PH.D. RESEARCH PROJECTS

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1. Stable homotopy groups from unstable homotopy groups

1.1. **Background.** The homotopy groups of a space form a Π-algebra, whereas the homotopy groups of a spectrum form a π_*-module, where $\pi_* = \pi_*(S^0)$ denotes the stable homotopy ring. One would like investigate the relationship between these two algebraic structures. How much information about stable homotopy operations can be recovered from unstable homotopy operations? More precisely, consider the diagram:

$$\begin{array}{ccc} \mathbf{Top}_* & \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} & \mathbf{Sp} \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ \mathbf{\Pi Alg} & \underset{\Omega^\infty}{\overset{\mathrm{Stab}}{\rightleftarrows}} & \mathbf{Mod}_{\pi_*} \end{array}$$

where the top row consists of the suspension spectrum functor Σ^∞ from (pointed) topological spaces to spectra and its right adjoint Ω^∞, the "zeroth space" functor. The bottom functor Ω^∞ is the restriction functor that makes unstable maps between spheres act via their stabilization, and it commutes with homotopy: $\Omega^\infty \pi_* = \pi_* \Omega^\infty$. The stabilization functor $\mathrm{Stab}\colon \mathbf{\Pi Alg} \to \mathbf{Mod}_{\pi_*}$ is the left adjoint of Ω^∞, and is the functor that comes as close as possible to making the diagram commute.

There is a comparison map $\mathrm{Stab}(\pi_* X) \to \pi_*^{\mathrm{st}} X = \pi_*(\Sigma^\infty X)$ for every space X, which is an isomorphism for free objects, i.e., wedges of spheres. Hence, one can obtain a spectral sequence converging to the stable homotopy groups $\pi_*^{\mathrm{st}} X$ via a Stover resolution of X by spheres, analogous to the Hurewicz spectral sequence of [Bla90].

1.2. **Project.** The goals of the project are the following.

(1) Study the properties of the functor Stab, tools to compute it, as well as its left derived functors.
(2) Construct a natural spectral sequence of the form
$$(L_* \mathrm{Stab})(\pi_* X) \Rightarrow \pi_*^{\mathrm{st}}(X)$$
computing the stable homotopy groups of a space X, starting from the derived functors of stabilization applied to the Π-algebra $\pi_* X$.
(3) Investigate convergence and vanishing lines in the spectral sequence.
(4) Apply the spectral sequence to Eilenberg–MacLane spaces $K(G, n)$.
(5) Apply the spectral sequence to some finite CW complexes.

This project was inspired by discussions with Haynes Miller and David Blanc.

References: [Sto90], [Bla90], [Bla94], [DKSS94].

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2. Operations on Quillen cohomology

2.1. **Background.** André–Quillen cohomology is a cohomology theory for commutative rings. It was introduced in the 1960s as a tool to solve problems in commutative algebra using methods from homotopy theory [And67] [Qui70]. Since then, it has found many applications in topology and in algebra [GS07]. The construction is available in any algebraic category $\mathcal{C}$, not just commutative rings. One can define the Quillen cohomology $\mathrm{HQ}^*(X; M)$ of an object X with coefficients in a Beck module M over X.

Quillen cohomology $\mathrm{HQ}^*(X; M)$ is more than a graded abelian group: it carries the additional structure of cohomology operations. These are not well understood in general, but have been studied in certain cases. For (supplemented) commutative $\mathbb{F}_2$-algebras, Quillen cohomology is equipped with a Lie bracket and Steenrod-type operations [Goe90]. Since Quillen cohomology is defined via simplicial methods, one expects a relationship between cohomology operations and homotopy operations, found in the homotopy $\pi_*(X_\bullet)$ of a simplicial object $X_\bullet$. Given a simplicial commutative $\mathbb{F}_2$-algebra $X_\bullet$, its homotopy $\pi_*(X_\bullet)$ is a commutative $\mathbb{F}_2$-algebra equipped with higher divided squares [Dwy80] [Goe90].

More generally, homotopy operations have been studied for algebras over an operad P [Fre00] [Fre09]. Fresse showed that the homotopy $\pi_*(X_\bullet)$ of a simplicial P-algebra is a P-algebra with divided symmetries, a structure described more explicitly by Ikonicoff [Iko20]. The example of commutative $\mathbb{F}_2$-algebras suggests that Koszul duality controls the relationship between homotopy and cohomology operations. For p odd, operations on Quillen cohomology of commutative $\mathbb{F}_p$-algebras have been computed as a graded $\mathbb{F}_p$-vector space by Arone, Brantner, and Mathew [AB21] [BM19]. In the spectral context, operations on topological André–Quillen cohomology of E_∞ $H\mathbb{F}_p$-algebras have been computed by Zhang [Zha25].

2.2. **Project.** The goals of the project are the following.

1. Study the algebraic theory that controls operations on Quillen cohomology. Determine conditions under which that theory is operadic.
2. Specialize to the case where the algebraic category $\mathcal{C}$ is the category of algebras over an operad P.
3. Investigate to what extent Koszul duality controls the relationship between homotopy and cohomology operations.
4. Work out explicit examples other than commutative $\mathbb{F}_2$-algebras.
5. Determine the relations among operations on Quillen cohomology of commutative $\mathbb{F}_p$-algebras for p odd, building on the results of [AB21] [BM19] [Zha25].

This project was inspired by discussions with Benoit Fresse, Lukas Brantner, and Adela Zhang.

References: [Qui67], [Qui70], [Dwy80], [Goe90], [GL95], [Fre00], [Fre09], [Pri70], [Pri10, §5], [AB21], [BM19], [Zha25].

3. The Künneth spectral sequence in triangulated categories

3.1. **Background.** The classical Künneth theorem in topology computes the homology of a product $X \times Y$ from the homology of the two spaces X and Y. For integral homology, there is a natural short exact sequence of abelian groups

$$0 \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \to H_n(X \times Y) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(X), H_j(Y)) \to 0$$

[Hat02, §3.B] [Bre97, §VI.1]. For a general commutative ring R, there is a natural spectral sequence of R-modules

$$E^2_{p,q} = \bigoplus_{i+j=q} \mathrm{Tor}_p^R(H_i(X;R), H_j(Y;R)) \Rightarrow H_{p+q}(X \times Y; R).$$

More generally, for an E_1 ring spectrum E and spectra X and Y, there is a natural spectral sequence of abelian groups

$$E^2_{p,q} = \mathrm{Tor}_p^{E_*}(E_*(X), E_*(Y))_q \Rightarrow E_{p+q}(X \wedge Y).$$

Yet more generally, for a right E-module M and a left E-module N, the Künneth spectral sequence has the form

$$E^2_{p,q} = \mathrm{Tor}_p^{E_*}(\pi_*(M), \pi_*(N))_q \Rightarrow \pi_{p+q}(M \wedge^{\mathbf{L}} N)$$

[EKMM97, Theorem IV.4.7]. There is also a generalization to stable ∞-categories [Lur17, Proposition 7.2.1.19].

3.2. **Project.** The Adams spectral sequence can be constructed in any triangulated category with a choice of projective class $\mathcal{P}$ that dictates which Adams resolutions are allowed [Chr98]. Similarly, the goal of the project is to develop the Künneth spectral sequence in a triangulated category relative to a projective class $\mathcal{P}$.

(1) In a symmetric monoidal triangulated category, construct a natural spectral sequence abutting to $\pi_*(X \otimes Y)$ whose E^2 term involves $\mathcal{P}$-relative derived functors.
(2) Find conditions under which the E^2 term has a homological-algebraic description, for instance the Tor groups
$$\mathrm{Tor}_p^{\pi_* \mathbb{1}}(\pi_* X, \pi_* Y)_q$$
over the graded endomorphism ring of the tensor unit $\mathbb{1}$.
(3) Investigate convergence of the spectral sequence as well as vanishing lines.
(4) Specialize the spectral sequence to familiar triangulated categories: modules over an E_∞ ring spectrum, (in particular) DG-modules over a commutative DG-algebra, equivariant stable homotopy categories, motivic stable homotopy categories, derived categories of quasi-coherent sheaves, stable module categories, etc.
(5) Compute some explicit examples.

This project was inspired by discussions with Sean Tilson.

References: [EKMM97, §IV.4], [Lur17, §7.2.1], [Sch12, §II.6], [Til18], [Chr98].

4. Realization of π_*-modules via Toda brackets

4.1. **Background.** Given a spectrum X, its homotopy groups $\pi_*(X)$ form a graded module over the graded ring $\pi_* = \pi_*(S)$ of stable homotopy groups of spheres. A class $\alpha \in \pi_k(S)$ acts by precomposition $\alpha^* \colon \pi_n(X) \to \pi_{n+k}(X)$ as in the diagram

$$S^{n+k} \xrightarrow{\Sigma^n \alpha} S^n \xrightarrow{x} X.$$

The π_*-module $\pi_*(X)$ only encodes primary homotopy operations. The spectrum X has richer higher order information, which is found in the mapping spaces $\mathrm{Map}(S^n, X)$ rather than just $\pi_0 \mathrm{Map}(S^n, X) = [S^n, X] = \pi_n(X)$. Taking the fundamental groupoids $\Pi_1 \mathrm{Map}(S^n, X)$ leads to the notion of *secondary homotopy groups*

$$\pi_{*,*}(X) = \left(\pi_{n,1}(X) \xrightarrow{\partial} \pi_{n,0}(X) \right)$$

developed by Baues and Muro [BM09, BM11]. The secondary homotopy groups of spheres $\pi_{*,*}(S)$ encode all 3-fold Toda brackets of maps between spheres (and a bit more information). Similarly, $\pi_{*,*}(X)$ encodes all secondary operations acting on $\pi_*(X)$, namely 3-fold Toda brackets of maps

$$S^{n+k+\ell} \xrightarrow{\Sigma^{n+k}\beta} S^{n+k} \xrightarrow{\Sigma^n \alpha} S^n \xrightarrow{x} X.$$

as subsets $\langle x, \alpha, \beta \rangle \subseteq [\Sigma S^{n+k+\ell}, X] = \pi_{n+k+\ell+1}(X)$.

4.2. **Project.** A lot is known about Toda brackets in $\pi_*(S)$ [Isa19, IWX23]. The project consists of leveraging that information to address the realizability of π_*-modules as the homotopy of spectra.

(1) Find families of π_*-modules that cannot be endowed with secondary operations, hence are non-realizable. Compare with [Sag08].
(2) Find families of π_*-modules such that endowing them with secondary operations is also a *sufficient* condition for realizability. In particular, investigate π_*-modules concentrated in certain degrees.

This project was inspired by discussions with Dan Isaksen.

References: [BM09, BM11], [Sag08], [Isa19, IWX23], [BF15].

5. Spectral track categories

5.1. **Background.** A **track category** is a category enriched in groupoids, which model homotopy 1-types. For example, starting from a topologically or simplicially enriched category $\mathcal{C}$, taking the fundamental groupoid of each mapping space $\mathcal{C}(X, Y)$ yields a track category $\Pi_1 \mathcal{C}$. Track categories were successfully used to encode secondary structure, notably secondary cohomology operations [Bau06, BJ11] and secondary homotopy groups [BM09, BM11]. However in a stable setting, the category $\mathcal{C}$ is spectrally enriched and the mapping spaces are obtained from the mapping spectra. Hence working with spectra retains more information. The goal is to develop categories enriched in *stable* 1-types.

5.2. **Project.** The project will use the model of stable 1-types by Picard groupoids, proved by Johnson and Osorno [JO12]. The objectives are the following.

(1) Find convenient lax monoidal functors for the connective cover of a spectrum and for the 1-truncation of a connective spectrum. This allows us to extract a spectral track category (i.e., a category enriched in Picard groupoids) from a spectral category, by changing the base of enrichment.
(2) Spell out the algebraic structure of a category enriched in Picard groupoids, and how to compute 3-fold Toda brackets in that structure.
(3) Exhibit how a 1-truncated DG-category $\mathcal{C}$ is a special kind of spectral track category. It was shown in [BF20] that $\mathcal{C}$ is a special kind of track category.
(4) Specialize to the full spectral subcategory of the stable homotopy category on finite products of Eilenberg–MacLane spectra $\Sigma^n H\mathbb{F}_p$. Compare with the work of Baues and Jibladze on secondary cohomology operations [Bau06, BJ11], cf. [BF20].
(5) Specialize to the spectral category of finite wedges of sphere spectra S^n. Compare with the work of Baues and Muro on secondary homotopy groups [BM09, BM11].
(6) Set up a similar construction of spectral 2-track categories, using algebraic models for stable 2-types [GJOS17, GJO19].

This project was inspired by discussions with Angélica Osorno.

References: [JO12, GJOS17, GJO19], [Bau06, BJ11, BF20], [BM09, BM11], [BP11], [BG21].

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