PH.D. RESEARCH PROJECTS

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1. Stable homotopy groups from unstable homotopy groups

1.1. **Background.** The homotopy groups of a space form a Π-algebra, whereas the homotopy groups of a spectrum form a π_*-module, where $\pi_* = \pi_*(S^0)$ denotes the stable homotopy ring. One would like investigate the relationship between these two algebraic structures. How much information about stable homotopy operations can be recovered from unstable homotopy operations? More precisely, consider the diagram:

$$\begin{array}{ccc} \mathbf{Top}_* & \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} & \mathbf{Sp} \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ \mathbf{\Pi Alg} & \underset{\Omega^\infty}{\overset{\mathrm{Stab}}{\rightleftarrows}} & \mathbf{Mod}_{\pi_*} \end{array}$$

where the top row consists of the suspension spectrum functor Σ^∞ from (pointed) topological spaces to spectra and its right adjoint Ω^∞, the "zeroth space" functor. The bottom functor Ω^∞ is the restriction functor that makes unstable maps between spheres act via their stabilization, and it commutes with homotopy: $\Omega^\infty \pi_* = \pi_* \Omega^\infty$. The stabilization functor $\mathrm{Stab}\colon \mathbf{\Pi Alg} \to \mathbf{Mod}_{\pi_*}$ is the left adjoint of Ω^∞, and is the functor that comes as close as possible to making the diagram commute.

There is a comparison map $\mathrm{Stab}(\pi_* X) \to \pi_*^{\mathrm{st}} X = \pi_*(\Sigma^\infty X)$ for every space X, which is an isomorphism for free objects, i.e., wedges of spheres. Hence, one can obtain a spectral sequence converging to the stable homotopy groups $\pi_*^{\mathrm{st}} X$ via a Stover resolution of X by spheres, analogous to the Hurewicz spectral sequence of [Bla90].

1.2. **Project.** The goals of the project are the following.

(1) Study the properties of the functor Stab, tools to compute it, as well as its left derived functors.

(2) Construct a natural spectral sequence of the form

$$(L_* \mathrm{Stab})(\pi_* X) \Rightarrow \pi_*^{\mathrm{st}}(X)$$

computing the stable homotopy groups of a space X, starting from the derived functors of stabilization applied to the Π-algebra $\pi_* X$.

(3) Investigate convergence and vanishing lines in the spectral sequence.

(4) Apply the spectral sequence to Eilenberg–MacLane spaces $K(G, n)$.

This project was inspired by discussions with Haynes Miller.

References: [Sto90], [Bla90], [Bla94], [DKSS94].

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2. Operations on Quillen cohomology

2.1. **Background.** André–Quillen cohomology is a cohomology theory for commutative rings. It was introduced in the 1960s as a tool to solve problems in commutative algebra using methods from homotopy theory [And67] [Qui70]. Since then, it has found many applications in topology and in algebra [GS07]. The construction is available in any algebraic category $\mathcal{C}$, not just commutative rings. One can define the Quillen cohomology $\mathrm{HQ}^*(X; M)$ of an object X with coefficients in a Beck module M over X.

Quillen cohomology $\mathrm{HQ}^*(X; M)$ is more than a graded abelian group: it carries the additional structure of cohomology operations. These are not well understood in general, but have been studied in certain cases. For (supplemented) commutative $\mathbb{F}_2$-algebras, Quillen cohomology is equipped with a Lie bracket and Steenrod-type operations [Goe90]. Since Quillen cohomology is defined via simplicial methods, one expects a relationship between cohomology operations and homotopy operations, found in the homotopy $\pi_*(X_\bullet)$ of a simplicial object $X_\bullet$. Given a simplicial commutative $\mathbb{F}_2$-algebra $X_\bullet$, its homotopy $\pi_*(X_\bullet)$ is a commutative $\mathbb{F}_2$-algebra equipped with higher divided squares [Dwy80] [Goe90].

More generally, homotopy operations have been studied for algebras over an operad P [Fre00] [Fre09]. Fresse showed that the homotopy $\pi_*(X_\bullet)$ of a simplicial P-algebra is a P-algebra with divided symmetries, a structure described more explicitly by Ikonicoff [Iko20]. The example of commutative $\mathbb{F}_2$-algebras suggests that Koszul duality controls the relationship between homotopy and cohomology operations.

2.2. **Project.** The goals of the project are the following.

(1) Study the algebraic theory that controls operations on Quillen cohomology. Determine whether that theory is operadic.
(2) Specialize to the case where the algebraic category $\mathcal{C}$ is the category of algebras over an operad P.
(3) Investigate to what extent Koszul duality controls the relationship between homotopy and cohomology operations.
(4) Work out explicit examples other than commutative $\mathbb{F}_2$-algebras.

This project was inspired by discussions with Benoit Fresse.

References: [Qui67], [Qui70], [Dwy80], [Goe90], [GL95], [Fre00], [Fre09].

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