e_{k_1} \wedge \ldots \wedge \widehat{e_{k_j}} \wedge \ldots \wedge e_{k_{i+1}}) \\ &= 0. \end{aligned}$$

Since $d^* f = f \circ d = 0$, the differential d^* in the cochain complex is zero. We obtain the Ext groups

$$\operatorname{Ext}^i_A(k,k) \cong k^{\binom{n}{i}}$$

for all $i \geqslant 0$. $\square$

In Proposition 4.3.7 we will show that $\operatorname{Ext}^*_A(k,k)$ is an exterior algebra over k on n generators in cohomological degree 1, i.e., $\operatorname{Ext}_A(k,k) = \Lambda_k(x_1^*, \ldots, x_n^*)$.

Chapter 4

Multiplicative structure of Ext-algebras

4.1 Multiplicative structure using endomorphism complexes

In this section we discuss how to calculate the multiplicative structure of Ext-algebras directly using an endomorphism complex. First, we need to fix some grading conventions for chain complexes.

Remark 4.1.1. *Let C be a chain complex and A an R-module. Applying the covariant functor* $\mathrm{Hom}_R(A, -)$ *to C yields a chain complex*

$$\mathrm{Hom}_R(A, C)$$

which is the same as the hom complex $\underline{\mathrm{Hom}}(A[0], C)$.

Applying the contravariant functor $\mathrm{Hom}_R(-, A)$ *to C yields a cochain complex*

$$\mathrm{Hom}_R(C, A).$$

The cochain complex $\mathrm{Hom}_R(C, A)$ *is:*

$$\begin{array}{lcccc} \text{degree} & -2 & -1 & 0 & 1 \\ \longrightarrow & \mathrm{Hom}_R(C_{-2}, A) \xrightarrow{\partial^*} & \mathrm{Hom}_R(C_{-1}, A) \xrightarrow{\partial^*} & \mathrm{Hom}_R(C_0, A) \xrightarrow{\partial^*} & \mathrm{Hom}_R(C_1, A) \longrightarrow \dots \end{array}$$

and the hom complex $\underline{\mathrm{Hom}}(C, A[0])$ *is:*

$$\begin{array}{lcccc} \text{degree} & 2 & 1 & 0 & -1 \\ \longrightarrow & \mathrm{Hom}_R(C_{-2}, A) \xrightarrow{-\partial^*} & \mathrm{Hom}_R(C_{-1}, A) \xrightarrow{\partial^*} & \mathrm{Hom}_R(C_0, A) \xrightarrow{-\partial^*} & \mathrm{Hom}_R(C_1, A) \longrightarrow \dots \end{array}$$

Up to signs in the differentials, the cochain complex $\mathrm{Hom}_R(C, A)$ *corresponds to the hom complex* $\underline{\mathrm{Hom}}(C, A[0])$ *via the following convention: a cochain complex* K^* *is viewed as a chain complex with reversed grading* $K^* = K_{-*}$. *In particular, a non-negatively graded cochain complex*

$$K^* = (0 \to K^0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \xrightarrow{d} \dots)$$

is identified with a non-positively graded chain complex

$$K_* = (0 \to K_0 \xrightarrow{d} K_{-1} \xrightarrow{d} K_{-2} \xrightarrow{d} \dots).$$

If the chain complex $C = (\dots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \to 0)$ *is non-negatively graded, then the hom complex* $\underline{\mathrm{Hom}}(C, A[0])$ *is non-positively graded:*

$$\begin{array}{lcccc} \text{degree} & 1 & 0 & -1 & -2 \\ & 0 \longrightarrow & \mathrm{Hom}_R(C_0, A) \longrightarrow & \mathrm{Hom}_R(C_1, A) \longrightarrow & \mathrm{Hom}_R(C_2, A) \longrightarrow \dots \end{array}$$

Proposition 4.1.2. *Let* $P_\bullet \to M$ *be a projective resolution of an* R*-module* M*.*

1. *The endomorphism complex* $\underline{\mathrm{Hom}}(P, P)$ *has homology groups*

$$H_n\underline{\mathrm{Hom}}(P, P) \cong \mathrm{Ext}_R^{-n}(M, M).$$

2. *The isomorphism of graded* R*-modules*

$$H_*\underline{\mathrm{Hom}}(P, P) \cong \mathrm{Ext}_R^{-*}(M, M)$$

is an isomorphism of graded R*-algebras, where the right-hand side is endowed with the Yoneda product.*

Proof. (1) Let $M[0]$ denote the chain complex with M concentrated in degree 0, and view the projective resolution as a quasi-isomorphism $f : P \xrightarrow{\sim} M[0]$ with the augmentation $d_0 : P_0 \to M$ in degree 0. Using Theorem 2.2.12, we have a quasi-isomorphism

$$f_* : \underline{\mathrm{Hom}}(P, P) \to \underline{\mathrm{Hom}}(P, M[0]).$$

Then, we obtain

$$H_n\underline{\mathrm{Hom}}(P, P) \xrightarrow[H_n(f_*)]{\cong} H_n\underline{\mathrm{Hom}}(P, M[0]).$$

We know that the chain complex $\underline{\mathrm{Hom}}(P, M[0])$ agrees with the cochain complex $\mathrm{Hom}_R(P, M)$ up to the regrading convention and signs in the differential, thus

$$H_n\underline{\mathrm{Hom}}(P, P) \cong H_n\underline{\mathrm{Hom}}(P, M[0]) \cong H^{-n}\,\mathrm{Hom}_R(P, M) \cong \mathrm{Ext}_R^{-n}(M, M).$$

(2) We can construct the Yoneda product using projective resolutions of the R-modules (see Remark 2.5.2). We have the Yoneda product using the projective resolution P of M:

$$\begin{array}{ccc}
H_n\underline{\mathrm{Hom}}(P,P)\otimes H_m\underline{\mathrm{Hom}}(P,P) & \longrightarrow & H_{m+n}\underline{\mathrm{Hom}}(P,P) \\
\downarrow \cong & & \downarrow \cong \\
H_n\underline{\mathrm{Hom}}(P,M[0])\otimes H_m\underline{\mathrm{Hom}}(P,M[0]) & \longrightarrow & H_{m+n}\underline{\mathrm{Hom}}(P,M[0]) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Ext}_R^{-n}(M,M)\otimes \mathrm{Ext}_R^{-m}(M,M) & \longrightarrow & \mathrm{Ext}_R^{-(m+n)}(M,M).
\end{array}$$

Here the top row is induced by the composition product on $\underline{\mathrm{Hom}}(P,P)$ (see Lemma 2.2.9 and Lemma 2.2.5) and the bottom row is the Yoneda product which is constructed using projective resolutions in Remark 2.5.2. Take $\beta \in \mathrm{Ext}_R^{-n}(M,M)$ and $\alpha \in \mathrm{Ext}_R^{-m}(M,M)$. Using the quasi-isomorphism $f_* : \underline{\mathrm{Hom}}(P,P) \to \underline{\mathrm{Hom}}(P,M[0])$ from part (1), we have corresponding homology classes $[\bar{\beta}] \in H_n\underline{\mathrm{Hom}}(P,P)$ and $[\bar{\alpha}] \in H_m\underline{\mathrm{Hom}}(P,P)$. The cycle $\bar{\alpha}$ is a chain map $\bar{\alpha} : P \to P$ of degree m and the cycle $\bar{\beta}$ is a chain map $\bar{\beta} : P \to P$ of degree n. The R-module maps $\bar{\alpha}_i : P_i \to P_{i+m}$, which are the graded pieces of $\bar{\alpha}$, make the parallelograms anticommute if m is odd and commute if m is even. The cocycle $d_0 \circ \bar{\alpha}_{-m} : P_{-m} \to M$ represents the element $\alpha \in \mathrm{Ext}_R^{-m}(M,M)$, where $d_0 : P_0 \to M$ is the augmentation. Hence the

R-module map $\bar{\alpha}_{-(m+n)} : P_{-(m+n)} \to P_{-n}$ is a valid choice for the construction in Remark 2.5.2. Then the composition

$$d_0 \circ \bar{\beta}_{-n} \circ \bar{\alpha}_{-(m+n)} : P_{-(m+n)} \to M$$

represents the Yoneda product $\beta\alpha \in \operatorname{Ext}_R^{-(m+n)}(M, M)$. Hence the above diagram commutes. □

Remark 4.1.3. *1. The negative grading* Ext^{-*} *is an artefact of our grading conventions; see Remark 4.1.1.*

2. One could also take an injective resolution $M \to I^\bullet$ *and obtain the* Ext*-algebra as the cohomology of the endomorphism cochain complex:* $\operatorname{Ext}_R(M, M) \cong H^* \underline{\operatorname{Hom}}(I, I)$.

Example 4.1.4. *Let* $A = \Lambda_k(x) \cong k[x]/(x^2)$ *be the exterior algebra over* k *on one generator. In Example 3.2.1, we computed the* Ext *groups:*

$$\operatorname{Ext}_A^i(k, k) = k \quad \text{for all} \quad i \geq 0$$

as the cohomology groups of the cochain complex $\operatorname{Hom}_A(P, k)$*. We also know that the quasi-isomorphism* $P \to k$ *induces a quasi-isomorphism between hom complexes*

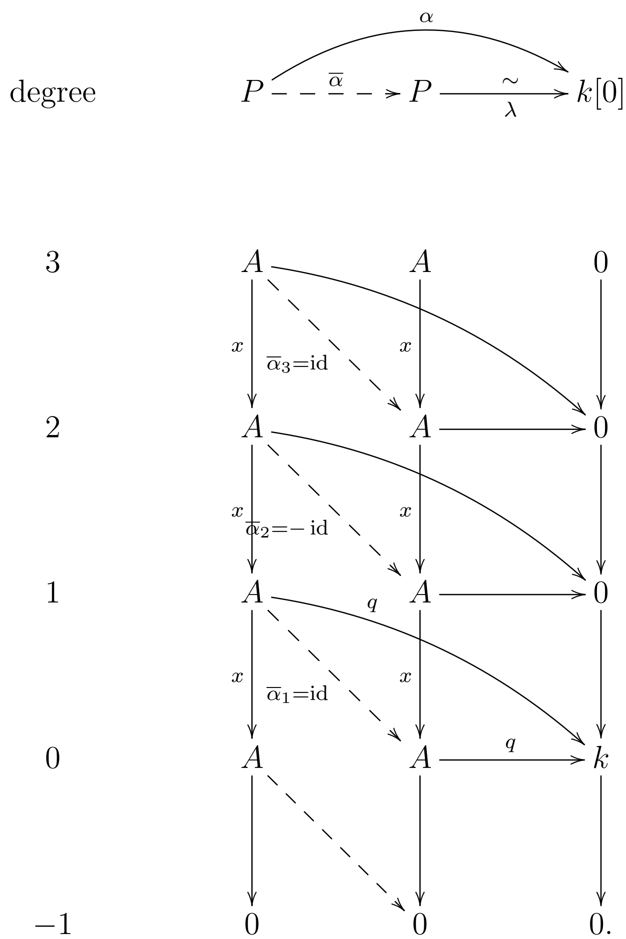
$$\underline{\operatorname{Hom}}(P, P) \to \underline{\operatorname{Hom}}(P, k[0]) \tag{4.4}$$

(see Theorem 2.2.12). We compute the multiplicative structure of the Ext*-algebra* $\operatorname{Ext}_A(k, k)$ *via the quasi-isomorphism* (4.4)*. Our claim is that the* Ext*-algebra is polynomial up to sign.*

Proof. Let $\alpha = q : A \to k$ be the canonical generator of $\mathrm{Ext}^1_A(k,k)$, where q is the quotient map. Here we identify the class α with its unique representative cocycle q. Via the regrading convention, we can view that as a (-1)-cycle in the hom complex $\alpha \in Z_{-1}\underline{\mathrm{Hom}}(P, k[0])$. We will lift the cycle α to a cycle $\bar{\alpha}$ in the endomorphism complex $\bar{\alpha} \in Z_{-1}\underline{\mathrm{Hom}}(P, P)$. More concretely, the cycle $\bar{\alpha}$ will be a chain map

$$\bar{\alpha} : P \to P$$

of degree -1, i.e., $\overline{\alpha}$ anti-commutes with the boundary maps (see Definition 2.2.8). We draw the following chain diagram whose parallelograms are anti-commutative:



The degree 2 piece of the chain map $\overline{\alpha}^2$ is:

$$-\operatorname{id} : A \to A.$$

Likewise the degree i piece of the chain map $\overline{\alpha}^i$ is:

$$(-1)^{\lfloor i/2 \rfloor} \operatorname{id} : A \to A.$$

The homology class $[\bar{\alpha}]^i$ is an element of $H_{-i}\underline{\mathrm{Hom}}(P, P)$. Since $H_{-i}\underline{\mathrm{Hom}}(P, P) \xrightarrow[\cong]{\lambda\circ-} H_{-i}\underline{\mathrm{Hom}}(P, k[0])$, the homology class $[\lambda \circ \bar{\alpha}^i] = \alpha^i$ can be viewed as an element of

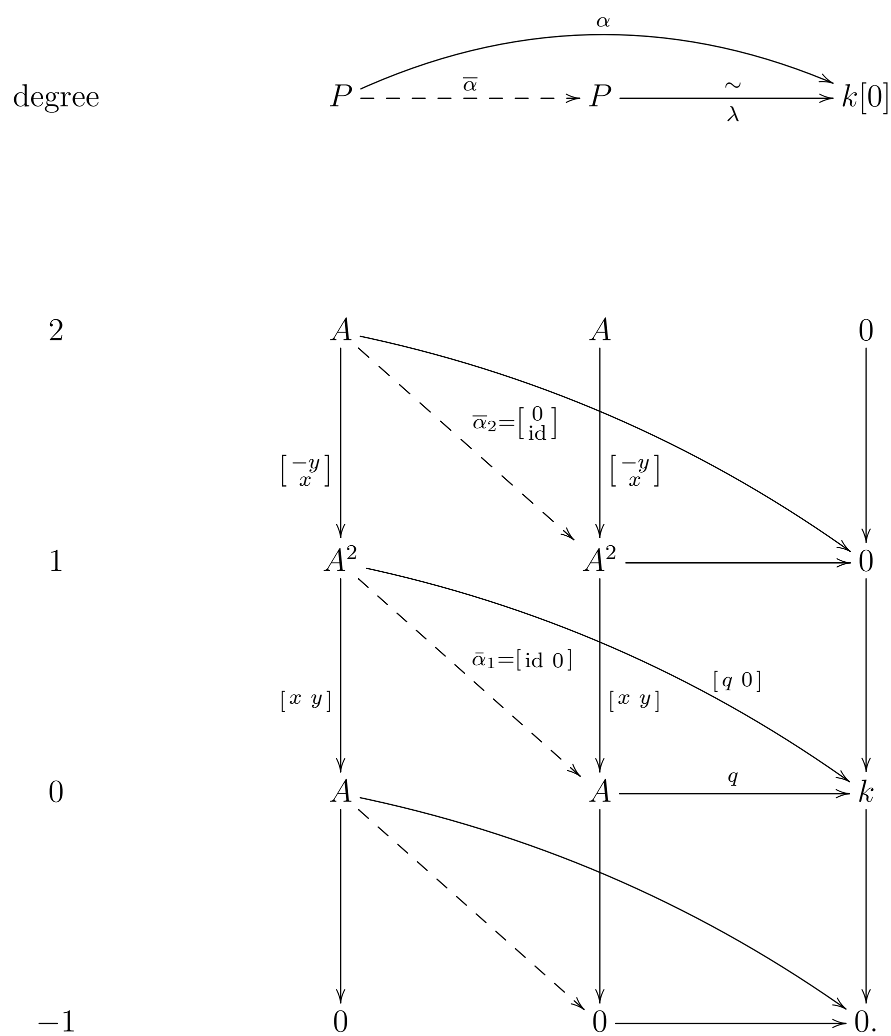
$H_{-i}\underline{\mathrm{Hom}}(P, k[0]) = \mathrm{Ext}^i_A(k,k) \cong k$. In Example 3.2.1, we found that the quotient map $q : A \to k$ is the generator of $\mathrm{Ext}^i_A(k,k)$. Since $\alpha^i = (-1)^{\lfloor i/2 \rfloor} q : A \to k$, the homology class α^i is a generator of $\mathrm{Ext}^i_A(k,k)$. Thus the Ext-algebra is polynomial up to sign. □

Example 4.1.5. *Let $A = k[x,y]$ be a polynomial algebra over k on two generators. In Example 3.1.4, we computed the* Ext *groups:*

$$\mathrm{Ext}^*_A(k,k) = \begin{cases} k, & * = 0, 2 \\ k^2, & * = 1 \\ 0, & \textit{otherwise.} \end{cases}$$

We compute the multiplicative structure of the Ext*-algebra* $\mathrm{Ext}^*_A(k,k)$ *via the quasi-isomorphism* (4.4)*. Our claim is that the* Ext*-algebra is exterior up to sign.*

Proof. Let $\alpha = \begin{bmatrix} q & 0 \end{bmatrix} : A^2 \to k$ and $\beta = \begin{bmatrix} 0 & q \end{bmatrix} : A^2 \to k$ be the canonical generators of $\mathrm{Ext}^1_A(k,k)$, where q is the quotient map $q : A \to k$. These cocycles can be viewed as (-1)-cycles in the hom complex, $\alpha, \beta \in Z_{-1}\underline{\mathrm{Hom}}(P, k[0])$. We will lift the cycles α and β to cycles $\bar{\alpha}$ and $\bar{\beta}$ in the endomorphism complex $\bar{\alpha}, \bar{\beta} \in Z_{-1}\underline{\mathrm{Hom}}(P, P)$. More concretely, the cycles $\bar{\alpha}, \bar{\beta}$ will be chain maps $\bar{\alpha}, \bar{\beta} : P \to P$ of degree -1. We draw the following chain complex diagram whose parallelograms are anti-commutative:



Using the above diagram and applying the same method we can find $\overline{\beta}_1 = \left[\, 0 \ \mathrm{id} \,\right]$: $A^2 \to A$ and $\overline{\beta}_2 = \left[\begin{smallmatrix} -\mathrm{id} \\ 0 \end{smallmatrix}\right] : A \to A^2$. We have

$$\overline{\alpha}_1 \circ \overline{\alpha}_2 = \overline{\beta}_1 \circ \overline{\beta}_2 = 0 : A \to A$$

and

$$\bar{\beta}_1 \circ \bar{\alpha}_2 = -\bar{\alpha}_1 \circ \bar{\beta}_2 = \mathrm{id} : A \to A,$$

which implies $\bar{\alpha}^2 = \bar{\beta}^2 = 0$ and $\bar{\alpha}\circ\bar{\beta} = -\bar{\beta}\circ\bar{\alpha}$. The homology class $[\bar{\beta}\circ\bar{\alpha}]$ is an element of $H_{-2}\underline{\mathrm{Hom}}(P,P)$. Since $H_{-2}\underline{\mathrm{Hom}}(P,P) \xrightarrow[\cong]{\lambda\circ -} H_{-2}\underline{\mathrm{Hom}}(P,k[0])$, the homology class $[\lambda\circ\bar{\beta}\circ\bar{\alpha}] = \beta\cdot\alpha$ can be viewed as an element of $H_{-2}\underline{\mathrm{Hom}}(P,k[0]) = \mathrm{Ext}^2_A(k,k) \cong k$. In Example 3.1.4, we found that the quotient map $q : A \to k$ is the generator of $\mathrm{Ext}^2_A(k,k)$. Since $\beta\cdot\alpha = q : A \to k$, the homology class $\beta\cdot\alpha$ is a generator of $\mathrm{Ext}^2_A(k,k)$. Thus the Ext-algebra is exterior up to sign. □

4.2 The tautological Koszul complex

Definition 4.2.1. *Let R be a commutative ring, M an R-module, and $\alpha : M \to R$ an R-module map. The* ***Koszul complex*** *associated to α is the exterior algebra on M*

$$K_*(\alpha) := \Lambda^*(M)$$

equipped with the differential $d : K_p(\alpha) \to K_{p-1}(\alpha)$ given by

$$d(x_1 \wedge \ldots \wedge x_p) = \sum_{j=1}^{p} (-1)^{j-1}\alpha(x_j) x_1 \wedge \ldots \wedge \widehat{x_j} \wedge \ldots \wedge x_p.$$

Remark 4.2.2. *Definition 4.2.1 generalizes Definition 3.3.1 as follows. Given elements $x_1, \ldots, x_d \in R$, take the free R-module $M = R^d$ and the linear functional*

$$\alpha = \left[\begin{smallmatrix} x_1 & x_2 & \cdots & x_d \end{smallmatrix}\right] : R^d \to R$$

using matrix notation. Then we have $K_*(x_1, \ldots, x_d) = K_*(\alpha)$.

Lemma 4.2.3. *The differential d satisfies $d^2 = 0$, which moreover satisfies the Leibniz rule. In other words, the Koszul complex $K(\alpha)$ is a DG-algebra.*

Proposition 4.2.4. *Let $\alpha : M \to R$ and $\beta : N \to R$ be R-module maps and consider the induced map*

$$[\,\alpha\ \beta\,] : M \oplus N \to R.$$

The Koszul complex $K([\,\alpha\ \beta\,])$ is isomorphic to the tensor product of Koszul complexes

$$K(\alpha) \otimes_R K(\beta).$$

More generally, given R-modules $M_1, \ldots, M_d$ and an R-module map

$$\alpha = [\,\alpha_1 \cdots \alpha_d\,] : M_1 \oplus \ldots \oplus M_d \to R,$$

the Koszul complex $K(\alpha)$ is isomorphic to the tensor product of complexes

$$K(\alpha_1) \otimes_R \ldots \otimes_R K(\alpha_d)$$

[Bou07, §9.3].

Definition 4.2.5. *Let k be a commutative ring and V a k-module. Consider the inclusion $V \cong S^1(V) \hookrightarrow S(V)$ and the map of $S(V)$-modules*

$$\alpha : S(V) \otimes_k V \to S(V)$$

induced by extension of scalars along the ring homomorphism $k \cong S^0(V) \hookrightarrow S(V)$, given by $\alpha(s \otimes v) = sv$. The map α is the restriction of the multiplication map

$S(V) \otimes S(V) \to S(V)$. *Via the isomorphism of graded* $S(V)$*-algebras*

$$\Lambda_{S(V)}(S(V) \otimes_k V) \cong S(V) \otimes_k \Lambda_k(V)$$

(see Proposition 2.8.4), the differential of the Koszul complex $K_*^{S(V)}(\alpha)$ *on the left-hand side corresponds to a differential*

$$d : S(V) \otimes_k \Lambda(V) \to S(V) \otimes_k \Lambda(V). \tag{4.5}$$

The resulting chain complex is called the ***tautological Koszul complex*** *of* V.

Let $(x_1 \dots x_p) \in S^p(V)$ *and* $(y_1 \wedge \dots \wedge y_q) \in \Lambda^q(V)$*. The differential of the tautological Koszul complex is given by*

$$d((x_1 \dots x_p) \otimes (y_1 \wedge \dots \wedge y_q)) = \sum_{j=1}^{q} (x_1 \dots x_p) y_j \otimes (y_1 \wedge \dots \wedge \hat{y_j} \wedge \dots \wedge y_q) \in S^{p+1}(V) \otimes_k \Lambda^{q-1}(V).$$

Note that the differential d *maps the summand* $S^p(V) \otimes_k \Lambda^q(V)$ *to* $S^{p+1}(V) \otimes_k \Lambda^{q-1}(V)$ [Bou07, §X.9.3].

Remark 4.2.6. 1. *In Definition 4.2.5, we followed Bourbaki's terminology to define the tautological Koszul complex* [Bou07]*. In* [Eis95]*, what Eisenbud called the tautological Koszul complex is the* $S(V)$*-linear dual of the chain complex* (4.5).

2. *The chain complex* (4.5) *decomposes as a direct sum of chain complexes*

$$0 \to S^0(V) \otimes_k \Lambda^n(V) \to S^1(V) \otimes_k \Lambda^{n-1}(V) \to \dots \to S^n(V) \otimes_k \Lambda^0(V) \to 0$$

indexed by $n \geq 0$.

Proposition 4.2.7. *Assume that the k-module V is finite free of rank r, that is $V \cong k^r$. The tautological Koszul complex of V is isomorphic to the Koszul complex $K(x_1, \ldots, x_r)$ associated to the generators x_i of the polynomial algebra $A = k[x_1, \ldots, x_r]$.*

In other words, the tautological Koszul complex provides a coordinate-free description of the Koszul complex that was used in Lemma 3.3.11.

Example 4.2.8. *Assume that the k-module V is finite free of rank 1 with basis element $x \in V$, that is $V \cong k$. The tautological Koszul complex of V is isomorphic to the Koszul complex $K(x)$ associated to the generator x of the polynomial algebra $A = k[x] \cong S(V)$. In Example 3.1.2, we calculated the first* Ext *group:*

$$y \in \operatorname{Ext}^1_A(k,k) \xleftarrow{\cong} \operatorname{Hom}_A(A,k) \cong k$$

and considered quotient map $y = q : A \to k$ as the canonical generator of this Ext *group. Using the tautological Koszul complex of $V \cong k$, we express the generator y as follows:*

$$y \in \operatorname{Hom}_{k[x]}(k[x] \otimes_k V, k) \cong \operatorname{Hom}_k(V,k) \cong V^*.$$

It implies that the generator y is in the dual module of V and we obtain $y(x) = 1$. Then the generator y is the dual of x, that is, $y = x^*$.

4.3 Multiplicative structure using the tautological Koszul complex

In this section we discuss the classical fact that the Ext-algebra over a polynomial algebra is exterior using the tautological Koszul complex $P(n) \otimes_k E(n)$, outlined in [Eis95, Exercise 17.21].

Given graded R-algebras A and B, the sign convention for the tensor product of graded R-algebras $A \otimes_R B$ is given by the Koszul sign rule:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2.$$

In [Eis95, §A2.3], Eisenbud proposed the convention of putting the generators of a symmetric algebra $S(V)$ in internal degree 2 and an exterior algebra $\Lambda(V)$ in internal degree 1. We use this sign convention throughout this section.

Lemma 4.3.1. *Assume that the k-module V is finite free. Let $V^* := \mathrm{Hom}_k(V, k)$ denote the k-linear dual of V and let $t \in V^* \otimes_k V$ be the element corresponding to id_V via the isomorphism*

$$V^* \otimes_k V \xrightarrow{\cong} \mathrm{Hom}_k(V, V)$$

cf. Remark 2.6.6. The element t can be viewed as an element of the algebra $\Lambda(V^*) \otimes_k S(V)$ via the inclusion of k-modules

$$V^* \otimes_k V = \Lambda^1(V^*) \otimes_k S^1(V) \hookrightarrow \Lambda(V^*) \otimes_k S(V).$$

In that algebra, the element t satisfies $t^2 = 0$, where $t^2 \in \Lambda^2(V^*) \otimes_k S^2(V)$.

Proof. Let $\{v_1, \ldots, v_r\}$ be a basis of V and $\{v_1^*, \ldots, v_r^*\}$ the dual basis of V^*. The element $t = \sum_{i=1}^{r} v_i^* \otimes v_i$. Consider

$$\begin{aligned} t^2 &= (\sum_{i=1}^{r} v_i^* \otimes v_i)(\sum_{j=1}^{r} v_j^* \otimes v_j) \\ &= \sum_{i,j=1}^{r} (-1)^{|v_i||v_j^*|}(v_i^* \wedge v_j^*) \otimes (v_i v_j) \in \Lambda^2(V^*) \otimes_k S^2(V) \\ &= \sum_{i,j=1}^{r} (v_i^* \wedge v_j^*) \otimes (v_i v_j), \text{ since } |v_i| = 2. \end{aligned}$$

In the case $i = j$, $v_i^* \wedge v_j^* = 0$ holds, which leaves only the terms with $i \neq j$:

$$\begin{aligned} t^2 &= \sum_{\substack{i,j=1 \\ i<j}}^{r} [(v_i^* \wedge v_j^*) \otimes (v_i v_j) + (v_j^* \wedge v_i^*) \otimes (v_j v_i)] \\ &= \sum_{\substack{i,j=1 \\ i<j}}^{r} [(v_i^* \wedge v_j^*) \otimes (v_i v_j) - (v_i^* \wedge v_j^*) \otimes (v_i v_j)] \quad = 0. \quad \square \end{aligned}$$

Lemma 4.3.2. *The $S(V)$-linear dual of the chain complex*

$$F_\bullet = \ldots \to S(V) \otimes_k \Lambda^{i+1}(V) \to S(V) \otimes_k \Lambda^i(V) \to \ldots$$

yields a cochain complex of $S(V)$-modules $F^* = \mathrm{Hom}_{S(V)}(F_\bullet, S(V))$, which is isomorphic to a cochain complex of the form

$$\ldots \to \Lambda^i(V^*) \otimes_k S(V) \to \Lambda^{i+1}(V^*) \otimes_k S(V) \to \ldots \tag{4.6}$$

with differential

$$\delta = \text{multiplication by } t.$$

Proof. We draw the following diagram to show that F^* is isomorphic to a cochain complex (4.6):

$$\begin{array}{ccc}
\mathrm{Hom}_{S(V)}(S(V) \otimes_k \Lambda^i(V), S(V)) & \xrightarrow{d^*} & \mathrm{Hom}_{S(V)}(S(V) \otimes_k \Lambda^{i+1}(V), S(V)) \\
\uparrow \cong & & \uparrow \cong \\
\mathrm{Hom}_k(\Lambda^i(V), S(V)) & \longrightarrow & \mathrm{Hom}_k(\Lambda^{i+1}(V), S(V)) \\
\uparrow \cong & & \uparrow \cong \\
\Lambda^i(V)^* \otimes_k S(V) & \longrightarrow & \Lambda^{i+1}(V)^* \otimes_k S(V) \\
\uparrow \cong & & \uparrow \cong \\
\Lambda^i(V^*) \otimes_k S(V) & \xrightarrow{\delta} & \Lambda^{i+1}(V^*) \otimes_k S(V),
\end{array}$$

using the isomorphisms from Propositions 2.7.7, 2.6.5, and 2.8.3. Let $\{e_1, \ldots, e_r\}$ be a basis of V and $\{e_1^*, \ldots, e_r^*\}$ the dual basis of V^*. Given indices $1 \leq j_1 < \ldots < j_i \leq r$, we denote the set $J = \{j_1, \ldots, j_i\}$. We write $e_J := e_{j_1} \wedge \ldots \wedge e_{j_i}$ and $e_J^* := e_{j_1}^* \wedge \ldots \wedge e_{j_i}^*$. Let $e_J^* \otimes s \in \Lambda^i(V^*) \otimes_k S(V)$. Using the isomorphism $\Lambda^i(V^*) \xrightarrow{\cong} \Lambda^i(V)^*$ from Proposition 2.7.7, we have $(e_J)^* \otimes s \in \Lambda^i(V)^* \otimes_k S(V)$. By the natural isomorphism $\Lambda^i(V)^* \otimes_k S(V) \xrightarrow{\cong} \mathrm{Hom}_k(\Lambda^i(V), S(V))$ from Proposition 2.6.5, we obtain the values of the corresponding map $\Lambda^i(V) \to S(V)$ on the basis elements:

$$(e_J)^*(e_J)s = 1 \cdot s = s$$

and

$$(e_J)^*(e_{k_1} \wedge \ldots \wedge e_{k_i})s = 0 \cdot s = 0,$$

where the indices $\{k_1, \ldots, k_i\}$ are different from J. Using the natural isomorphism from Proposition 2.8.3, given $\beta = se_J^* \in \mathrm{Hom}_k(\Lambda^i(V), S(V))$ the corresponding map via extension of scalars $\beta' \in \mathrm{Hom}_{S(V)}(S(V) \otimes \Lambda^i(V), S(V))$ is

$$\beta'(s' \otimes y) = s'\beta(y).$$

Given indices $1 \leq k_1 < \ldots < k_{i+1} \leq r$, we denote the set $K = \{k_1, \ldots, k_{i+1}\}$. The differential d^* of the cochain complex in the top of the diagram is given by

$$\begin{aligned} d^*(\beta')(s' \otimes e_K) =&(\beta' \circ d)(s' \otimes e_K) \\ =&\beta'\left(d(s' \otimes e_K)\right) \\ =&\beta'\left(\sum_{\ell=1}^{i+1}(-1)^{\ell-1}s'e_{k_\ell} \otimes (e_{k_1} \wedge \ldots \wedge \widehat{e_{k_\ell}} \wedge \ldots \wedge e_{k_{i+1}})\right) \\ =&s'\sum_{\ell=1}^{i+1}(-1)^{\ell-1}e_{k_\ell}\beta\left(e_{k_1} \wedge \ldots \wedge \widehat{e_{k_\ell}} \wedge \ldots \wedge e_{k_{i+1}}\right). \end{aligned}$$

In the case $J \nsubseteq K$, the indices $\{k_1, \ldots, k_{\ell-1}, k_{\ell+1}, \ldots, k_{i+1}\}$ are different from J, so we obtain

$$d^*(\beta')(s' \otimes e_K) = s'\sum_{\ell=1}^{i+1}(-1)^{\ell-1}e_{k_\ell}\beta\left(e_{k_1} \wedge \ldots \wedge \widehat{e_{k_\ell}} \wedge \ldots \wedge e_{k_{i+1}}\right) = 0.$$

In the case $J \subseteq K$, the set K is of the form $K = J \cup \{k_m\}$, which yields

$$\begin{aligned} d^*(\beta')(s' \otimes e_K) =&s'\sum_{\substack{\ell=1 \\ \ell\neq m}}^{i+1}(-1)^{\ell-1}e_{k_\ell}\beta\left(e_{k_1} \wedge \ldots \wedge \widehat{e_{k_\ell}} \wedge \ldots \wedge e_{k_{i+1}}\right) + s'(-1)^{m-1}e_{k_m}\beta(e_J) \\ =&0 + s'(-1)^{m-1}e_{k_m}\beta(e_J) \\ =&s'(-1)^{m-1}e_{k_m}s. \end{aligned}$$

From Proposition 2.8.3, given $d^*(\beta') = \alpha' \in \mathrm{Hom}_{S(V)}(S(V) \otimes \Lambda^i(V), S(V))$ the corresponding map $\alpha \in \mathrm{Hom}_k(\Lambda^i(V), S(V))$ is given by

$$\alpha(e_K) = \alpha'(1 \otimes e_K).$$

Using Proposition 2.6.5, we have

$$\sum_{1\leq k_1<k_2<\ldots<k_{i+1}\leq r} (e_K)^* \otimes \alpha(e_K) \in \Lambda^{i+1}(V)^* \otimes_k S(V).$$

Since the set K ranges over all subsets of $\{1,\ldots,\text{r}\}$ of cardinality $i+1$ and in the case $J \nsubseteq K$, $\alpha(e_K) = 0$, we take the set K as $K = J \cup \{k'\}$ for some $k' \in \{1,\ldots,r\} \setminus J$. We denote $k' = k_m$ to indicate the position of k' in the set K. We have

$$e_K^* = e_{J\cup\{k'\}}^* = (-1)^{m-1}(e_{k'}^* \wedge e_J^*)$$

and

$$\alpha(e_K) = \alpha(e_{J\cup\{k'\}}) = \alpha'(1 \otimes e_{J\cup\{k'\}}) = (-1)^{m-1}e_{k'}s.$$

Then we obtain

$$\begin{aligned}\sum_{k'\in\{1,\ldots,r\}\setminus J} (e_K)^* \otimes \alpha(e_K) &= \sum_{k'\in\{1,\ldots,r\}\setminus J} (e_{J\cup k'})^* \otimes (-1)^{m-1}e_{k'}s \\ &= \sum_{k'\in\{1,\ldots,r\}\setminus J} (e_{k'} \wedge e_J)^* \otimes e_{k'}s.\end{aligned}$$

Using Proposition 2.7.7,

$$\sum_{k'\in\{1,\ldots,r\}\setminus J} e_{k'}^* \wedge e_J^* \otimes e_{k'}s \in \Lambda^{i+1}(V^*) \otimes S(V).$$

Hence the differential δ of the cochain complex in the bottom of the diagram is

$$\delta(e_J^* \otimes s) = \sum_{k'\in\{1,\ldots,r\}\setminus J} e_{k'}^* \wedge e_J^* \otimes e_{k'}s.$$

Let us show that the differential δ is multiplication by t where $t \in \Lambda(V^*) \otimes_k S(V)$ as follows:

$$\begin{aligned} t(e_J^* \otimes s) &= \sum_{\ell=1}^{r} (e_\ell^* \otimes e_\ell)(e_J^* \otimes s) \\ &= \sum_{\ell=1}^{r} (-1)^{|e_\ell||e_J^*|} e_\ell^* \wedge e_J^* \otimes e_\ell s \\ &= \sum_{\ell=1}^{r} e_\ell^* \wedge e_J^* \otimes e_\ell s, \text{ since } |e_\ell| = 2. \end{aligned}$$

Since $e_\ell^* \wedge e_J^* = 0$ in the case $\ell \in J$, the range of ℓ is $\{1, \ldots, r\} \setminus J$, which yields

$$t(e_J^* \otimes s) = \sum_{\ell \in \{1,\ldots,r\} \setminus J} e_\ell^* \wedge e_J^* \otimes e_\ell s. \qquad \square$$

Lemma 4.3.3. *Let $C^\bullet$ be a cochain complex of R-modules that has the structure of a graded R-algebra such that the differential $d : C^i \to C^{i+1}$ is given by multiplication by some fixed element $t \in C^1$, that is:*

$$d(x) = tx$$

for all $x \in C^i$. The cocycles $Z(C) \subseteq C$ form a subalgebra of C. Note that if C was unital to begin with, $Z(C)$ need not be unital, cf. Lemma 2.2.5.

Proof. Let $\alpha, \beta \in Z(C)$ be two cocycles of C. Since the differential d is given by multiplication by some fixed element $t \in C^1$, we obtain

$$d(\alpha\beta) = t(\alpha\beta) = (t\alpha)\beta = d(\alpha)\beta = 0,$$

so that $\alpha\beta \in Z(C)$ is a cocycle. $\square$

Notation 4.3.4. *When working with coalgebras, we use Sweedler's notation* [LV12, §1.2.1]*. In a coalgebra C with comultiplication $\Delta : C \to C \otimes C$, the comultiplication of $x \in C$ is given by*

$$\Delta(x) = \sum_{i=1}^{n} x^i_{(1)} \otimes x^i_{(2)}$$

*for $x^i_{(1)}$ and $x^i_{(2)} \in C$. In **Sweedler's notation**, this is abbreviated to*

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

Omitting the summation symbols, we write

$$\Delta(x) = x_{(1)} \otimes x_{(2)}.$$

Proposition 4.3.5 is found in [Eis95, Exercise A3.28] and we follow the sketch of proof in the section "Hints and Solutions for Selected Exercises" in [Eis95].

Proposition 4.3.5. *Let $R \twoheadrightarrow k$ be a quotient ring homomorphism. Let*

$$P_\bullet = \ldots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \to 0$$

be a projective resolution of k as an R-module, with $P_0 = R$ and the augmentation $P_0 \twoheadrightarrow k$ being the quotient map. Assume that each projective R-module P_i is finitely generated. Taking the R-linear dual yields a cochain complex of R-modules $P^* := \mathrm{Hom}_R(P_\bullet, R)$.

Assume that P^* satisfies the setup of Lemma 4.3.3. More precisely, P^* has the structure of a graded R-algebra such that the differential $d^* : P^i \to P^{i+1}$ is given by *multiplication by some fixed element $t \in P^1$, that is:*

$$d^*(x) = tx$$

for all $x \in P^i$.

Consider the graded k-algebra $P^* \otimes_R k$ together with the canonical cochain map

$$\theta : P^* \otimes_R k \to \operatorname{Hom}_R(P, k).$$

Since each projective R-module P_i is finitely generated, the cochain map θ is an isomorphism. The map of graded k-modules given by the two equal composites in the diagram

$$\begin{array}{ccc} Z(P^* \otimes_R k) & \xrightarrow[\cong]{Z(\theta)} & Z(\operatorname{Hom}_R(P,k)) \\ {\scriptstyle quotient}\downarrow & & \downarrow{\scriptstyle quotient} \\ H(P^* \otimes_R k) & \xrightarrow[\cong]{H(\theta)} & H(\operatorname{Hom}_R(P,k)) = \operatorname{Ext}_R(k,k) \end{array}$$

is an algebra homomorphism, where $\operatorname{Ext}_R(k,k)$ is endowed with Yoneda product.

Proof. Part 1. Defining the comultiplication.

Let $\mu : P^* \otimes P^* \to P^*$ be the multiplication on the cochain complex of R-modules $P^* := \operatorname{Hom}_R(P_\bullet, R)$. From the commutativity of the following diagram, we interpret the multiplication of P^* as a comultiplication $\Delta : P \to P \otimes P$ on P:

$$\begin{array}{ccccc} P^{**} & \xrightarrow{\mu^*} & (P^* \otimes P^*)^* & \xleftarrow[\cong]{\alpha} & P^{**} \otimes P^{**} \\ {\scriptstyle \sigma}\uparrow{\scriptstyle \cong} & & & & {\scriptstyle \cong}\uparrow{\scriptstyle \sigma\otimes\sigma} \\ P & & \xrightarrow{\Delta} & & P \otimes P, \end{array}$$

see the isomorphisms from Lemma 2.6.8 and Lemma 2.6.11. Let $x \in P$. Using the isomorphism $P \xrightarrow{\cong} P^{**}$ from Lemma 2.6.8, we have $\mathrm{ev}_x \in P^{**}$. Applying the map μ^*, we obtain $\mu^*(\mathrm{ev}_x) \in (P^* \otimes P^*)^*$. For all $f, g \in P^*$, we have

$$\mu^*(\mathrm{ev}_x)(f \otimes g) = (f \cdot g)(x) \in R, \tag{4.7}$$

where $f \cdot g = \mu(f \otimes g)$ denotes the product in the algebra P^*. Using Notation 4.3.4, given $x \in P$ we have $\Delta(x) = x_{(1)} \otimes x_{(2)} \in P \otimes P$. From the isomorphism in Lemma 2.6.8, we obtain $\mathrm{ev}_{x_{(1)}} \otimes \mathrm{ev}_{x_{(2)}} \in P^{**} \otimes P^{**}$. The isomorphism from Lemma 2.6.11 yields $\alpha(\mathrm{ev}_{x_{(1)}} \otimes \mathrm{ev}_{x_{(2)}}) \in (P^* \otimes P^*)^*$. For all $f, g \in P^*$, we have

$$\alpha(\mathrm{ev}_{x_{(1)}} \otimes \mathrm{ev}_{x_{(2)}})(f \otimes g) = f(x_{(1)})g(x_{(2)}) \in R. \tag{4.8}$$

Equating (4.7) and (4.8), we find that the comultiplication $\Delta(x) = x_{(1)} \otimes x_{(2)}$ is characterized by

$$(f \cdot g)(x) = f(x_{(1)})g(x_{(2)}),$$

for all $f, g \in P^*$. Let $|x| = n$ and $i + j = n$, then the piece of $\Delta(x)$ in the summand $P_i \otimes_R P_j$ is characterized by

$$(f \cdot g)(x) = f(x_{(1),i})g(x_{(2),j}),$$

for all $f \in P_i^*$ and $g \in P_j^*$.

Part 2. Relating the comultiplication to the boundary map.

Let $f : P_m \to R$ and $g : P_n \to R$ be two cocycles and consider the composite

$$h : P_{m+n} \xrightarrow{\Delta} \bigoplus_{i+j=m+n} P_i \otimes P_j \xrightarrow{\text{project}} P_m \otimes P_n \xrightarrow{f \otimes g} R \otimes R \cong R.$$

Let $x \in P_{m+n}$. After composing Δ and the projection map, we obtain $x_{(1),m} \otimes x_{(2),n} \in P_m \otimes P_n$. Then we have

$$x_{(1),m} \otimes x_{(2),n} \mapsto f(x_{(1),m}) \otimes g(x_{(2),n}) \mapsto f(x_{(1),m})g(x_{(2),n}) = (f \cdot g)(x) \in R. \tag{4.9}$$

Hence the function $h = f \cdot g$. We show that the condition on d^* guarantees the commutativity of the diagrams:

$$\begin{array}{ccccc} P_{m+n+1} & \xrightarrow{\Delta} & \bigoplus\limits_{i+j=m+n+1} P_i \otimes P_j & \xrightarrow{\text{project}} & P_{m+1} \otimes P_n \\ \downarrow{\scriptstyle d} & & & & \downarrow{\scriptstyle d\otimes 1} \\ P_{m+n} & \xrightarrow{\Delta} & \bigoplus\limits_{i+j=m+n} P_i \otimes P_j & \xrightarrow{\text{project}} & P_m \otimes P_n. \end{array}$$

We have two R-module maps $P_{m+n+1} \to P_m \otimes P_n$ defined by the above diagram. Since both R-modules in the source and target are finitely generated projective, it suffices to show that the R-linear duals of the two maps are equal to conclude that the above diagram commutes. Since the projective R-modules P_m and P_n are finitely generated, $P_m^* \otimes P_n^*$ is naturally isomorphic to $(P_m \otimes P_n)^*$. Let $f \in P_m^*$ and $g \in P_n^*$. First, we consider the dual of the map which is given by the top of the diagram:

$$P_{m+n+1} \xrightarrow{\Delta} \bigoplus_{i+j=m+n+1} P_i \otimes P_j \xrightarrow{\text{project}} P_{m+1} \otimes P_n \xrightarrow{d\otimes 1} P_m \otimes P_n \xrightarrow{f\otimes g} R \otimes R \cong R.$$

Let $x \in P_{m+n+1}$. After composing Δ and the projection map, we obtain $x_{(1),m+1} \otimes x_{(2),n} \in P_{m+1} \otimes P_n$. Applying the composition of $d \otimes 1$ and $f \otimes g$, we obtain

$$(f \otimes g) \circ (d \otimes 1)(x_{(1),m+1} \otimes x_{(2),n}) = ((f \circ d) \otimes g)(x_{(1),m+1} \otimes x_{(2),n}) = d^*(f)(x_{(1),m+1}) \otimes g(x_{(2),n}) \in R \otimes R.$$

Then we have

$$d^*(f)(x_{(1),m+1}) \otimes g(x_{(2),n}) \mapsto d^*(f)(x_{(1),m+1})g(x_{(2),n}) = (d^*(f) \cdot g)(x) \in R.$$

Second, we consider the dual of the map which is given by the bottom of the diagram:

$$P_{m+n+1} \xrightarrow{d} P_{m+n} \xrightarrow{\Delta} \bigoplus_{i+j=m+n} P_i \otimes P_j \xrightarrow{\text{project}} P_m \otimes P_n \xrightarrow{f \otimes g} R \otimes R \cong R.$$

In Equation (4.9), we showed that $x \in P_{m+n}$ maps to $(f \cdot g)(x) \in R$. After composing with the map d we have

$$(f \cdot g) \circ d = d^*(f \cdot g).$$

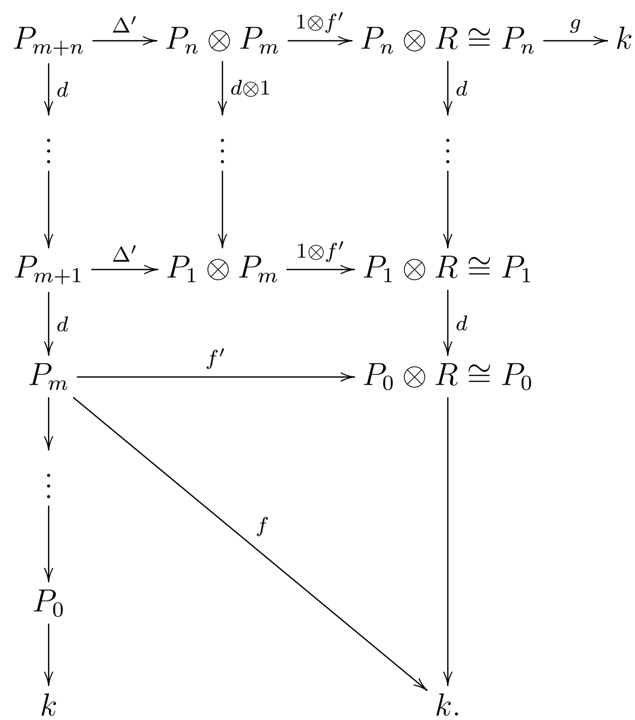
We are comparing two functions $d^*(f) \cdot g$ and $d^*(f \cdot g)$. By the condition on P^*, we obtain

$$d^*(f \cdot g) = t \cdot (f \cdot g) = (tf) \cdot g = d^*(f) \cdot g.$$

Thus the diagram commutes.

Part 3. Representing the Yoneda products.

Let $\alpha = [f] \in \mathrm{Ext}_R^m(k,k)$ be represented by a cocycle $f : P_m \to k$ and $\beta = [g] \in \mathrm{Ext}_R^n(k,k)$ be represented by a cocycle $g : P_n \to k$. Let $\Delta' : P_{m+n} \to P_n \otimes P_m$ denote the composition of Δ and the projection map. Since P_m is projective and the map $P_0 \to k$ is onto, there exists a lift $\alpha_0 = f' : P_m \to P_0$ of f to P_0. We obtain the commutative diagram:



The maps $\alpha_i = (1 \otimes f') \circ \Delta' : P_{m+i} \to P_i$ satisfy $d \circ \alpha_i = \alpha_{i-1} \circ d$. In particular for $i = n$, the map α_n is

$$(1 \otimes f') \circ \Delta' : P_{m+n} \to P_n.$$

By Remark 2.5.2, the composition $g \circ \alpha_n$:

$$g \circ ((1 \otimes f') \circ \Delta') : P_{m+n} \to k$$

represents the Yoneda product $\beta\alpha$ in $\mathrm{Ext}_R^{m+n}(k,k)$. By Equation (4.9), the product $g \cdot f \in P^* \otimes k$ is the composite

$$g \cdot f : P_{m+n} \xrightarrow{\Delta} \bigoplus_{i+j=m+n} P_i \otimes P_j \xrightarrow{\text{project}} P_n \otimes P_m \xrightarrow{g\otimes f} k \otimes k \cong k,$$

which yields

$$g \cdot f = g \circ ((1 \otimes f') \circ \Delta') : P_{m+n} \to k.$$

Thus $g \cdot f$ is a representative of the Yoneda product

$$[g \cdot f] = [g][f] = \beta\alpha \in \operatorname{Ext}_R^{m+n}(k,k). \qquad \square$$

Remark 4.3.6. *1. If R is Noetherian, then the R-module k (being finitely generated) admits a resolution by finitely generated projective R-modules* [Wei94, Definition 3.3.9].

2. If multiplication on P^* is commutative or graded-commutative, then the cohomology $H(P^* \otimes_R k)$ inherits a graded k-algebra structure. In that case, the conclusion is equivalent to $H(\theta)$ being an algebra homomorphism. If $H(\theta)$ is moreover an isomorphism, the conclution is that the multiplication on $H(P^* \otimes_R k) \cong \operatorname{Ext}_R(k,k)$ agrees with the Yoneda product.

Proposition 4.3.7. *Let k be a commutative ring and let V a finite free k-module of rank r, i.e., $V \cong k^r$. The* Ext *algebra over a polynomial algebra is exterior:*

$$\operatorname{Ext}_{S(V)}(k,k) \cong \Lambda(V^*).$$

Proof. We use the tautological Koszul complex $F_\bullet = S(V) \otimes_k \Lambda(V)$ to prove this isomorphism. By applying the functor $- \otimes_{S(V)} k$ to the cochain complex $F^* = \operatorname{Hom}_{S(V)}(F_\bullet, S(V))$ we get a cochain complex of k-modules:

$$\operatorname{Hom}_{S(V)}(F_\bullet, S(V)) \otimes_{S(V)} k \xrightarrow{\cong} \operatorname{Hom}_{S(V)}(F_\bullet, k).$$

We compute Ext groups using the k-module cochain complex $\mathrm{Hom}_{S(V)}(F_\bullet, k)$. Note that composition of $S(V)$-module maps $S(V) \otimes_k \Lambda^{i+1}(V) \to S(V) \otimes_k \Lambda^i(V) \to k$ is zero, as argued in the proof of Lemma 3.3.11. Hence the cochain complex $\mathrm{Hom}_{S(V)}(F_\bullet, k)$ has zero differential. By definition of Ext, the cohomology groups are the Ext groups:

$$H^i(\mathrm{Hom}_{S(V)}(F_\bullet, k)) \cong \mathrm{Ext}^i_{S(V)}(k, k).$$

The algebra $F^* \cong \Lambda(V^*) \otimes_k S(V)$ has differential $\delta =$ multiplication by t (see Lemma 4.3.2). Then the algebra $F^* \otimes_{S(V)} k \cong \Lambda(V^*)$ has zero differential. Hence the cohomology of $F^* \otimes_{S(V)} k$ is given by

$$H(F^* \otimes_{S(V)} k) = F^* \otimes_{S(V)} k \cong \Lambda(V^*).$$

Since F^* satisfies Proposition 4.3.5, the algebra structure on F^* is compatible with the Ext-algebra. We obtain an algebra isomorphism

$$H(F^* \otimes_{S(V)} k) \cong \mathrm{Ext}_{S(V)}(k, k) \cong \Lambda(V^*). \qquad \square$$

Remark 4.3.8. *We constructed the Yoneda product using endomorphism complexes (see Proposition 4.1.2) and projective resolutions of the R-modules (see Remark 2.5.2). In Remark 4.1.1, the cochain complex $\mathrm{Hom}_R(P, k)$ is isomorphic to the hom complex $\underline{\mathrm{Hom}}(P, k[0])$ up to a sign in the differential and in Remark 2.5.2, there is a sign discrepancy between parallelograms that commute versus anticommute (see Definition 2.2.8). Hence the Yoneda product using endomorphism complexes agrees with* *the Yoneda product using projective resolutions up to a sign.*

Chapter 5

A proof using Koszul duality for algebras

In this chapter, we provide a proof that the Ext-algebra over a polynomial algebra is exterior and vice versa (Proposition 5.4.3). The main references of this chapter are [PP05] and [Wu16], both of which assume that the ground ring k is a field. In [Pos21], Koszul duality for algebras is developed over an arbitrary ground ring k, working with k-bimodules. We will always assume that k is commutative and work with k-modules instead of bimodules. Some of the material in this chapter is found in [Pos21, §10.1]; we fill in some details.

5.1 Quadratic algebras

Facts about quadratic algebras are from [PP05] and [Wu16, §4.2].

Definition 5.1.1. *Let k be a commutative ring. A **quadratic algebra** $A = \bigoplus_{n\in\mathbb{N}} A_n$*

over k is a graded algebra that satisfies $A_0 = k$ and admits a presentation

$$A \cong T(A_1)/I$$

where $T(A_1)$ is the tensor algebra of A_1 and I is the two-sided ideal generated by a module $R \subseteq A_1 \otimes_k A_1$ of homogeneous elements of degree two in $T(A_1)$.

Observe that the module R is a quadratic relation set $R = (\ker \alpha) \cap (A_1 \otimes_k A_1)$ where $\alpha : T(A_1) \to A$ is the canonical map defined by $m_1 \otimes \ldots \otimes m_i \mapsto m_1 \ldots m_i$ for all $i \in \mathbb{N}_+$.

Example 5.1.2. *Here are some examples of quadratic algebras.*

1. *The tensor algebra $T(V)$ is quadratic since the tensor algebra can be represented as $T(V) = T(V)/(0)$.*

2. *The symmetric algebra $S(V)$ is quadratic since the symmetric algebra is the quotient of the tensor algebra $S(V) = T(V)/I$, where I is generated by $\{x \otimes y - y \otimes x \mid x, y \in V\}$.*

3. *The exterior algebra $\Lambda(V)$ is quadratic since the exterior algebra is the quotient of the tensor algebra $\Lambda(V) = T(V)/I$, where I is generated by $\{x \otimes x \mid x \in V\}$.*

5.2 Koszul algebras

Facts about Koszul algebras are taken from [Wu16, §4.1].

Definition 5.2.1. *Let k be a commutative ring. A* ***Koszul algebra*** $A = \bigoplus_{n\in\mathbb{N}} A_n$ *is a graded algebra that satisfies* $A_0 = k$ *and* $A_0 \cong A/\bigoplus_{n>0} A_n$ *considered as a graded A-module admits a graded projective resolution:*

$$\ldots \to P^2 \to P^1 \to P^0 \to A_0 \to 0$$

such that P^i is generated as a $\mathbb{Z}$-graded A-module by its degree i component, i.e., $P^i = \bigoplus_{j\in\mathbb{Z}} P^i_j$ *(decomposition into graded pieces) satisfies* $P^i = AP^i_i$ *(see* [PP05, §2.1]*).*

Notation 5.2.2. *Let R be a $\mathbb{Z}$-graded ring and* $M = \bigoplus_{i\in\mathbb{Z}} M_i$ *be a $\mathbb{Z}$-graded R-module. The notation*

$$M(n) = \bigoplus_{j\in\mathbb{Z}} M(n)_j$$

denotes the same R-module M with shifted grading $M(n)_j = M_{j+n}$ [Wu16, Notation 2.3.11].

Definition 5.2.3. *Let A be a graded algebra and M, N a pair of graded A-modules. The* ***graded homomorphism functor*** *is*

$$\mathrm{Hom}_A(M, N) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}^j_A(M, N),$$

where $\mathrm{Hom}^j_A(M, N)$ *is the module of all homomorphisms of graded A-modules that decrease the degree by j, that is,*

$$f : M_k \to N_{k-j}.$$

We denote by

$$\operatorname{Ext}^i_A(M,N) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Ext}^{i,j}_A(M,N)$$

the i^{th} *derived functor of the graded homomorphism functor. The first grading* i *is called the homological grading and the second* j *is called the internal one* [PP05, §1.1].

Example 5.2.4. *Consider the polynomial ring* $A = k[x,y] = \bigoplus_{j=0}^{\infty} k[x,y]_j$ *which is graded by degree of homogeneous polynomials. Using Notation 5.2.2 we obtain the graded version* $P_\bullet$ *of the (length 2) free resolution of* k *as a* $k[x,y]$*-module from Example 3.1.4:*

$$0 \to A(-2) \xrightarrow{\left[\begin{smallmatrix}-y\\ x\end{smallmatrix}\right]} A^2(-1) \xrightarrow{[x\ y]} A \to k \to 0.$$

Note that as graded A*-modules,* A *is generated in degree 0,* $A^2(-1)$ *is generated in degree 1 and* $A(-2)$ *is generated in degree 2. This free resolution implies that* $k[x,y]$ *is Koszul.*

We compute the bigraded Ext *groups,* $\operatorname{Ext}^{i,j}_A(k,k)$ *in graded* $k[x,y]$*-modules. The cochain complex of graded* k*-modules* $\operatorname{Hom}_A(P_\bullet, k)$ *is*

$$0 \to \operatorname{Hom}_A(A,k) \xrightarrow{d} \operatorname{Hom}_A(A^2(-1),k) \xrightarrow{d} \operatorname{Hom}_A(A(-2),k) \to 0.$$

Note that $d = 0$, $\operatorname{Hom}_A(A,k) \cong k(0)$, $\operatorname{Hom}_A(A^2(-1),k) \cong k^2(-1)$ *and* $\operatorname{Hom}_A(A(-2),k) \cong k(-2)$. *Thus the cochain complex has the form:*

$$0 \to k(0) \xrightarrow{0} k^2(-1) \xrightarrow{0} k(-2) \to 0.$$

The Ext *groups are concentrated in diagonal bidegrees* $\mathrm{Ext}_A^{i,i}(k,k)$:

$$\mathrm{Ext}_A^{i,i}(k,k) = \begin{cases} k, & i = 0 \\ k^2, & i = 1 \\ k, & i = 2 \\ 0, & \textit{otherwise.} \end{cases}$$

We obtain $\mathrm{Ext}_A^{i,j}(k,k) = 0$ *for* $i \neq j$. *A graded algebra* A *should satisfy this condition to be Koszul* [PP05, §2.1 Definition 1(a)].

Example 5.2.5. *Let* k *be a commutative ring and* V *a finitely generated projective* k*-module. The symmetric algebra* $S(V)$ *and the exterior algebra* $\Lambda(V)$ *are Koszul* [PP05, §2.1 Example 1].

Proposition 5.2.6. *If the algebra* A *is Koszul, then the algebra* A *is quadratic* [Wu16, Proposition 4.2.12].

5.3 Koszul dual algebra of a quadratic algebra

Facts about the Koszul dual algebra of a quadratic algebra are from [Wu16, §5.1].

Definition 5.3.1. *Let* $T(V)$ *be the tensor algebra of* V *over* k. *The* ***orthogonal submodule*** *of any submodule* $W \subseteq V^{\otimes j}$ *is defined as*

$$W^{\perp} = \{u \in (V^*)^{\otimes j} \mid u(w) = 0 \ \text{ for all } \ w \in W\},$$

where $u(w)$ denotes the pairing from Lemma 2.7.6. The symbol $\langle W \rangle$ denotes the two-sided ideal of $T(V)$ generated by W, i.e.,

$$\langle W \rangle = T(V)WT(V)$$

and the symbol $\langle W^{\perp} \rangle$ denotes the two-sided ideal of $T(V^*)$ generated by $W^{\perp}$, i.e.,

$$\langle W^{\perp} \rangle = T(V^*)W^{\perp}T(V^*).$$

Lemma 5.3.2. *Let k be a commutative ring. Assume that V is finitely generated projective and $W \subseteq V \otimes V$ is a submodule such that $V \otimes V/W$ is projective. Then the following hold.*

1. *W and $V \otimes V/W$ are also finitely generated projective.*

2. *There is a natural isomorphism $W \xrightarrow{\cong} (W^{\perp})^{\perp}$.*

Proof. (1) Since V is finitely generated projective, there exists a k-module Q such that $V \oplus Q \cong k^r$ is finite free. Consider

$$\begin{aligned} k^{rr} &\cong (V \oplus Q) \otimes (V \oplus Q) \\ &\cong (V \otimes_k V) \oplus (V \otimes_k Q) \oplus (Q \otimes_k V) \oplus (Q \otimes_k Q). \end{aligned}$$

Thus $V \otimes_k V$ is finitely generated projective. Since $V \otimes V$ is finitely generated and a quotient of a finitely generated module is finitely generated, $V \otimes V/W$ is finitely generated.

We have a short exact sequence of k-modules:

$$0 \longrightarrow W \hookrightarrow V \otimes V \twoheadrightarrow V \otimes V/W \longrightarrow 0\,. \tag{5.10}$$

Since $V \otimes V/W$ is projective and the map $r : V \otimes V \to V \otimes V/W$ is surjective, there exists a map $s : V \otimes V/W \to V \otimes V$ such that $r \circ s = \mathrm{id}_{V\otimes V/W}$. Then, the above sequence (5.10) is split and we obtain $V \otimes V \cong W \oplus (V \otimes V/W)$. Following the fact that $V \otimes V$ is finitely generated projective, its direct summand W is also finitely generated projective.

(2) We have a long exact sequence from applying $\mathrm{Hom}_k(-, k)$ to the sequence (5.10):

$$0 \longrightarrow \mathrm{Hom}_k(V \otimes V/W, k) \longrightarrow \mathrm{Hom}_k(V \otimes V, k) \longrightarrow \mathrm{Hom}_k(W, k)$$

$$\xrightarrow{\delta} \mathrm{Ext}^1_k(V \otimes V/W, k) \longrightarrow \mathrm{Ext}^1_k(V \otimes V, k) \longrightarrow \mathrm{Ext}^1_k(W, k) \xrightarrow{\delta} \dots$$

Since $V \otimes V/W$ is projective over k, we have $\mathrm{Ext}^1_k(V \otimes V/W, k) = 0$. We get the short exact sequence:

$$0 \longrightarrow (V \otimes V/W)^* \longrightarrow (V \otimes V)^* \longrightarrow W^* \longrightarrow 0.$$

Using the isomorphism $V^* \otimes V^* \cong (V \otimes V)^*$ (see Lemma 2.6.10) and the definition of $W^\perp$ in Definition 5.3.1,

$$W^\perp \xrightarrow{\cong} \mathrm{Hom}_k(V \otimes V/W, k) = (V \otimes V/W)^*.$$

Then, we have

$$0 \longrightarrow W^\perp \longrightarrow V^* \otimes V^* \longrightarrow W^* \longrightarrow 0. \tag{5.11}$$

We apply $\mathrm{Hom}_k(-, k)$ to the sequence (5.11):

$$0 \to (W^\perp)^\perp \to V^{**} \otimes V^{**} \to (W^\perp)^* \to 0.$$

Since V and W are finitely generated projective, the natural maps $V \to V^{**}$ and $W \to W^{**}$ are isomorphisms (see Lemma 2.6.8). Hence via the identification $V \otimes V \cong V^{**} \otimes V^{**}$, we get

$$W = (W^{\perp})^{\perp}. \qquad \square$$

Definition 5.3.3. *Let $A = T(V)/\langle W \rangle$ be a quadratic algebra, where $\langle W \rangle$ is generated by the quadratic relation set $W \subseteq V^{\otimes 2}$. The* ***Koszul dual algebra*** *of A is defined as*

$$A^! = T(V^*)/\langle W^{\perp} \rangle.$$

Note that $A^!$ is a quadratic algebra. We can express $A^!$ as a graded algebra

$$A^! = \bigoplus_{n \in \mathbb{N}} A^!_n$$

with $A^!_1 = V^*$ [Wu16, Remark 5.1.6].

Remark 5.3.4. *Given a quadratic presentation $A = T(V)/\langle W \rangle$, we have $A_1 = V$ and $A_2 = V \otimes V/W$. Assume that V is finitely generated projective and $W \subseteq V \otimes V$ is a submodule such that $V \otimes V/W$ is projective. We write $A^!$ in a way that is independent of the quadratic presentation of A as follows:*

$$A^! = T(A_1^*)/\langle A_2^* \rangle,$$

where $\alpha^* : A_2^* \hookrightarrow A_1^* \otimes A_1^* \xrightarrow{\cong} (A_1 \otimes A_1)^*$ is the dual of the multiplication map $\alpha : A_1 \otimes A_1 \to A_2$, cf. [Pos21, Proposition 1.2]. *Note that α^* is injective since α is surjective.*

Proposition 5.3.5. *Let $A = T(V)/\langle W\rangle$ be a Koszul algebra. Assume that V is finitely generated projective and $W \subseteq V \otimes V$ is a submodule such that $V \otimes V/W$ is projective. There is a natural isomorphism $(A^!)^! \cong A$.*

Proof. Using Lemma 5.3.2, we have

$$(A^!)^! = T(V^{**})/\langle (W^{\perp})^{\perp}\rangle \cong T(V)/\langle W\rangle = A. \qquad \square$$

The following proposition is found in [Wu16, Example 5.1.8] in the case where k is a field.

Proposition 5.3.6. *Let V be a finitely generated projective k-module. The symmetric algebra $S(V)$ and exterior algebra $\Lambda(V)$ are Koszul dual to each other.*

Proof. We consider the symmetric algebra of V, $S(V) = T(V)/\langle W_s\rangle$, where $\langle W_s\rangle$ is generated by $W_s = \{x \otimes y - y \otimes x \mid x, y \in V\}$ and the exterior algebra of V^*, $\Lambda(V^*) = T(V^*)/\langle W_e\rangle$, where $\langle W_e\rangle$ is generated by $W_e = \{\delta \otimes \delta \mid \delta \in V^*\}$. For any $\delta \in V^*$ and $x, y \in V$, we have

$$(\delta \otimes \delta)(x \otimes y - y \otimes x) = \delta(x)\delta(y) - \delta(y)\delta(x) = 0.$$

This implies that

$$W_e \subseteq W_s^{\perp}.$$

Now we show the reverse inclusion $W_s^{\perp} \subseteq W_e$. First, we assume that V is finite free. Let $\{e_1, \ldots, e_r\}$ be a basis of V and $\{e_1^*, \ldots, e_r^*\}$ be the dual basis of V^*. Let

$\varphi \in W_s^\perp \subseteq V^* \otimes V^*$. We express φ in terms of the basis elements $e_i^* \otimes e_j^*$ of $V^* \otimes V^*$ as follows:

$$\varphi = \sum_{i,j=1}^{r} \alpha_{ij}(e_i^* \otimes e_j^*)$$

where $\alpha_{ij} \in k$. Using Lemma 2.7.3, the basis of W_e in terms of the basis elements $\{e_i^* \otimes e_j^* \mid 1 \leq i, j \leq r\}$ is

$$\{e_i^* \otimes e_j^* + e_j^* \otimes e_i^* \mid 1 \leq i < j \leq r\} \cup \{e_i^* \otimes e_i^* \mid 1 \leq i \leq r\}.$$

The element φ pairs trivially with all of W_s, in particular:

$$\begin{aligned} \varphi(e_k \otimes e_\ell - e_\ell \otimes e_k) &= 0 \\ \sum_{i,j=1}^{r} \alpha_{ij}(e_i^* \otimes e_j^*)(e_k \otimes e_\ell - e_\ell \otimes e_k) &= 0 \\ \sum_{i,j=1}^{r} \alpha_{ij}(e_i^* \otimes e_j^*)(e_k \otimes e_\ell) - \sum_{i,j=1}^{r} \alpha_{ij}(e_i^* \otimes e_j^*)(e_\ell \otimes e_k) &= 0 \\ \alpha_{k\ell} - \alpha_{\ell k} &= 0 \quad \text{by} \quad \text{Proposition 2.7.7} \\ \alpha_{k\ell} &= \alpha_{\ell k} \end{aligned}$$

where $1 \leq k < \ell \leq r$. We obtain

$$\begin{aligned} \varphi &= \sum_{i=1}^{r} \alpha_{ii}(e_i^* \otimes e_i^*) + \sum_{1 \leq i < j \leq r} \alpha_{ij}(e_i^* \otimes e_j^*) + \sum_{1 \leq i < j \leq r} \alpha_{ji}(e_j^* \otimes e_i^*) \\ &= \sum_{i=1}^{r} \alpha_{ii}(e_i^* \otimes e_i^*) + \sum_{1 \leq i < j \leq r} \alpha_{ij}(e_i^* \otimes e_j^* + e_j^* \otimes e_i^*). \end{aligned}$$

Hence, φ is a k-linear combination of the basis of W_e, i.e., φ lies in W_e. Thus, we have

$$W_s^\perp \subseteq W_e$$

and therefore

$$W_s^{\perp} = W_e.$$

We deduce

$$(S(V))^! = T(V^*)/\langle W_s^{\perp}\rangle = T(V^*)/\langle W_e\rangle = \Lambda(V^*)$$

and

$$(\Lambda(V^*))^! = (S(V))^{!!} \cong S(V).$$

In the case where V is finite free, the natural inclusion $W_e \subseteq W_s^{\perp}$ is an isomorphism. Using the fact that a finitely generated projective module is a retract of a finite free module, we have the same natural isomorphism between W_e and $W_s^{\perp}$ (see Lemma 2.6.3). Thus, we get the same above results when V is finitely generated projective. □

Example 5.3.7. *Let $A = k[x, y]$ be a polynomial algebra in two variables. We present the algebra A as a quotient of a tensor algebra, $A = T(V)/\langle R\rangle$ where $V = k \oplus k$ is the free module of rank 2 generated by x and y and the set $R = \{x \otimes y - y \otimes x\}$. The Koszul dual of A is $A^! = T(V^*)/\langle R^{\perp}\rangle$ where $R^{\perp} = \{x^* \otimes y^* + y^* \otimes x^*, x^* \otimes x^*, y^* \otimes y^*\}$. Thus*

$$A^! = \Lambda(x^*, y^*),$$

i.e., the exterior algebra on the dual generators.

5.4 Multiplicative structure using Koszul duality for algebras

In this section we prove the classical facts that the Ext-algebra over a symmetric algebra is exterior and the Ext-algebra over an exterior algebra is polynomial using Koszul duality for algebras as outlined in [Tam18].

Theorem 5.4.1. *Let k be a commutative ring and $A = T(V)/\langle W\rangle$ a quadratic algebra. Assume that V is finitely generated projective and $W \subseteq V \otimes V$ is a submodule such that $V \otimes V/W$ is projective. If A is a Koszul algebra, then the* Ext*-algebra* $\mathrm{Ext}_A(k,k)$ *is isomorphic to the Koszul dual algebra $A^!$, i.e.,*

$$\mathrm{Ext}_A(k,k) \cong A^!$$

[PP05, §2.1 Definition 1]*.*

Remark 5.4.2. *In* [Pos21, Proposition 2.23]*, the author states that the* Ext*-algebra is isomorphic to the opposite of the Koszul dual algebra $A^!$, considering the definition of the natural pairing between $V \otimes V$ and $V^* \otimes V^*$ by*

$$(f \otimes g)(u \otimes v) = g(u)f(v),$$

where $f, g \in V^*$ and $u, v \in V$ [Pos21, Lemma 1.1, §3.4]*. This different pairing formula leads to a different definition of the Koszul dual algebra $A^!$, namely the opposite of the quadratic dual algebra $A^!$ defined in* [PP05, §1.2]*. In* [PP05, §2.1 Definition 1]*, it* *is stated that the* Ext-*algebra is isomorphic to the Koszul dual algebra* $A^!$ *as a result of defining the natural pairing between* $V \otimes V$ *and* $V^* \otimes V^*$ *by*

$$(f \otimes g)(u \otimes v) = f(u)g(v),$$

where $f, g \in V^*$ *and* $u, v \in V$. *In Lemma 2.6.10, we used the pairing formula which is used in* [PP05, §1.2] *to define the natural pairing between* $M \otimes N$ *and* $M^* \otimes N^*$.

Proposition 5.4.3. *Let* k *be a commutative ring and* V *a finite free* k*-module.*

1. *The* Ext*-algebra over a symmetric algebra is exterior:*

$$\mathrm{Ext}_{S(V)}(k, k) \cong \Lambda(V^*).$$

2. *The* Ext*-algebra over an exterior algebra is polynomial:*

$$\mathrm{Ext}_{\Lambda(V^*)}(k, k) \cong S(V).$$

Proof. The symmetric algebra $S(V)$ and exterior algebra $\Lambda(V^*)$ are Koszul. Using Theorem 5.4.1 and Proposition 5.3.6, we have

$$\mathrm{Ext}_{S(V)}(k, k) \cong (S(V))^! = \Lambda(V^*)$$

and

$$\mathrm{Ext}_{\Lambda(V^*)}(k, k) \cong (\Lambda(V^*))^! = S(V).$$ □

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