Modular Operads from Moduli Spaces

Riemann Surfaces: Compact, Connected 1-dimensional C manifolds
2-dim IR manifold oriented

[Classification of Surfaces:]
Any 2-dim, connected, compact IR-manifold is homeomorphic to the connected sum of some
1. Spheres \( \leadsto \text{ Sphere} \)
2. Tori \( g=0 \)
3. \( \mathbb{RP}^2 \to \text{ Not orientable} \) \( g=3 \)

Topologically, Riemann Surfaces are the connected sum of \( g \) tori (\( g=0 \) is taken to be the sphere)

Q: What are the possible \( \mathbb{RP}^2 \) for a given genus \( g \)?

Moduli Spaces
Points in space = Isomorphism Classes
Maps into space = Families of objects

ex. Triangles
Moduli Space of Triangles

Isomorphisms: Similarity
\[
\begin{align*}
\triangle \sim \triangle & \quad \text{perimeter } P = \frac{\alpha + \beta + \gamma}{\text{perimeter } = 1} \\
& \quad \text{perimeter } = 1
\end{align*}
\]

Now consider \( (a, b, c) \in \mathbb{R}^3 \mid a \leq b \leq c, a + b + c = 1, c < a + b \)
\[
\begin{align*}
\triangle & \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \\
& \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)
\end{align*}
\]

Consider Families of Triangles

1. Parameterize families by topological spaces

2. Two families are equivalent if we have a continuous isomorphism between them

\[
\begin{align*}
\Delta I \quad \text{I} \rightarrow \text{X} = \text{same family}
\end{align*}
\]

\[
\begin{align*}
\Delta & \quad \text{I} = \text{I} \quad \text{I}
\end{align*}
\]

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Automorphisms Prevent
Fine Moduli Spaces
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Moduli Stack
Moduli Space of Riemann Surfaces

Appropriate notion: \( R^S \) with marked points

\( \mathcal{M}_{g,n} \) is the moduli space of \( R^S \) of genus \( g \) with \( n \) marked points

\[ \begin{align*}
\text{ex. } \mathcal{M}_{0,3} & \quad \text{ genus } 0 \rightarrow \text{ Riemann Sphere} \\
y = \frac{ax + b}{cx + d} & \quad \text{ ad- \& bc \neq 0 } (a, b, c, d)
\end{align*} \]

\[ \mathcal{M}_{0,3} = \{ x \} \rightarrow \text{ Actual Moduli Space} \]

\[ \begin{align*}
\text{ex. } \mathcal{M}_{0,4} & \quad (x^2 + 1) \\
p_1 \rightarrow & \quad 0 \\
p_2 \rightarrow & \quad 1 \\
p_3 \rightarrow & \quad \infty
\end{align*} \]

\[ \mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \]

\[ \mathcal{M}_{0,n} = \mathcal{M}_{0,4} \setminus \Delta \]

Compactification:

\( \mathcal{M}_{g,n} \) is the Deligne–Mumford compactification of \( \mathcal{M}_{g,n} \)

Consider \( R^S \) with marked points and nodal points with the condition:

\[ 2g + n - 2 > 0 \quad \text{ [Stability]} \]

Finite automorphism group
String Theory

World line \{ S \} \subset \text{time} \quad \text{Particle}

String \text{time} \quad \text{World sheet}

Path of the particle in spacetime

World sheets = Riemann Surfaces with marked points

\[ -\infty < t \rightarrow \infty \]

Interactions of Strings:

2 Types

1. \( S \to S \)

2. \( S \to S \to S \)

Example:

\[ \infty \quad \infty \quad \infty \]

So, \( M_n \) are important for string theory

Compactification:

Universe (spacetime)

\( M \) Symplectic manifold

Let \( \mathcal{E} \) be a \( \mathbb{R}^8 \) and \( X \) be a symplectic manifold

\[ \mathcal{E} \to X \]

Pseudoholomorphic

\( 5 \) Compactification \to \text{Stable maps} \]
Moduli Spaces: $M_{g,n}$

Dual Graph

$\mathcal{H}_{1,2}$

Modular Operad

Ordinary (symmetric) operads: $\mathcal{Y}$

Modular operads:

Cyclic Operads:

$\Sigma_3:\begin{array}{c} 1 \rightarrow \square \rightarrow 0 \\ \text{Symmetry}
\end{array}$

$C_q = \langle x \rangle$

Cyclic operads don’t distinguish between inputs / outputs

Composition: $O(n) \otimes O(m) \rightarrow O(n+m-2)$
Modular Operad

Composition: $O(g, n) @ O(g', n') \rightarrow O(g + g', n + n' - 2)$

Contraction: $O(g, n) \rightarrow O(g + 1, n - 2)$

Indexing: \[ \{ O(g, n) \mid 2g + n \geq 2 \} \]

if $2g + n - 2 < 0$

then $O(g, n) = 0$

Composition:

\[ A, g = 1 \leftarrow \begin{array}{c} \uparrow \\ \downarrow \\ A, g = 3 \end{array} \rightarrow \begin{array}{c} \uparrow \\ \downarrow \\ B, g = 2 \end{array} \]

Contraction:

\[ A, g \]

\[ A, g + 1 \]

ex. $\{ \overline{M}_{g, n} \}$ is a modular operad

ex. $\{ \overline{H}, \overline{M}_{g, n} \}$ modular operad

$\sqrt{Abelian}$ over this $= Cohomology$ Field theories

$\Rightarrow$
Opeads, Cyclic, Modular

Symmetric Opead

\[ \Sigma \]

Object: \( \mathbb{C} \) \( \mathbb{C} \)

Morphism: bijective

\( O : \Sigma \to C \)

\( O([n]) = O(n) \)

\( O([i, j]) = S_n \) actions

\( O : \text{Fin} \to C \)

Need to define \( O(X) \); \( X \) : finite set

\( |X| = n \)

\[ O(X) = \left( \bigoplus_{f : [n] \to X} O(n) \right) S_n \]

\( \alpha \cdot x = x \)

\( T \) is a tree

\[ O(T) = \bigotimes_{\text{vertices in } T} O(\text{input } T) \]

\[ MO = \colim_T O(T) \]

\( O(9, 1) \)

1. Rooted trees \( \to \) Symmetric Opeads
2. Unrooted trees \( \to \) Cyclic Opeads
3. Dual graph \( \to \) Modular Opeads
Cohomological Field Theory

In physics $\rightarrow$ Correlation functions

$$\langle x, y \rangle$$
probability of a particle going from $x \rightarrow y$

$$\langle x_1, \ldots, x_n \rangle$$ all of $x_i$ are related

$$H^\bullet(X) \cong H^\bullet(\overline{M}_{g,n}(X, \beta))$$

Operations of modular operad $H^\bullet(\overline{M}_{g,n})$
become correlation functions