UNDERGRADUATE RESEARCH PROJECTS

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1. Cohomology of graded Lie algebras

Lie algebras appear in many areas of mathematics, such as representation theory, differential geometry, topology, and deformation theory. Given a Lie algebra L and a module M over L, there is a notion of cohomology groups $H^*(L; M)$ of L with coefficients in M. This can be generalized to graded Lie algebras.

Goals.

(1) Learn the basics about Lie algebras and their cohomology.
(2) Familiarize oneself with existing software to compute Lie algebra cohomology[1].
(3) Write a program to compute the cohomology of a *graded* Lie algebra with coefficients in a module. Compute some examples.

My motivation. These cohomology groups appear in the classification of rational homotopy types X having prescribed homotopy groups $\pi_* X$ endowed with Whitehead products. This structure is called a *rational* Π-*algebra*.

References. [CE48], [Kos50], [HS97, §VII], [Wei94, §7], [NR66].

Date: December 26, 2020.

[1]See for instance the overview here:
`https://mathoverflow.net/questions/43665/compute-lie-algebra-cohomology`

2. The homology of filtered dg-algebras

Differential graded algebras (*dg-algebras* for short) are important both in algebra and in topology. Given a dg-algebra A, one often wants to compute its homology H_*A. A filtration on A yields a spectral sequence that computes H_*A starting from the homology of the filtration quotients.

Goals.

(1) Learn the basics about dg-algebras, dg-modules, and their homology.
(2) Learn about the spectral sequence computing the homology of a filtered dg-algebra. Work out some small examples.
(3) Write code to compute the homology of a dg-algebra and the spectral sequence associated to a filtration.

My motivation. I want tools to compute examples of May's convergence theorem, about the behavior of Massey products in the spectral sequence.

References. [GM03, §V.3], [May69], [Wei94, 4.5.2, 5.4.8, 6.5.11] [LV12, §1.5], [BMR14].

3. Topological data analysis using persistent homology

Big data sets can be difficult to analyze using traditional statistics. Topology provides tools to describe the shape of big data sets that work well in high dimension and are robust to noise. The main tool in topological data analysis is persistent homology, which was developed in the early 2000s.

Goals.

(1) Learn the basics about the Rips and Čech complexes and persistent homology[2].
(2) Familiarize oneself with software[3] to compute persistent homology, such as JavaPlex [MS19, §7].
(3) Use those computations to analyze a real-life data set.

My motivation. I know the basics of persistent homology from a theoretical perspective. I would like some assistance with the computational implementations.

References. [Wei11], [Car09], [Ghr08], [EH08], [EH10, §VII], [Ede14, §11], [MS19].

[2]See this delightful introduction by Matthew Wright: `https://www.youtube.com/watch?v=2PSqWBIrn90`

[3]See an overview here: `https://people.maths.ox.ac.uk/otter/PH_programs`

4. Categorical aspects of graphs

Graph theorists solve actual problems about graphs: counting problems, extremal problems, colorings, embeddings, etc. One can also view graphs as an algebraic structure and study the category of all graphs, instead of properties of individual graphs. One then runs into several variants of graphs, for instance:

- Are the edges directed or undirected?
- Are loops allowed?
- Are multiple edges allowed?
- Is the collapsing of edges allowed?
- Are the vertices ordered?

These variants yield different categories with different properties. The goal of the project is to sort out how those different categories are related.

Goals.

(1) Describe the various categories of graphs, with inclusion and forgetful functors between them.
(2) Describe various adjoint pairs between those categories.
(3) Describe limits and colimits in those categories, in particular examples that are not preserved by certain functors.

My motivation. Graphs and graph-like structures appear in topology (e.g. simplicial complexes, covering spaces) and in homotopy theory (e.g. free resolutions of categories, operads). A guide to the variants of graphs would come in handy. On the expository front, it will illustrate through concrete examples some concepts of category theory and universal algebra that may appear dry.

References. [Rie17, 1.1.3(vi), 3.5.iii, 5.5.7(iv), 6.2.i], [AR94, 1.2(3), 1.50(5), 2.57(3), 3.20(4), 3.35(2)], [ARV11, 1.16, 1.23, 4.6(4), 10.2(2), 10.10(2), 11.18, 11.20].

5. Stable module categories

Representation theory of groups studies groups by examining how they act on other objects. A **representation** of a group G over a field k is a k-vector space on which G acts. This is the same as a (left) module over the group algebra kG. When the characteristic of k divides the order of G, kG-modules behave very differently, which is the subject of modular representation theory. We will focus on the case $\mathrm{char}(k) = p$ and G is a p-group, that is, with order $|G| = p^k$ for some $k \geq 1$.

The kG-modules that are not projective are of particular interest. The **stable module category** $\mathrm{StMod}(kG)$ is defined by quotienting out the homomorphisms that factor through a projective module. This category is an example of a triangulated category.

Goals.

(1) Learn the basics about stable module categories, notably their triangulated structure.
(2) Familiarize oneself with existing code to compute maps in the stable module category[4].
(3) Write code (in GAP or Sage) for further computations in the stable module category. The following features are desirable:
   - Compute the cofiber of a map, that is, extend any map to an exact triangle.
   - Test whether a candidate triangle is exact.
   - Test whether two exact triangles are isomorphic.
   - Compute all the cofiber fill-ins starting from a commutative square.

My motivation. Stable module categories provide good testing ground for some of my work on triangulated categories.

References. [Ben98, §2.1], [Car96, §5], [CTVEZ03, §2.6], [CF17, Appendix A], [Wei94, §10.2], [Nee01, §1].

[4]Peter Webb has some GAP code for group representations, some of which is useful for the stable module category: `https://www-users.math.umn.edu/~webb/GAPfiles/grouprepstutorial.html`
Dan Christensen provided some additional GAP code.

6. The tensor product of Beck modules

Definition 6.1. Let $\mathcal{C}$ be an algebraic category and X an object in $\mathcal{C}$. A **Beck module** over X is an abelian group object in the slice category $\mathcal{C}/X$.

The category $(\mathcal{C}/X)_{\mathrm{ab}}$ of Beck modules over X is an abelian category. This notion of module recovers familiar notions of module in various categories, as illustrated in the following table.

Category	**Object**	**Beck modules over the object**
abelian groups	A	abelian groups
groups	G	G-modules in the usual sense
rings	R	R-bimodules
commutative rings	R	R-modules in the usual sense
Lie algebras	$\mathfrak{G}$	$\mathfrak{G}$-modules in the usual sense

Since the Lawvere theory of abelian groups is commutative, abelian group objects in a category admit a tensor product [Bor94, §3.10]. In particular, the category $(\mathcal{C}/X)_{\mathrm{ab}}$ of Beck modules over X has a tensor product.

Goals.

(1) Learn the basics about the tensor product $\otimes_R$ of modules over a (commutative) ring R.
(2) Familiarize oneself with Beck modules and how they recover familiar notions of modules in the five categories listed above.
(3) Work out the tensor product of Beck modules in the five categories listed above, and show that it agrees with the usual notion of tensor product. Work out other examples.

My motivation. In an algebraic category $\mathcal{C}$, the Quillen cohomology $\mathrm{HQ}^*(X; M)$ of an object X with coefficients in a Beck module M is always available. I want to investigate the Quillen homology $\mathrm{HQ}_*(X; M) := \pi_*(\mathbb{L}_X \otimes M)$ of X with coefficients in M, which requires understanding the tensor product of Beck modules.

References. [Bec03], [Bar02, §6], [Qui67, §II.5], [Qui70, §2–4], [GS07, §4.4], [Fra15, §1–2].

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