Triangulated categories and other n-angulations

Sebastian H. Martensen

Norwegian University of Science and Technology

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Overview

1. Triangulated categories
   - History
   - Definition
   - Example: The stable category of a Frobenius category
   - A few properties
2. n-angulated categories
   - Definition
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3. Main example
   - $(n-2)$-cluster tilting categories
   - Concrete examples

History

Introduced independently by

- Puppe (1962): studying stable homotopy theory.
- Verdier (1963): studying algebraic geometry (derived categories).

Definition

Let $\mathcal{T}$ be an additive category equipped with an additive autoequivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and a class of diagrams, Δ, in $\mathcal{T}$ of the form

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$$

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is in Δ.

(c) Δ is closed under isomorphisms of diagrams.

Definition

(T2) For all distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ the left and right rotations are also in Δ:

$$Y \xrightarrow{\quad g \quad} Z \xrightarrow{\quad h \quad} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

$$\Sigma^{-1} Z \xrightarrow{-\Sigma^{-1} h} X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad g \quad} Z$$

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(T3) Given two distinguished triangles and a commutative square there exists a morphism φ_3 making the entire diagram commute.

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \Sigma A_1 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \Sigma B_1 \end{array}$$

Definition

(T4) (Octahedral axiom) For three distinguished triangles constituting the rows in the diagram below such that the two leftmost squares commute, there is a distinguished triangle filling in the dashed arrows and making the entire diagram commute.

$$
\begin{array}{ccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \Sigma A_1 \\
\| & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \theta_1} & & \| \\
A_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \Sigma A_1 \\
\downarrow{\scriptstyle \alpha_1} & & \| & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\
A_2 & \xrightarrow{\gamma_1} & B_2 & \xrightarrow{\gamma_2} & C_3 & \xrightarrow{\gamma_3} & \Sigma A_2 \\
 & & & & \downarrow{\scriptstyle (\Sigma\alpha_2)\gamma_3} & & \\
 & & & & \Sigma A_3 & &
\end{array}
$$

Example: The stable category of a Frobenius category

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See blackboard $\longrightarrow$

A few properties

For a morphism $g\colon Y \to Z$ in an triangulated category sitting in a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

has weak kernel (fibre) $f\colon X \to Y$ and weak cokernel (cofibre) $h\colon Z \to \Sigma X$.

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Note: All objects in a distinguished triangle "sees" the other two.

A few properties

Given an object T and a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

we obtain a long exact sequence of groups

$$\to \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, Z) \to \mathrm{Hom}(T, \Sigma X) \to \mathrm{Hom}(T, \Sigma Y) \to$$

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- Inspired by subjects in representation theory, namely *cluster tilting theory* and Osamu Iyama's *higher Auslander-Reiten theory*.
- Axiomatization of categories naturally inhabited by "shadows" of bounded long exact sequences of a fixed length but with no "shadows" of short exact sequences.

Preliminaries

Let $\mathcal{C}$ be an additive category with an automorphism $\Sigma\colon \mathcal{C} \to \mathcal{C}$ and $n \geqslant 3$ an integer. A sequence of objects and morphisms in $\mathcal{C}$ of the form

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is called an *n-Σ-sequence*.
The *left* and *right rotations* of the sequence are the n-Σ-sequences

$$A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma A_1 \xrightarrow{(-1)^n \Sigma\alpha_1} \Sigma A_2$$

and

$$\Sigma^{-1} A_n \xrightarrow{(-1)^n \Sigma^{-1}\alpha_n} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} A_n$$

respectively.

Definition (Geiss, Keller, & Oppermann)

Let $\mathscr{N}$ denote a collection of n-Σ-sequences in $\mathcal{C}$. The triple $(\mathcal{C}, \Sigma, \mathscr{N})$ is an *n-angulated category* if they satisfy the following four axioms:

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(N3) Given the solid part of the commutative diagram

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B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & B_n & \xrightarrow{\beta_n} & \Sigma B_1 .
\end{array}$$

with rows in $\mathscr{N}$, the dotted morphisms exist and constitute a morphism of n-Σ-sequence.

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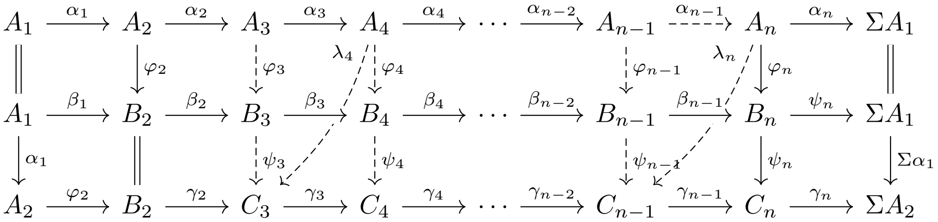

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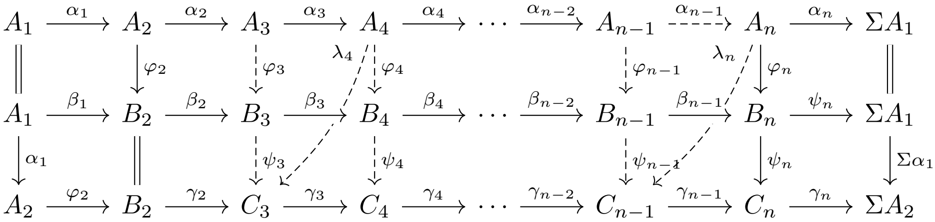

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Definition (Arentz-Hansen, Bergh, & Thaule)

the n-Σ-sequence

$$A_3 \xrightarrow{\begin{pmatrix} \alpha_3 \\ \varphi_3 \end{pmatrix}} A_4 \oplus B_3 \xrightarrow{\begin{pmatrix} -\alpha_4 & 0 \\ \varphi_4 & -\beta_3 \\ \lambda_4 & \psi_3 \end{pmatrix}} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\begin{pmatrix} -\alpha_5 & 0 & 0 \\ -\varphi_5 & -\beta_4 & 0 \\ \lambda_5 & \psi_4 & \gamma_3 \end{pmatrix}} A_6 \oplus B_5 \oplus C_4$$

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belongs to $\mathscr{N}$.

The collection $\mathscr{N}$ is an *n-angulation* of the category $\mathcal{C}$ relative to the automorphism Σ, and the elements of $\mathscr{N}$ are called *n-angles*.

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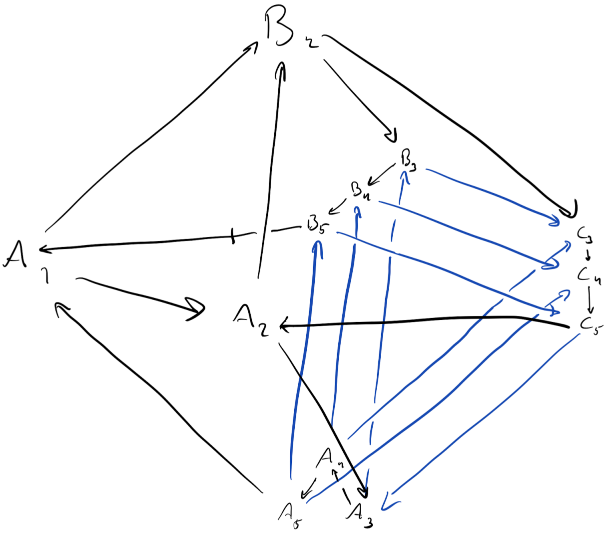
Note: 3-angulated $\Leftrightarrow$ triangulated.

It is still an octahedron!



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Lemma (Geiss, Keller, & Oppermann)

Let $(\mathcal{C}, \Sigma, \mathscr{N})$ be a pre-n-angulated category. If $X_\bullet$ and $Y_\bullet$ are weakly isomorphic exact n-Σ-sequences, $X_\bullet \in \mathscr{N}$ if and only if $Y_\bullet \in \mathscr{N}$.

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Lemma

Weakly isomorphic $\Longrightarrow$ homotopy equivalent as “chain complexes” in $\mathcal{C}$.

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Let $(\mathcal{C}, \Sigma, \mathscr{N})$ be a pre-n-angulated category. Then

$$\text{mod}\, \mathcal{C} := \{M : \mathcal{C}^{\text{op}} \to \boldsymbol{Ab} \mid M = \text{Cok}(\text{Hom}(-, X) \to \text{Hom}(-, Y)) \text{ for some } X, Y \text{ in } \mathcal{C}\}$$

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is an abelian Frobenius category. Furthermore, if $\mathcal{C}$ has split idempotents, $\mathcal{C}$ sits in $\operatorname{mod} \mathcal{C}$ *as the full subcategory of injective objects via the identification*

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If $\mathcal{C}$ is a k-category such that

$$\operatorname{Hom}_k(\operatorname{Hom}_{\mathcal{C}}(X, Y), k) \cong \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-d} X, Y)$$

for all X, Y in $\mathcal{C}$, the category $\mathbf{S}(\mathcal{C})$ is a $(nd-1)$-*Calabi–Yau category.*

Examples

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- Cluster tilting subcategories of triangulated categories.
- Exotic n-angulated categories.
- That's it.

Main example: $(n-2)$-cluster tilting categories

Definition

Let $\mathcal{T}$ be a triangulated category with suspension Σ_3. A full subcategory $\mathcal{C}$ of $\mathcal{T}$ is called *d-cluster tilting* if it is *functorially finite* (see blackboard) and

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Due to Iyama, although he called them maximal $(d-1)$-orthogonal subcategories.

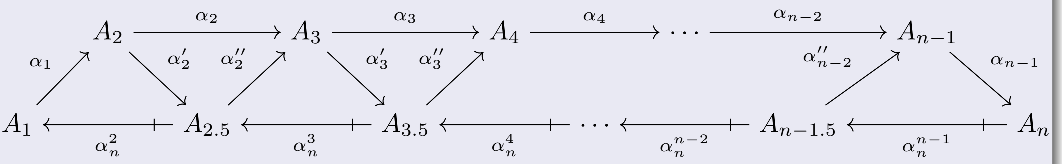
Main example: $(n-2)$-cluster tilting categories

Theorem (Geiss, Keller, Oppermann)

Let $(\mathcal{T}, \Sigma, \Delta)$ be a triangulated category with an $(n-2)$-cluster tilting subcategory $\mathcal{C}$ such that $\Sigma^{n-2}\mathcal{C} \simeq \mathcal{C}$. Then $(\mathcal{C}, \Sigma^{n-2}, \mathscr{N})$ is an n-angulated category, where $\mathscr{N}$ is the class of all sequences

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} \Sigma^{n-2}A_1$$

such that there exists a diagram



with $A_i \in \mathcal{T}$ for $i \notin \mathbb{Z}$, such that $(\alpha''_i, \alpha'_{i+1}, \alpha_n^{i+1}) \in \Delta$ for $i \in \{1, \ldots, n-2\}$, where $\alpha''_1 := \alpha_1$ and $\alpha'_{n-1} := \alpha_{n-1}$, and such that $\alpha_i = \alpha''_i \alpha'_i$ for $i \in \{2, \ldots, n-2\}$ and $\alpha_n = (\Sigma^{n-3}\alpha_n^2)(\Sigma^{n-4}\alpha_n^3)\cdots(\Sigma\alpha_n^{n-2})\alpha_n^{n-1}$.

Concrete examples

Let A be a finite dimensional k-algebra of global dimension at most $n-2$ and consider the autoequivalence

$$\mathbb{S} := - \otimes_A \operatorname{Hom}_k(A, k).$$

If mod A has a $(n-2)$-*cluster tilting object*, we get a $(n-2)$-cluster tilting subcategory of $D^b(A)$ given by

$$\mathcal{C} = \operatorname{add}\{\mathbb{S}^i A[-(n-2)i] \mid i \in \mathbb{Z}\}.$$

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Let A be a finite dimensional k-algebra of global dimension at most $n-2$ and consider the autoequivalence

$$\mathbb{S} := - \otimes_A \operatorname{Hom}_k(A, k).$$

If mod A has a $(n-2)$-*cluster tilting object*, we get a $(n-2)$-cluster tilting subcategory of $D^b(A)$ given by

$$\mathcal{C} = \operatorname{add}\{\mathbb{S}^i A[-(n-2)i] \mid i \in \mathbb{Z}\}.$$

Quiver example (see blackboard).

Final remarks

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Thank you for your attention!

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