ENRICHED MODEL CATEGORIES AND THE DOLD-KAN CORRESPONDENCE

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Abstract

The work we present in this thesis is an application of the monoidal properties of the Dold–Kan correspondence and is constituted of two main parts. In the first one, we observe that by a theorem of Christensen and Hovey, the category of non-negatively graded chain complexes of left R-modules has a model structure, called the Hurewicz model structure, where the weak equivalences are the chain homotopy equivalences. Hence, the Dold–Kan correspondence induces a model structure on the category of simplicial left R-modules and some properties, notably it is monoidal. In the second part, we observe that changing the enrichment of an enriched, tensored and cotensored category along the Dold–Kan correspondence does not preserve the tensoring nor the cotensoring. Thus, we generalize this observation to any weak monoidal Quillen adjunction and we give an insight of which properties are preserved and which are weakened after changing the enrichment of an enriched model category along a right weak monoidal Quillen adjoint.

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Dedication

To my lovely daughters

Léana, *Sophia* and *Daphne*

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Transparency Statement

I declare that no AI-assisted technology has been used in the preparation of the thesis.

Chapter 1

Introduction

1.1 Motivation

On the one hand, for a nice bicomplete abelian category $\mathcal{A}$, two of the well-known standard model structures on the category $\mathrm{Ch}(\mathcal{A})$ of unbounded chain complexes of objects of $\mathcal{A}$ are the projective model structure, for which weak equivalences are quasi-isomorphisms and fibrations are degreewise epimorphisms, and the Hurewicz model structure, for which weak equivalences are chain homotopy equivalences and fibrations are degreewise split epimorphisms. May and Ponto [MP12, Chapter 18] gave a clear description of these two model structures along with their monoidal properties when $\mathcal{A}$ is the category Mod_R of left R-modules for a given ring R. The Hurewicz model structure on the category $\mathrm{Ch}(R)$ of unbounded chain complexes of left R-modules was introduced by Golasiński and Gromadzki [GG82], using work of Kamps [Kam78]. It was later constructed using different methods in [Col99], [SV02], [CH02], [Wil13, §I], and [Gil15]. Quillen [Qui67, §II.4] introduced the projective model structure on the category $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ of non-negatively graded chain complexes of objects of $\mathcal{A}$, that is with quasi-isomorphisms as weak equivalences and degreewise monomorphisms with degreewise projective cokernel as cofibrations. One of the motivations of the first part of this work was to complete this square, that is to give an explicit description of the Hurewicz model structure on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. David White [Whi17] pointed out that such a model structure can be obtained using work of Christensen and Hovey [CH02], which was kindly confirmed to us by Dan Christensen. In [CH02], given a bicomplete abelian category $\mathcal{A}$ with a projective class, Christensen and Hovey construct a model structure on $\mathrm{Ch}(\mathcal{A})$ that reflects the homological algebra of the projective class in the sense that it encodes the Ext groups and more general derived functors. A general description of such a model structure on the category $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ of non-negatively graded chain complexes of objects of $\mathcal{A}$ under mild conditions is given by [CH02, Corollary 6.4].

On the other hand, for a $\mathcal{V}$-model category $\mathcal{C}$ and a strong monoidal Quillen adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$, the category $G_*\mathcal{C}$ obtained by changing the enrichment along G is a $\mathcal{W}$-model category [Rie14, Chapter 3 and § 11.3]. But, the Dold–Kan correspondence, as an instance of weak monoidal Quillen adjunctions, is not strong [SS03]. In this case, we are wondering which properties are preserved and which ones might be weakened.

1.2 Main results

In the first part of this work, we show that the Hurewicz model structure on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ satisfies similar properties, in particular it is monoidal, as its analogue on unbounded chain complexes $\mathrm{Ch}(\mathcal{A})$ and the proofs follow mostly from what is done in [MP12, Chapter 18]. The mixed model structure is obtained from the projective and the Hurewicz model structures on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, and **Proposition 3.2.5** and **Proposition 3.2.7** give the enrichment relations between those three structures. Our main contribution in this part is showing that the Hurewicz model structure on the category sMod_R of simplicial R-modules is monoidal whenever R is commutative, see **Proposition 3.3.4**. We also characterize the fibrations and cofibrations respectively in terms of homotopy lifting and homotopy extension, see **Proposition 3.3.3**.

In the second part, we observe in **Proposition 4.1.4** that, for monoidal categories $\mathcal{V}$ and $\mathcal{W}$, after a change of enrichment along the lax monoidal adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$, a necessary condition for a $\mathcal{W}$-enriched category $G_*\mathcal{V}$ to admit a tensoring or a cotensoring over $\mathcal{W}$ is for F to be strong monoidal. But this is not the case for the Dold–Kan correspondence [SS03]. We will thus define weaker versions of the tensoring and the cotensoring in **Definition 4.3.1**, both preserved by any weak monoidal Quillen adjunction. We will also show that other properties of enriched model categories, such as the presence of a model structure on the underlying category and SM7, are preserved after the change of enrichment along a weak monoidal Quillen adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$ under the condition that $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$. Hence, we give some equivalent formulations of the unit axiom in **Corollary 4.4.2** and **Proposition 4.4.3**, we introduce the notion of weak $\mathcal{V}$-model category $\mathcal{C}$ in **Definition 4.5.1**, and we show that in this case, $G_*\mathcal{C}$ is a weak $\mathcal{W}$-model category in **Theorem 4.5.3**.

1.3 Organization

The thesis is organized as follows. In Chapter 2, we will recall some background material on enriched model categories, on monoidal properties of the Dold–Kan correspondence and on projective classes. In Chapter 3, we will describe the Hurewicz model structure on $\mathrm{Ch}_{\geq 0}(R)$ as an instance of the Christensen–Hovey setup and we will show that Dold–Kan correspondence transfers the existing monoidal model structure on $\mathrm{Ch}_{\geq 0}(R)$ to a monoidal model structure on sMod_R. Finally, in Chapter 4, we will introduce the notions of weak $\mathcal{V}$-adjunction, weak $\mathcal{V}$-tensoring, weak $\mathcal{V}$-cotensoring and weak $\mathcal{V}$-model category, and we will show that the three latter ones are preserved by the change of enrichment along a weak monoidal Quillen adjunction. In the last chapter, namely Chapter 5, we give some directions for related future work.

Chapter 2

Preliminaries

The chapter is organized as follows. In Section 2.1, we define and give some of the properties of monoidal and enriched categories. In Section 2.2, we discuss some properties of tensored and cotensored categories. In Section 2.3, we describe some monoidal properties of the Dold–Kan correspondence. In Section 2.4, we define and give some of the properties of monoidal and enriched model categories. In Section 2.5, we describe some facts about model categories and the associated homotopy categories. Finally, in Section 2.6, we define the notion of projective class and give some examples.

2.1 Monoidal categories and enriched categories

Most of the material we review here can be found in [Hov99, Chapter 4], [MP12, Chapter 16], [Rie14, Chapter 3] and [Bor94, § 6].

Definition 2.1.1. *A* ***monoidal category*** *is a category $\mathcal{V}$ equipped with*

- *a bifunctor* $-\otimes- : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$, *called the tensor product,*
- *an object* $\mathbb{1}\in\mathcal{V}$, *called the tensor unit,*
- *a natural isomorphism* $\alpha_{x,y,z} : (x\otimes y)\otimes z \xrightarrow{\cong} x\otimes(y\otimes z)$, *called the associator,*
- *two natural isomorphisms* $\lambda_x : \mathbb{1}\otimes x \xrightarrow{\cong} x$ *and* $\rho_x : x\otimes\mathbb{1} \xrightarrow{\cong} x$, *respectively called left and right unitor,*

satisfying both the triangle and pentagon identities respectively given as follows:

$$\begin{tikzcd}
(x\otimes\mathbb{1})\otimes y \arrow[rr, "\alpha_{x,\mathbb{1},y}"] \arrow[dr, "\rho_x\otimes y"'] & & x\otimes(\mathbb{1}\otimes y) \arrow[dl, "x\otimes\lambda_y"] \\
& x\otimes y, &
\end{tikzcd}$$

$$\begin{tikzcd}
& (w\otimes x)\otimes(y\otimes z) \arrow[dr, "\alpha_{w,x,y\otimes z}"] & \\
((w\otimes x)\otimes y)\otimes z \arrow[ur, "\alpha_{w\otimes x,y,z}"] \arrow[d, "\alpha_{w,x,y}\otimes z"'] & & w\otimes(x\otimes(y\otimes z)) \\
(w\otimes(x\otimes y))\otimes z \arrow[rr, "\alpha_{w,x\otimes y,z}"'] & & w\otimes((x\otimes y)\otimes z). \arrow[u, "w\otimes\alpha_{x,y,z}"']
\end{tikzcd}$$

A monoidal category $\mathcal{V}$ is said to be ***symmetric*** *if it is also equipped with a specified natural isomorphism* $B_{x,y} : x\otimes y \to y\otimes x$, *called braiding, satisfying the hexagon* *identity given by*

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "\alpha_{x,y,z}"] \arrow[d, "B_{x,y\otimes z}"'] & x \otimes (y \otimes z) \arrow[r, "B_{x,y\otimes z}"] & (y \otimes z) \otimes x \arrow[d, "\alpha_{y,z,x}"] \\
(y \otimes x) \otimes z \arrow[r, "\alpha_{y,x,z}"'] & y \otimes (x \otimes z) \arrow[r, "y\otimes B_{x,z}"'] & y \otimes (z \otimes x)
\end{tikzcd}$$

and the identity $B_{y,x} \circ B_{x,y} = \mathrm{id}_{x\otimes y}$.

Definition 2.1.2. *For a symmetric monoidal category $\mathcal{V}$, a* ***$\mathcal{V}$-category*** *(a.k.a. a* ***$\mathcal{V}$-enriched category****) $\mathcal{C}$ consists of*

- *a hom-object $\underline{\mathcal{C}}(x, y) \in \mathcal{V}$, for any objects $x, y \in \mathcal{C}$,*
- *a morphism $1_x : \mathbb{1} \to \underline{\mathcal{C}}(x, x)$ in $\mathcal{V}$, for any object $x \in \mathcal{C}$,*
- *a morphism $\circ : \underline{\mathcal{C}}(y, z) \otimes \underline{\mathcal{C}}(x, y) \to \underline{\mathcal{C}}(x, z)$ in $\mathcal{V}$, for any objects $x, y, z \in \mathcal{C}$,*

such that the following diagrams in $\mathcal{V}$ commute for all $x, y, z, w \in \mathcal{C}$:

$$\begin{tikzcd}
\underline{\mathcal{C}}(z,w) \otimes \underline{\mathcal{C}}(y,z) \otimes \underline{\mathcal{C}}(x,y) \arrow[r, "\mathrm{id}\otimes\circ"] \arrow[d, "\circ\otimes\mathrm{id}"'] & \underline{\mathcal{C}}(z,w) \otimes \underline{\mathcal{C}}(x,z) \arrow[d, "\circ"] \\
\underline{\mathcal{C}}(y,w) \otimes \underline{\mathcal{C}}(x,y) \arrow[r, "\circ"'] & \underline{\mathcal{C}}(x,w),
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathcal{C}}(x,y) \otimes \mathbb{1} \arrow[r, "\mathrm{id}\otimes 1_x"] \arrow[dr, "\cong", "\rho"'] & \underline{\mathcal{C}}(x,y) \otimes \underline{\mathcal{C}}(x,x) \arrow[d, "\circ"] \\
& \underline{\mathcal{C}}(x,y),
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathcal{C}}(y,y) \otimes \underline{\mathcal{C}}(x,y) \arrow[d, "\circ"'] & \mathbb{1} \otimes \underline{\mathcal{C}}(x,y) \arrow[l, "1_y\otimes\mathrm{id}"'] \arrow[dl, "\cong"', "\lambda"] \\
\underline{\mathcal{C}}(x,y). &
\end{tikzcd}$$

Definition 2.1.3 (Closed symmetric monoidal category)**.** *A symmetric monoidal category $\mathcal{V}$ is* ***closed*** *if for all object $v \in \mathcal{V}$, the tensor product functor $- \otimes v : \mathcal{V} \to \mathcal{V}$*

has a right adjoint functor $[v, -] : \mathcal{V} \to \mathcal{V}$*,*

$$- \otimes v : \mathcal{V} \rightrightarrows \mathcal{V} : [v, -] \, ,$$

that is, for any $u, v, w \in \mathcal{V}$*, we have a natural bijection*

$$\mathcal{V}(u \otimes v, w) \cong \mathcal{V}(u, [v, w]), \tag{2.1.1}$$

natural in all arguments. The objects $[v, w]$ *is called the* ***internal hom*** *of* v *and* w*.*

Proposition 2.1.4. *[Rie14, Remark 3.3.9] The hom-set* $\mathcal{V}(-, -)$ *in the isomorphism* (2.1.1) *can be replaced with the internal hom, denoted* $\underline{\mathcal{V}}(-, -) := [-, -]$*, to obtain an analogous isomorphism*

$$\underline{\mathcal{V}}(u \otimes v, w) \cong \underline{\mathcal{V}}(u, \underline{\mathcal{V}}(v, w)),$$

in $\mathcal{V}$*.*

Definition 2.1.5. *For two monoidal categories* $(\mathcal{V}, \otimes, \mathbb{1}_{\mathcal{V}})$ *and* $(\mathcal{W}, \otimes, \mathbb{1}_{\mathcal{W}})$*, a* ***lax monoidal functor*** *is a functor* $G : \mathcal{V} \to \mathcal{W}$ *together with*

- *a morphism* $\eta : \mathbb{1}_{\mathcal{W}} \to G(\mathbb{1}_{\mathcal{V}})$*, called the lax monoidal unit,*
- *a natural transformation* μ*, given by* $\mu_{x,y} : G(x) \otimes G(y) \to G(x \otimes y)$ *for any objects* $x, y \in \mathcal{V}$*, called the lax monoidal transformation,*

satisfying the following conditions.

(i) ***Associativity****: For objects $x, y, z \in \mathcal{V}$, the following diagram in $\mathcal{W}$ commutes*

$$\begin{tikzcd}
(G(x) \otimes G(y)) \otimes G(z) \arrow[r, "\alpha"] \arrow[d, "\mu_{x,y} \otimes G(z)"'] & G(x) \otimes (G(y) \otimes G(z)) \arrow[d, "G(x) \otimes \mu_{y,z}"] \\
G(x \otimes y) \otimes G(z) \arrow[d, "\mu_{x \otimes y, z}"'] & G(x) \otimes G(y \otimes z) \arrow[d, "\mu_{x, y \otimes z}"] \\
G((x \otimes y) \otimes z) \arrow[r, "G(\alpha)"'] & G(x \otimes (y \otimes z)).
\end{tikzcd}$$

(ii) ***Unitality****: For objects $x \in \mathcal{V}$, the following diagram in $\mathcal{W}$ commutes*

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{W}} \otimes G(x) \arrow[r, "\eta \otimes G(x)"] \arrow[d, "\lambda_{G(x)}"'] & G(\mathbb{1}_{\mathcal{V}}) \otimes G(x) \arrow[d, "\mu_{\mathbb{1}_{\mathcal{V}}, x}"] \\
G(x) & G(\mathbb{1}_{\mathcal{V}} \otimes x). \arrow[l, "G(\lambda_x)"]
\end{tikzcd}$$

A lax monoidal functor G is said to be ***strong monoidal*** *if the transformations μ and η are isomorphisms. An* ***oplax monoidal functor*** *$F : \mathcal{W} \to \mathcal{V}$ is defined analogously except that the structure arrows are going in the opposite direction.*

The following lemma introduce one of the most important tools used in our forth chapter.

Lemma 2.1.6 (Change of base)**.** *[Bor94, Proposition 6.4.3] In the presence of any lax monoidal functor $G : \mathcal{V} \to \mathcal{W}$, any $\mathcal{V}$-category $\mathcal{C}$ has an associated $\mathcal{W}$-category, denoted $G_*\mathcal{C}$, with the same objects and the hom-objects given by $\underline{G_*\mathcal{C}}(x, y) := G(\underline{\mathcal{C}}(x, y))$.*

Proposition 2.1.7. *A change of enrichment along a composite $(GH)_*$ is the composite of change of enrichments G_*H_*.*

Proof. Change of enrichment along GH is given by:

$$\begin{aligned}\underline{(GH)_*\mathcal{C}}(x,y) &= (GH)\underline{\mathcal{C}}(x,y)\\ &= G(H\underline{\mathcal{C}}(x,y))\\ &= G\underline{H_*\mathcal{C}}(x,y)\\ &= \underline{(G_*H_*)\mathcal{C}}(x,y). \qquad \square\end{aligned}$$

Lemma 2.1.8. *[Bor94, Proposition 6.4.2] The functor of underlying sets $\mathcal{V}(\mathbb{1},-) : \mathcal{V} \to$ **Set** is lax monoidal.*

Hence, one can change the base of enrichment of any $\mathcal{V}$-category $\mathcal{C}$ from $\mathcal{V}$ to the category **Set** of sets and obtain the underlying category of $\mathcal{C}$ as defined below.

Definition 2.1.9. *The* ***underlying category*** *$\mathcal{C}_0$ of a $\mathcal{V}$-category $\mathcal{C}$ has the same objects and has hom-sets $\mathcal{C}(x,y) := \mathcal{V}(\mathbb{1},\underline{\mathcal{C}}(x,y))$, for any $x,y \in \mathcal{C}$. The identities $\mathrm{id}_x \in \mathcal{C}(x,x)$ are defined to be the specified morphisms $1_x \in \mathcal{V}(\mathbb{1},\underline{\mathcal{C}}(x,x))$. The composition is defined hom-wise by the following arrow in* ***Set***

$$\begin{tikzcd}
\mathcal{C}(y,z)\otimes\mathcal{C}(x,y) \arrow[rr, dashed] \arrow[d, equal] & & \mathcal{C}(x,z) \arrow[d, equal] \\
\mathcal{V}(\mathbb{1},\underline{\mathcal{C}}(y,z))\otimes\mathcal{V}(\mathbb{1},\underline{\mathcal{C}}(x,y)) \arrow[r, "\mu"'] & \mathcal{V}(\mathbb{1},\underline{\mathcal{C}}(y,z)\otimes\underline{\mathcal{C}}(x,y)) \arrow[r, "\mathcal{V}(\mathbb{1},\circ)"'] & \mathcal{V}(\mathbb{1},\underline{\mathcal{C}}(x,z)),
\end{tikzcd}$$

where the first arrow is the lax monoidal transformation of $\mathcal{V}(\mathbb{1},-)$ and the second one is obtained by applying $\mathcal{V}(\mathbb{1},-)$ to the composition morphism for $\mathcal{C}$.

Lemma 2.1.10. *If $(\mathcal{V},\otimes,\mathbb{1})$ is a closed symmetric monoidal category, then the underlying category of the $\mathcal{V}$-category $\mathcal{V}$ is the unenriched category $\mathcal{V}$ itself.*

Definition 2.1.11. *A* ***$\mathcal{V}$-functor*** *$F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-categories is given by an object map $\mathcal{C} \ni x \mapsto Fx \in \mathcal{D}$ together with morphisms $\underline{\mathcal{C}}(x,y) \xrightarrow{F_{x,y}} \underline{\mathcal{D}}(Fx,Fy)$ in $\mathcal{V}$, for any $x, y \in \mathcal{C}$, such that the following diagrams commute, for any $x, y, z \in \mathcal{C}$*

$$\begin{tikzcd}
\underline{\mathcal{C}}(y,z) \otimes \underline{\mathcal{C}}(x,y) \arrow[r, "\circ"] \arrow[d, "F_{y,z} \otimes F_{x,y}"'] & \underline{\mathcal{C}}(x,z) \arrow[d, "F_{x,z}"] \\
\underline{\mathcal{D}}(Fy,Fz) \otimes \underline{\mathcal{D}}(Fx,Fy) \arrow[r, "\circ"'] & \underline{\mathcal{D}}(Fx,Fz),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "1_x"] \arrow[dr, "1_{Fx}"'] & \underline{\mathcal{C}}(x,x) \arrow[d, "F_{x,x}"] \\
& \underline{\mathcal{D}}(Fx,Fx).
\end{tikzcd}$$

The following definition can be found in [Kel82, § 1.3] or [Bor94, Corollary 6.4.4]; see also [Rie14, Exercise 3.5.3].

Definition 2.1.12. *The* ***underlying functor*** *$F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ of a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ is defined on objects by F as follows*

$$\begin{tikzcd}
\mathrm{Ob}(\mathcal{C}_0) \arrow[r, "F_0"] \arrow[d, equal] & \mathrm{Ob}(\mathcal{D}_0) \arrow[d, equal] \\
\mathrm{Ob}(\mathcal{C}) \arrow[r, "F"] & \mathrm{Ob}(\mathcal{D})
\end{tikzcd}$$

and on hom-sets, it is given by the following composite

$$\begin{tikzcd}
x \to y \arrow[d, equal] & & Fx \to Fy \arrow[d, equal] \\
\mathbb{1} \to \underline{\mathcal{C}}(x,y) & & \mathbb{1} \to \underline{\mathcal{C}}(x,y) \xrightarrow[F_{x,y}]{} \underline{\mathcal{D}}(Fx,Fy),
\arrow[from=1-1, to=1-3, mapsto, "F_0"]
\end{tikzcd}$$

for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

Definition 2.1.13. *A* ***$\mathcal{V}$-natural transformation*** *$\alpha : F \Rightarrow G$ between a pair of $\mathcal{V}$-functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ consists of a morphism $\alpha_x : \mathbb{1} \to \underline{\mathcal{D}}(Fx, Gx)$ in $\mathcal{V}$ for* *each $x \in \mathcal{C}$ such that for all $x, y \in \mathcal{C}$ the following diagram commutes*

$$\begin{tikzcd}
\underline{\mathcal{C}}(x,y) \arrow[r, "F_{x,y}"] \arrow[d, "G_{x,y}"'] & \underline{\mathcal{D}}(Fx,Fy) \arrow[d, "(\alpha_y)_*"] \\
\underline{\mathcal{D}}(Gx,Gy) \arrow[r, "(\alpha_x)^*"'] & \underline{\mathcal{D}}(Fx,Gy),
\end{tikzcd}$$

where $(\alpha_x)^*$ and $(\alpha_y)_*$ are respectively given by

$$(\alpha_x)^* : \underline{\mathcal{D}}(Gx,Gy) \cong \underline{\mathcal{D}}(Gx,Gy) \otimes \mathbb{1} \xrightarrow{\mathrm{id}\otimes\alpha_x} \underline{\mathcal{D}}(Gx,Gy) \otimes \underline{\mathcal{D}}(Fx,Gx) \xrightarrow{\circ} \underline{\mathcal{D}}(Fx,Gy),$$

$$(\alpha_y)_* : \underline{\mathcal{D}}(Fx,Fy) \cong \mathbb{1} \otimes \underline{\mathcal{D}}(Fx,Fy) \xrightarrow{\alpha_y\otimes\mathrm{id}} \underline{\mathcal{D}}(Fy,Gy) \otimes \underline{\mathcal{D}}(Fx,Fy) \xrightarrow{\circ} \underline{\mathcal{D}}(Fx,Gy).$$

The (Enriched) Yoneda Lemma is a major tool used in some of the proofs in Chapter 4.

Proposition 2.1.14 (Yoneda Lemma). *Let $\mathcal{C}$ be a locally small category, with category of functors denoted $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$. For any functor $X \in \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$, there is a canonical isomorphism $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(y(c), X) \cong X(c)$ between the set of natural transformations from the representable functor $y(c)$, given by*

$$y : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$$
$$c \mapsto y(c) = \mathcal{C}(c, -),$$

to X, and the value of X at c.

Proposition 2.1.15 (Enriched Yoneda Lemma). *Given a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{V}$ and an object $x \in \mathcal{C}$, the set of $\mathcal{V}$-natural transformations $\alpha : \underline{\mathcal{C}}(x, -) \to F$ is in natural* *bijection with the set of elements of* $F(x) \cong \mathcal{V}(\mathbb{1}, F(x))$*, that is, the set of morphisms* $\mathbb{1} \to F(x)$*, obtained by the composition* $\mathbb{1} \xrightarrow{1_x} \underline{\mathcal{C}}(x,x) \xrightarrow{\alpha_x} F(x)$.

The (Enriched) Yoneda Lemma yields the following.

Lemma 2.1.16. *Let* $\mathcal{C}$ *be a* $\mathcal{V}$*-category. The following statements are equivalent.*

1. *The objects* $x, y \in \mathcal{C}$ *are isomorphic as objects of* $\mathcal{C}$.
2. *The representable functors* $\mathcal{C}(x,-), \mathcal{C}(y,-) : \mathcal{C} \rightrightarrows \mathbf{Set}$ *are naturally isomorphic.*
3. *The underlying functors of the representable functors* $\underline{\mathcal{C}}(x,-), \underline{\mathcal{C}}(y,-) : \mathcal{C} \rightrightarrows \mathcal{V}$ *are naturally isomorphic.*
4. *The representable* $\mathcal{V}$*-functors* $\underline{\mathcal{C}}(x,-), \underline{\mathcal{C}}(y,-) : \mathcal{C} \rightrightarrows \mathcal{V}$ *are* $\mathcal{V}$*-isomorphic.*

Definition 2.1.17. *A* ***$\mathcal{V}$-equivalence of categories*** *is given by a* $\mathcal{V}$*-functor* $F : \mathcal{C} \to \mathcal{D}$ *satisfying the following properties.*

- ***Essentially surjective****, i.e., every* $d \in \mathcal{D}$ *is isomorphic (in* $\mathcal{D}_0$*) to some* Fc.
- ***$\mathcal{V}$-Fully faithful****, i.e., for each* $x, y \in \mathcal{C}$*, the map* $F_{x,y} : \underline{\mathcal{C}}(x,y) \to \underline{\mathcal{D}}(Fx, Fy)$ *is an isomorphism in* $\mathcal{V}$.

Definition 2.1.18. *A* ***$\mathcal{V}$-adjunction*** *consists of* $\mathcal{V}$*-functors* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *together with*

- *$\mathcal{V}$-natural isomorphisms $\underline{\mathcal{D}}(Fc, d) \cong \underline{\mathcal{C}}(c, Gd)$ in $\mathcal{V}$, for any $c \in \mathcal{C}$ and $d \in \mathcal{D}$, or equivalently*

- *$\mathcal{V}$-natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \implies GF$ (unit) and $\varepsilon : FG \implies \mathrm{id}_{\mathcal{D}}$ (counit) satisfying the triangle identities*

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F. \end{tikzcd}$$

2.2 Tensored and cotensored categories

Background material on tensored and cotensored categories can be found in [Kel82, § 1], [Bor94, § 6.5] and [Rie14, § 3.7].

Definition 2.2.1.

- *A $\mathcal{V}$-category $\mathcal{C}$ is* ***tensored*** *over $\mathcal{V}$ if for any $v \in \mathcal{V}$ and $x \in \mathcal{C}$, there is an object $x \otimes v \in \mathcal{C}$ together with isomorphisms*

$$\underline{\mathcal{C}}(x \otimes v, y) \cong \underline{\mathcal{V}}(v, \underline{\mathcal{C}}(x, y)), \text{ for all } y \in \mathcal{C}.$$

- *A $\mathcal{V}$-category $\mathcal{C}$ is* ***cotensored*** *over $\mathcal{V}$ if for any $v \in \mathcal{V}$ and $y \in \mathcal{C}$, there is an object $y^v \in \mathcal{C}$ together with isomorphisms*

$$\underline{\mathcal{C}}(x, y^v) \cong \underline{\mathcal{V}}(v, \underline{\mathcal{C}}(x, y)), \text{ for all } x \in \mathcal{C}.$$

Remark 2.2.2. *The functor $x \otimes - : \mathcal{V} \to \mathcal{C}$ admits a right adjoint $\underline{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{V}$ by the enriched Yoneda Lemma.*

Lemma 2.2.3. *Let* $(\mathcal{V}, \otimes, \mathbb{1})$ *be a closed symmetric monoidal category and suppose* $\mathcal{C}$ *is a tensored* $\mathcal{V}$*-category. Then the tensoring is unital and associative, i.e., there exist natural isomorphisms*

$$x \otimes \mathbb{1} \cong x, \;\; x \otimes (u \otimes v) \cong (x \otimes u) \otimes v, \;\; \textit{for all } u, v \in \mathcal{V}, x \in \mathcal{C}.$$

The following statement is found in [Rie14, Proposition 3.7.10]. Since the proof was left as an exercise, we fill in some details here.

Proposition 2.2.4. *Suppose* $\mathcal{C}$ *and* $\mathcal{D}$ *are tensored and cotensored* $\mathcal{V}$*-categories and* $F : \mathcal{C}_0 \rightleftarrows \mathcal{D}_0 : G$ *is an adjunction between the underlying categories (an unenriched adjunction). Then the data of the following determines the other:*

(i) *a* $\mathcal{V}$*-adjunction* $\underline{\mathcal{D}}(Fc, d) \underset{\cong}{\xrightarrow{\varphi}} \underline{\mathcal{C}}(c, Gd)$,

(ii) *a* $\mathcal{V}$*-functor* F *together with natural isomorphisms* $F(c \otimes v) \cong F(c) \otimes v$,

(iii) *a* $\mathcal{V}$*-functor* G *together with natural isomorphisms* $G(d^v) \cong (Gd)^v$.

Proof. (i) $\Longrightarrow$ (ii) We have

$$\begin{aligned} \underline{\mathcal{D}}(F(c \otimes v), d) &\cong \underline{\mathcal{C}}(c \otimes v, Gd) \text{ by hypothesis} \\ &\cong \underline{\mathcal{V}}(v, \underline{\mathcal{C}}(c, Gd)) \text{ by enriched tensor-hom adjunction} \\ &\cong \underline{\mathcal{V}}(v, \underline{\mathcal{D}}(Fc, d)) \text{ by hypothesis} \\ &\cong \underline{\mathcal{D}}(F(c) \otimes v, d) \text{ by enriched tensor-hom adjunction.} \end{aligned}$$

Therefore, $F(c \otimes v) \cong F(c) \otimes v$ by the enriched Yoneda Lemma and the naturality is given by the naturality of the $\mathcal{V}$-adjunction $\underline{\mathcal{D}}(Fc, d) \cong \underline{\mathcal{C}}(c, Gd)$.

(ii) $\Longrightarrow$ (i) We want to enriched the functor $G : \mathcal{D}_0 \to \mathcal{C}_0$ as

$$G_{d,d'} : \underline{\mathcal{D}}(d, d') \overset{?}{\dashrightarrow} \underline{\mathcal{C}}(Gd, Gd') \cong \underline{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \underline{\mathcal{C}}(Gd, Gd')), \text{ for all } d, d' \in \mathcal{D}.$$

Hence, the map $G_{d,d'}$ is given by the following diagram. For any $v \in \mathcal{V}$, we have

$$\begin{tikzcd}
\mathcal{V}(v, \underline{\mathcal{D}}(d, d')) \arrow[r, dashed, "(G_{d,d'})_*"] \arrow[d, "\text{tensoring } \cong"'] & \mathcal{V}(v, \underline{\mathcal{C}}(Gd, Gd')) \arrow[d, "\cong \text{ tensor-hom adjunction}"] \\
\mathcal{D}(d \otimes v, d') \arrow[rdd, "(\varepsilon_d \otimes \mathrm{id})^*"'] & \mathcal{C}(Gd \otimes v, Gd') \arrow[d, "\cong \text{ unenriched adjunction } F \dashv G"] \\
 & \mathcal{D}(F(Gd \otimes v), d') \arrow[d, "\cong \text{ since } F(c \otimes v) \cong Fc \otimes v"] \\
 & \mathcal{D}(FGd \otimes v, d').
\end{tikzcd}$$

Also, we have

$$\begin{aligned}
\mathcal{V}(v, \underline{\mathcal{D}}(Fc, d)) &\cong \mathcal{D}(Fc \otimes v, d) \text{ by the unenriched adjunction } F \dashv G \\
&\cong \mathcal{D}(F(c \otimes v), d) \text{ by hypothesis} \\
&\cong \mathcal{C}(c \otimes v, Gd) \text{ by the unenriched adjunction } F \dashv G \\
&\cong \mathcal{C}(v, \underline{\mathcal{C}}(c, Gd)) \text{ by the definition of tensoring.}
\end{aligned}$$

Therefore, $\underline{\mathcal{D}}(Fc, d) \cong \underline{\mathcal{C}}(c, Gd)$ by the Yoneda Lemma and the naturality is given by the naturality of the isomorphism $F(c \otimes v) \cong Fc \otimes v$.

(i) $\Longrightarrow$ (iii) We have

$$
\begin{aligned}
\underline{\mathcal{D}}(c, G(d^v)) &\cong \underline{\mathcal{C}}(Fc, d^v) \text{ by hypothesis}\\
&\cong \underline{\mathcal{V}}(Fc \otimes v, d) \text{ by the definition of tensoring}\\
&\cong \underline{\mathcal{V}}(F(c \otimes v), d) \text{ by hypothesis since } (i) \Leftrightarrow (ii)\\
&\cong \underline{\mathcal{C}}(c \otimes v, Gd) \text{ by the unenriched adjunction } F \dashv G\\
&\cong \underline{\mathcal{C}}(c, (Gd)^v) \text{ by the definition of cotensoring.}
\end{aligned}
$$

Therefore, $G(d^v) \cong (Gd)^v$ by the Yoneda Lemma and the naturality is given by the naturality of the $\mathcal{V}$-adjunction $\underline{\mathcal{D}}(Fc, d) \cong \underline{\mathcal{C}}(c, Gd)$ and the isomorphism $F(c \otimes v) \cong F(c) \otimes v$.

(iii) $\Longrightarrow$ (i) We want to enriched the functor $G : \mathcal{D}_0 \to \mathcal{C}_0$ as

$$G_{d,d'} : \underline{\mathcal{D}}(d, d') \overset{?}{\dashrightarrow} \underline{\mathcal{C}}(Gd, Gd') \cong \underline{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \underline{\mathcal{C}}(Gd, Gd')), \text{ for all } d, d' \in \mathcal{D}.$$

Hence, the map $G_{d,d'}$ is given by the following diagram. For any $v \in \mathcal{V}$, we have

$$\begin{tikzcd}
\mathcal{V}(v, \underline{\mathcal{D}}(d, d')) \arrow[r, dashed, "(G_{d,d'})_*"] \arrow[d, "\text{cotensoring } \cong"'] & \mathcal{V}(v, \underline{\mathcal{C}}(Gd, Gd')) \arrow[d, "\cong \text{ cotensoring}"] \\
\mathcal{D}(d, d'^v) \arrow[rdd, "(\varepsilon_d \otimes \mathrm{id})^*"'] & \mathcal{C}(Gd, (Gd')^v) \arrow[d, "\cong \text{ since } (Gd')^v \cong G(d'^v)"] \\
 & \mathcal{D}(Gd, G(d'^v)) \arrow[d, "\cong \text{ unenriched adjunction } F \dashv G"] \\
 & \mathcal{D}(FGd, d'^v).
\end{tikzcd}$$

Also, we have

$$\begin{aligned}\mathcal{V}(v, \underline{\mathcal{D}}(Fc, d)) &\cong \mathcal{D}(Fc, d^v) \text{ by definition of cotensoring}\\ &\cong \mathcal{C}(c, G(d^v)) \text{ by the unenriched adjunction } F \dashv G\\ &\cong \mathcal{C}(c, (Gd)^v) \text{ by hypothesis}\\ &\cong \mathcal{V}(v, \underline{\mathcal{C}}(c, Gd)) \text{ by definition of cotensoring.}\end{aligned}$$

Therefore, $\underline{\mathcal{D}}(Fc, d) \cong \underline{\mathcal{C}}(c, Gd)$ by the Yoneda Lemma and the naturality is given by the naturality of the isomorphism $G(d^v) \cong (Gd)^v$. □

Theorem 2.2.5. *Suppose we have an adjunction* $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$ *between closed symmetric monoidal categories such that the left adjoint* F *is strong monoidal. Then for any tensored and cotensored* $\mathcal{V}$*-category* $\mathcal{C}$*, the* $\mathcal{W}$*-category* $G_*\mathcal{C}$ *becomes canonically tensored and cotensored over* $\mathcal{W}$*, given respectively by*

$$x \otimes w := x \otimes Fw \text{ and } x^w := x^{Fw}, \text{ for any } x \in \mathcal{C}, w \in \mathcal{W}.$$

Corollary 2.2.6. *The strong monoidal adjunction* $F \dashv G$ *of* ***Theorem 2.2.5*** *is a* $\mathcal{W}$*-adjunction with respect to the induced* $\mathcal{W}$*-category structure on* $\mathcal{V}$*, i.e., on* $G_*\mathcal{V}$*.*

In the following proposition we see how to transport the tensoring and the cotensoring along an equivalence of categories.

Proposition 2.2.7. *Let* $\mathcal{D}$ *be a* $\mathcal{V}$*-enriched category tensored and cotensored over a closed symmetric monoidal category* $\mathcal{V}$*, and an equivalence of categories* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *.*

(a) We can transport the tensoring from $\mathcal{D}$ to $\mathcal{C}$ by

$$c \otimes v := G(Fc \otimes v), \text{ for all } c \in \mathcal{C} \text{ and } v \in \mathcal{V}.$$

(b) We can transport the cotensoring from $\mathcal{D}$ to $\mathcal{C}$ by

$$c^v := G((Fc)^v), \text{ for all } c \in \mathcal{C} \text{ and } v \in \mathcal{V}.$$

2.3 Monoidal properties of the Dold–Kan correspondence

Here, we recall the definition of the tensor product and the hom complex of chain complexes of R-modules. Before, let us review the following about R-modules.

Lemma 2.3.1. *1. For a commutative ring R, the category of R-modules Mod_R endowed with the usual tensor product over R and the R-module structure on its hom set, $(\mathrm{Mod}_R, \otimes_R, R, \mathrm{Hom}_R)$, is a closed symmetric monoidal category.*

2. For an arbitrary ring R, the category of left R-modules Mod_R is enriched, tensored, and cotensored over the category Ab of abelian groups. Here, for an abelian group A and a left R-module M, the action of R on the tensoring $M \otimes_{\mathbb{Z}} A$ is given by:

$$r(m \otimes a) := (rm) \otimes a, \text{ for all } r \in R, m \in M \text{ and } a \in A.$$

Similarly, the action of R on the cotensoring $\mathrm{Hom}_{\mathbb{Z}}(A, M)$ is given by:

$$(rf)(a) := r(f(a)), \text{ for all } r \in R, f \in \mathrm{Hom}_{\mathbb{Z}}(A, M) \text{ and } a \in A.$$

Definition 2.3.2. *For a commutative ring R, the category of chain complexes of R-modules $\mathrm{Ch}(R)$ is endowed with a* ***tensor product*** *defined as follows:*

$$(X \otimes Y)_n := \bigoplus_{i \in \mathbb{Z}} X_i \otimes_R Y_{n-i}, \quad \text{with} \quad d(x \otimes y) := d(x) \otimes y + (-1)^{|x|} x \otimes d(y).$$

Definition 2.3.3. *For an arbitrary ring R, the category of chain complexes of left R-modules $\mathrm{Ch}(R)$ is endowed with a* ***hom complex*** *defined as follows:*

$$\underline{\mathrm{Hom}}_{\mathrm{Ch}(R)}(X,Y)_n := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{i+n}), \quad \text{with} \quad (df)(x) := d(f(x)) - (-1)^{|f|} f(d(x))$$

which is a chain complex of abelian groups. If R is commutative, then the hom complex is a chain complex of R-modules.

Lemma 2.3.4. *For a commutative ring R, the category $\mathrm{Ch}(R)$ endowed with the tensor product $\otimes$ and the hom complex $\underline{\mathrm{Hom}}_{\mathrm{Ch}(R)}$, $(\mathrm{Ch}(R), \otimes, R[0], \underline{\mathrm{Hom}}_{\mathrm{Ch}(R)})$, is a closed symmetric monoidal category. Here the tensor unit $R[0]$ is the chain complex with R concentrated in degree 0.*

Definition 2.3.5. *The* ***good truncation functor*** *$\tau_{\geq 0} : \mathrm{Ch}(R) \to \mathrm{Ch}_{\geq 0}(R)$ is defined by*

$$(\tau_{\geq 0} C)_n := \begin{cases} C_n, & \text{if } n \geq 1 \\ \ker(d_0), & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

for any complex $C \in \mathrm{Ch}(R)$.

Lemma 2.3.6. *1. For a commutative ring R, the category $\mathrm{Ch}_{\geq 0}(R)$ endowed with the tensor product $\otimes$ and the hom complex $\underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)} := \tau_{\geq 0}(\underline{\mathrm{Hom}}_{\mathrm{Ch}(R)})$, $(\mathrm{Ch}_{\geq 0}(R), \otimes, R[0], \underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)})$, is a closed symmetric monoidal category.*

2. For an arbitrary ring R, the category $\mathrm{Ch}_{\geq 0}(R)$ is enriched, tensored and cotensored over the category $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ of non-negatively graded chain complexes of abelian groups.

- *The enrichment is given by the hom complex $\underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}(X, Y)$ in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ as defined in item (1.) above, for any complexes X and Y in $\mathrm{Ch}_{\geq 0}(R)$.*
- *The tensoring is given by $X \otimes K$ in $\mathrm{Ch}_{\geq 0}(R)$ as defined in **Definition 2.3.2**, where X is seen as a complex of abelian groups with the R-action induced by the action described in item (2.) of **Lemma 2.3.1**, for any complexes X in $\mathrm{Ch}_{\geq 0}(R)$ and K in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$.*
- *The cotensoring is given by $X^K := \underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(K, X)$ in $\mathrm{Ch}_{\geq 0}(R)$ as defined in item (1.) above, where X is seen as a complex of abelian groups with the R-action induced by the action described in item (2.) of **Lemma 2.3.1**, for any complexes X in $\mathrm{Ch}_{\geq 0}(R)$ and K in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$.*

Let us recall the definition of a simplicial object. Read more on simplicial objects in [Wei94, § 8.1].

Definition 2.3.7. *Let Δ be the simplex category, that is, the category whose objects* *are the finite ordered sets* $[n] = \{0 < 1 < \cdots < n\}$ *for integers* $n \geq 0$*, and whose morphisms are non-decreasing functions. A* ***simplicial object*** A *in a category* $\mathcal{A}$ *is a contravariant functor from* Δ *to* $\mathcal{A}$*, that is,* $A : \Delta^{\mathrm{op}} \to \mathcal{A}$*. We denote by* $s\mathcal{A}$ *the category of simplicial objects in* $\mathcal{A}$*.*

The following can be found in [GJ99, §II.2].

Lemma 2.3.8. *1. For a commutative ring* R*, the category* sMod_R *of simplicial* R*-modules has a closed symmetric monoidal structure. The tensor product is given by the degreewise tensor product defined* $(A \otimes B)_n := A_n \otimes_R B_n$*, for any simplicial* R*-modules* A *and* B*. The constant simplicial* R*-module* $c(R)$ *is the tensor unit, that is,* $c(R) \otimes A \cong A$*. The internal hom of simplicial* R*-modules* A *and* B *is the simplicial* R*-module* $\underline{\mathrm{Hom}}_{\mathrm{sMod_R}}(A, B)$ *given in degree* n *by the* R*-module*

$$\underline{\mathrm{Hom}}_{\mathrm{sMod}_R}(A, B)_n := \mathrm{Hom}_{\mathrm{sMod}_R}(A \otimes R\Delta^n, B).$$

2. For an arbitrary ring R*, the category* sMod_R *of simplicial left* R*-modules is enriched, tensored and cotensored over the category* sAb *of simplicial abelian groups.*

- *The tensoring is given by* $A \otimes K$ *in* sMod_R *as defined in item (1.) above, where* A *is seen as a simplicial abelian group, for any objects* A *in* sMod_R *and* K *in* sAb*.*

- *The enrichment is given by* $\underline{\mathrm{Hom}}_{\mathrm{sMod}_R}(A, B)_n := \mathrm{Hom}_{\mathrm{sMod}_R}(A \otimes \mathbb{Z}\Delta^n, B)$ *in* sAb*, for any objects A and B in* sMod_R*.*
- *The cotensoring is given by* $A^K := \underline{\mathrm{Hom}}_{\mathrm{sAb}}(K, A)$ *in* sMod_R *as defined in item (1.) above, where A is seen as a simplicial abelian group, for any objects A in* sMod_R *and K in* sAb.

Definition 2.3.9. • *The* ***standard interval*** *in simplicial sets is* $\Delta^1 := \mathrm{Hom}_\Delta(-, [1])$*. It comes with three structure maps: inclusion at* 0 *given by* $d^1 : \Delta^0 \to \Delta^1$*, inclusion at* 1 *given by* $d^0 : \Delta^0 \to \Delta^1$ *and the collapse map* $s^0 : \Delta^1 \to \Delta^0$*.*

- *The* ***standard interval*** *in simplicial left R-modules is given by $R\Delta^1$. It comes with structure maps which are obtained by applying the free left R-module functor* $R : \mathbf{sSet} \to \mathrm{sMod}_R$ *to the structure maps for Δ^1, so we have inclusion at* 0*,* $\iota_0 = R(d^1) : R\Delta^0 \to R\Delta^1$*, inclusion at* 1*,* $\iota_1 = R(d^0) : R\Delta^0 \to R\Delta^1$*, and the collapse map* $\mathrm{coll}_{R\Delta^0} = R(s^0) : R\Delta^1 \to R\Delta^0$*.*
- *The* ***standard cylinder*** *of A is the simplicial left R-module given by $A \otimes R\Delta^1$. It comes with structure maps: inclusion at* 0*,* $\iota_0 = \mathrm{id}_A \otimes R(d^1) : A \to A \otimes R\Delta^1$*, inclusion at* 1*,* $\iota_1 = \mathrm{id}_A \otimes R(d^0) : A \to A \otimes R\Delta^1$*, and the collapse map* $\mathrm{coll}_A = \mathrm{id}_A \otimes \mathrm{coll}_{R\Delta^0} : A \otimes R\Delta^1 \to A$*, since $R\Delta^0$ is the constant simplicial left R-module $c(R)$, which is the tensor unit.*

Definition 2.3.10. *Let $A \in \mathrm{sMod}_R$.*

- *The **unnormalized chain complex** (a.k.a. **Moore complex**) of A is the chain complex $C(A)$ with the same graded left R-module as A, i.e., $C(A)_n := A_n$, and with the differential $\partial_n : C(A)_n \to C(A)_{n-1}$ given by the alternating sum of the face maps $\partial_n := \sum_{i=0}^{n}(-1)^i d_i$.*

- *The **normalized chain complex** of A is the subcomplex $N(A) \hookrightarrow C(A)$ given by the joint kernel $N(A)_n := \cap_{i=0}^{n-1}\ker(d_i)$ and with the differential $\partial_n : N(A)_n \to N(A)_{n-1}$ given by $\partial_n := (-1)^n d_n$.*

- *The **degenerate subcomplex** of a $C(A)$ is the subcomplex $D(A) \hookrightarrow C(A)$ defined in degree $n \geq 0$ by the submodule $D(A)_n \subseteq C(A)_n$ generated by the degenerate n-simplices and given by $D(A)_n := \displaystyle\bigoplus_{i=0}^{n-1} s_i(A_{n-1})$.*

Definition 2.3.11. *Let $C \in \mathrm{Ch}_{\geq 0}(R)$. The **denormalization** of C is the simplicial R-module $\Gamma(C)$ defined as follows. In degree n, $\Gamma(C)$ is given by*

$$\Gamma(C)_n := \bigoplus_{[n]\twoheadrightarrow[k]} C_k,$$

where the direct sum is indexed by surjective maps $[n] \twoheadrightarrow [k]$ in the simplex category Δ. For a map $f : [m] \to [n]$ in Δ, we have the corresponding map $f^* : \Gamma(C)_n \to \Gamma(C)_m$, which by definition is

$$f^* : \bigoplus_{[n]\twoheadrightarrow[k]} C_k \to \bigoplus_{[m]\twoheadrightarrow[r]} C_r,$$

whose restriction to the summand labelled by the surjective map $\delta : [n] \twoheadrightarrow [k]$ is given by the composition

$$C_k \xrightarrow{d^*} C_s \hookrightarrow \bigoplus_{[m] \twoheadrightarrow [r]} C_r,$$

where $[m] \overset{t}{\twoheadrightarrow} [s] \xrightarrow{d} [k]$ *is the epi-mono factorization of* $[m] \xrightarrow{f} [n] \xrightarrow{\delta} [k]$. *Here, the map*

$$C_s \hookrightarrow \bigoplus_{[m] \twoheadrightarrow [r]} C_r$$

is the inclusion into the summand C_s labelled by the surjective map $t : [m] \twoheadrightarrow [s]$ in Δ and the map $d^* : C_k \to C_s$, corresponding to the monomorphism $d : [s] \to [k]$ in Δ, is defined by the formula

$$d^* = \begin{cases} \mathrm{id} & \textit{if } d \textit{ is of form } \mathrm{id} : [k] \to [k] \\ \partial_k & \textit{if } d \textit{ is of form } d^k : [k-1] \to [k] \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 2.3.12. *The* ***normalization*** $N : \mathrm{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$ *and the* ***denormalization*** $\Gamma : \mathrm{Ch}_{\geq 0}(R) \to \mathrm{sMod}_R$ *are functors.*

Theorem 2.3.13. *The pair of functors* $N : \mathrm{sMod}_R \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} \mathrm{Ch}_{\geq 0}(R) : \Gamma$, *given by the normalization N and the denormalization Γ, form an equivalence of categories called the* ***Dold–Kan correspondence****.*

Definition 2.3.14. *The* ***Eilenberg–Zilber map*** *(a.k.a.* ***shuffle map****) is the natural* *transformation of chain complexes*

$$\mathrm{EZ} : C(A) \otimes C(B) \to C(A \otimes B), \text{ for any } A, B \in \mathrm{sMod}_R,$$

defined by

$$\mathrm{EZ}(a \otimes b) := \sum_{(\mu,\nu)} \mathrm{sign}(\mu, \nu) \cdot s_\nu(a) \otimes s_\mu(b) \ \in C(A \otimes B)_{p+q} = A_{p+q} \otimes B_{p+q},$$

with simplices $a \in A_p$ *and* $b \in B_q$*, where the sum is taken over all* (p, q)*-shuffles, i.e., permutations of the set* $\{1, ..., p+q\}$ *which leave the first* p *elements and the last* q *elements in the natural order. A shuffle is of the form* $(\mu, \nu) = (\mu_1, ..., \mu_p, \nu_1, ..., \nu_q)$ *and the corresponding degeneracy maps are* $s_\mu = s_{\mu_p-1} \cdots s_{\mu_1-1}$ *and* $s_\nu = s_{\nu_q-1} \cdots s_{\nu_1-1}$*.*

Proposition 2.3.15. *The Eilenberg–Zilber map* $\mathrm{EZ} : C(A) \otimes C(B) \to C(A \otimes B)$ *makes the unnormalized chain complex functor* $C : \mathrm{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$ *lax monoidal, and induces the chain map* $\mathrm{EZ} : N(A) \otimes N(B) \to N(A \otimes B)$ *defined by* $\mathrm{EZ}([a] \otimes [b]) = [\mathrm{EZ}(a \otimes b)]$*, which makes* $N : \mathrm{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$ *lax monoidal.*

Definition 2.3.16. *The* ***Alexander–Whitney map*** *is the natural transformation of chain complexes*

$$\mathrm{AW} : C(A \otimes B) \to C(A) \otimes C(B), \text{ for any } A, B \in \mathrm{sMod}_R,$$

defined by

$$\mathrm{AW}(a \otimes b) = \bigoplus_{p+q=n} \tilde{d}^p(a) \otimes d_0^q(b),$$

for simplices $a \in A_n$ and $b \in B_n$, where $\tilde{d}^p : A_{p+q} \to A_p$ is the front face and $d_0^q : B_{p+q} \to B_q$ is the back face, induced respectively by the injective monotone maps $\tilde{\delta}^p : [p] \to [p+q]$ and $\delta_0^q : [q] \to [p+q]$ defined by $\tilde{\delta}^p(i) = i, i = 0, ..., p$ and $\delta_0^q(j) = p + j, j = 0, ..., q$.

Proposition 2.3.17. *The Alexander–Whitney map* $\mathrm{AW} : C(A \otimes B) \to C(A) \otimes C(B)$ *makes the unnormalized chain complex functor* $C : \mathrm{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$ *oplax monoidal, and induces the chain map* $\mathrm{AW} : N(A \otimes B) \to N(A) \otimes N(B)$ *defined by*

$$\mathrm{AW}\left([a \otimes b]\right) = \sum_{p+q=n} \left[\tilde{d}^p(a) \otimes d_0^q(b)\right], \text{ for } a \otimes b \in C(A \otimes B),$$

which makes $N : \mathrm{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$ *oplax monoidal.*

Proposition 2.3.18. *For simplicial R-modules A and B, the chain complex $N(A) \otimes N(B)$ is a deformation retract of $N(A \otimes B)$ given by the following two diagrams*

$$\begin{tikzcd} N(A) \otimes N(B) \arrow[r, "\mathrm{EZ}"] \arrow[rr, bend right=40, "\mathrm{id}"', ""{name=I}] & N(A \otimes B) \arrow[r, "\mathrm{AW}"] \arrow[to=I, equal] & N(A) \otimes N(B), \end{tikzcd} \qquad \begin{tikzcd} N(A \otimes B) \arrow[r, "\mathrm{AW}"] \arrow[rr, bend right=40, "\mathrm{id}"', ""{name=J}] & N(A) \otimes N(B) \arrow[r, "\mathrm{EZ}"] \arrow[to=J, Rightarrow] & N(A \otimes B). \end{tikzcd}$$

That is, the composite $\mathrm{AW} \circ \mathrm{EZ}$ *is the identity, and the composite* $\mathrm{EZ} \circ \mathrm{AW}$ *is naturally chain homotopic to the identity.*

Proposition 2.3.19. *The multiplicative map* $\mu : \Gamma(C) \otimes \Gamma(D) \to \Gamma(C \otimes D)$, *defined by the composition*

$$\begin{tikzcd} \Gamma(C) \otimes \Gamma(D) \arrow[r, "\mu"] \arrow[d, "\eta"', "\cong"] & \Gamma(C \otimes D) \\ \Gamma N(\Gamma(C) \otimes \Gamma(D)) \arrow[r, "\Gamma(\mathrm{AW})"'] & \Gamma(N\Gamma(C) \otimes N\Gamma(D)), \arrow[u, "\Gamma(\epsilon \otimes \epsilon)"', "\cong"] \end{tikzcd}$$

where $\eta : A \overset{\cong}{\to} \Gamma N(A)$ *and* $\epsilon : N\Gamma(C) \overset{\cong}{\to} C$ *are any natural isomorphisms exhibiting* N *and* Γ *as adjoint equivalences, defines a lax monoidal structure on* Γ*.*

Nice examples of computations involving Eilenberg–Zilber EZ and Alexander–Whitney AW maps can be found in [Opa21, Chapter 3].

Lemmas 2.3.4, **2.3.6** and **2.3.8** can be extended to the case where the category of R-modules Mod_R is replaced by an abelian category $\mathcal{A}$ as given by the following two propositions.

Proposition 2.3.20. *For a bicomplete closed symmetric monoidal abelian category* $\mathcal{A}$*, we have that*

1. $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ *is a closed symmetric monoidal category,*
2. $\mathrm{Ch}(\mathcal{A})$ *is a closed symmetric monoidal category,*
3. $s\mathcal{A}$ *is a closed symmetric monoidal category.*

Proposition 2.3.21. *For a bicomplete abelian category* $\mathcal{A}$*, we have that*

1. $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ *is enriched, tensored and cotensored over* $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$*,*
2. $\mathrm{Ch}(\mathcal{A})$ *is enriched, tensored and cotensored over* $\mathrm{Ch}(\mathbb{Z})$*,*
3. $s\mathcal{A}$ *is enriched, tensored and cotensored over* sAb*.*

Remark 2.3.22. *All the results in Sections 3.1, 3.2 and 3.3 are true for any abelian category $\mathcal{A}$ satisfying the assumptions of* ***Propositions 2.3.20*** *or* ***2.3.21****. But mostly we focus in Chapter 3 on the case $\mathcal{A} = \mathrm{Mod}_R$ for convenience.*

2.4 Monoidal model categories and enriched model categories

Most of the material we review here can be found in [Hov99, Chapter 4], [MP12, Chapter 16] and [Rie14, Chapter 11]. Let us start by recalling the definition of a model category.

Definition 2.4.1. *A* ***model category*** *is a category $\mathcal{C}$ with three distinguished classes of maps: weak equivalences, cofibrations, and fibrations, satisfying:*

- ***MC1:*** *Small limits and colimits exist in $\mathcal{C}$.*
- ***MC2:*** *(2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.*
- ***MC3:*** *(Retracts) If g is a morphism belonging to one of the distinguished classes, and f is a retract of g, then f belongs to the same distinguished class.*
- ***MC4:*** *(Lifting) Define a map to be a* ***trivial cofibration*** *(a.k.a.* ***acyclic cofibration****) if it is both a cofibration and a weak equivalence. Similarly, define a map to be a* ***trivial fibration*** *(a.k.a.* ***acyclic fibration****) if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting*

property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations, as given respectively by the following diagrams:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"', "\sim"] & E \arrow[d, two heads, "p"] \\ Y \arrow[r, "g"'] \arrow[ur, dotted, "\exists h"'] & B, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & E \arrow[d, two heads, "p", "\sim"'] \\ Y \arrow[r, "g"'] \arrow[ur, dotted, "\exists h"] & B. \end{tikzcd}$$

- ***MC5:*** *(Factorization) Any morphism f factors in two ways, a cofibration followed by a trivial fibration, and a trivial cofibration followed by a fibration.*

Definition 2.4.2. *A* ***monoidal model category*** *is a model category $\mathcal{V}$ equipped with the structure of a closed symmetric monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ such that the following two compatibility conditions are satisfied.*

(i) ***Pushout-product axiom:*** *For every pair of cofibrations $i : a \to b$ and $k : x \to y$ both in $\mathcal{V}$, their pushout-product $i \square k$, that is, the induced morphism out of the pushout given by $(a \otimes y) \amalg_{a \otimes x} (b \otimes x) \xrightarrow{i \square k} b \otimes y$, is itself a cofibration in $\mathcal{V}$, which moreover is acyclic if i or k is.*

(ii) ***Unit axiom:*** *For every cofibrant object $x \in \mathcal{V}$ and every cofibrant replacement of the tensor unit $q : Q\mathbb{1} \to \mathbb{1}$ (i.e. a weak equivalence with cofibrant source) in $\mathcal{V}$, the resulting morphism $x \otimes Q\mathbb{1} \xrightarrow{x \otimes q} x \otimes \mathbb{1} \xrightarrow{\cong} x$ is a weak equivalence.*

Remark 2.4.3. *[MP12, Lemma 16.4.5] The pushout-product axiom is equivalent to the following axiom.*

Pullback-power axiom: *For a cofibration $i : a \to b$ and a fibration $p : x \to y$ in $\mathcal{V}$, their pullback-power (i^*, p_*), that is, the induced morphism into the pullback given by $[b, x] \xrightarrow{(i^*,p_*)} [a, x] \times_{[a,y]} [b, y]$, is a fibration in $\mathcal{V}$, which moreover is acyclic if i or p is.*

Definition 2.4.4. *Let $\mathcal{V}$ be a monoidal model category. A **$\mathcal{V}$-model category** is a $\mathcal{V}$-category $\mathcal{C}$, which is tensored and cotensored over $\mathcal{V}$, with the structure of a model category on the underlying category $\mathcal{C}_0$ such that the following two compatibility conditions are satisfied.*

(i) ***External pushout-product axiom:*** *For every pair of cofibrations $i : x \to y$ in $\mathcal{C}_0$ and $k : a \to b$ in $\mathcal{V}$, their pushout-product $i\Box k$, that is, the induced morphism out of the pushout given by $(x \otimes b) \amalg_{x\otimes a} (y \otimes a) \xrightarrow{i\Box k} y \otimes b$, is itself a cofibration in $\mathcal{C}_0$, which moreover is acyclic if i or k is.*

(ii) ***External Unit axiom:*** *For every cofibrant object x in $\mathcal{C}_0$ and every cofibrant replacement of the tensor unit $q : Q\mathbb{1} \to \mathbb{1}$ in $\mathcal{V}$, the resulting morphism $x \otimes Q\mathbb{1} \xrightarrow{x\otimes q} x \otimes \mathbb{1} \xrightarrow{\cong} x$ is a weak equivalence in $\mathcal{C}_0$.*

Remark 2.4.5. *[MP12, Lemma 16.4.5] The external pushout-product axiom is equivalent to the following two axioms.*

- ***External pullback-power axiom:*** *For a cofibration $i : a \to b$ in $\mathcal{V}$ and a fibration $p : x \to y$ in $\mathcal{C}_0$, their pullback-power (i^*, p_*), that is, the induced* *morphism into the pullback given by* $x^b \xrightarrow{(i^*,p_*)} x^a \times_{y^a} y^b$*, is a fibration in* $\mathcal{C}_0$*, which moreover is acyclic if* i *or* p *is.*

- ***SM7**: For a cofibration* $i : a \to b$ *and a fibration* $p : x \to y$ *both in* $\mathcal{C}_0$*, their pullback-power* (i^*, p_*)*, that is, the induced morphism into the pullback given by* $\underline{\mathcal{C}}(b, x) \xrightarrow{(i^*,p_*)} \underline{\mathcal{C}}(a, x) \times_{\underline{\mathcal{C}}(a,y)} \underline{\mathcal{C}}(b, y)$*, is a fibration in* $\mathcal{V}$*, which moreover is acyclic if* i *or* p *is.*

Lemma 2.4.6. *For any* $\mathcal{V}$*-category* $\mathcal{C}$ *equipped with a model structure satisfying the external pushout-product axiom, we have the following for any* $x, y \in \mathcal{C}$ *and* $v \in \mathcal{V}$*.*

1. *The tensor* $x \otimes v \in \mathcal{C}$ *is cofibrant, whenever* x *and* v *are cofibrant.*

2. *The cotensor* $x^v \in \mathcal{C}$ *is fibrant, whenever* x *is fibrant and* v *is cofibrant.*

3. *The hom object* $\underline{\mathcal{C}}(x, y) \in \mathcal{V}$ *is fibrant, whenever* x *is cofibrant and* y *is fibrant.*

Definition 2.4.7. *For* $\mathcal{V}$ *and* $\mathcal{W}$ *two model categories, a pair* $F : \mathcal{V} \rightleftarrows \mathcal{W} : G$ *of adjoint functors is a* ***Quillen adjunction*** *if the following equivalent conditions are satisfied.*

1. F *preserves cofibrations and acyclic cofibrations.*

2. G *preserves fibrations and acyclic fibrations.*

3. F *preserves cofibrations and* G *preserves fibrations.*

4. F preserves acyclic cofibrations and G preserves acyclic fibrations.

Definition 2.4.8. *For $\mathcal{V}$ and $\mathcal{W}$ two monoidal model categories, a* ***lax (or weak) monoidal Quillen adjunction*** *is a Quillen adjunction $(F \dashv G) : \mathcal{W} \rightleftarrows \mathcal{V}$ between the underlying model categories, equipped with the structure of a lax monoidal functor on G with respect to the underlying monoidal categories such that the induced structure of an oplax monoidal functor on F satisfies the following properties.*

(i) For all cofibrant objects $x, y \in \mathcal{W}$ the oplax monoidal transformation $F(x \otimes y) \xrightarrow[\sim]{\delta_{x,y}} F(x) \otimes F(y)$ is a weak equivalence in $\mathcal{V}$.

(ii) For some (hence any) cofibrant replacement of the tensor unit $q : Q\mathbb{1}_{\mathcal{W}} \to \mathbb{1}_{\mathcal{W}}$ in $\mathcal{W}$, the composition $F(Q\mathbb{1}_{\mathcal{W}}) \xrightarrow{F(q)} F(\mathbb{1}_{\mathcal{W}}) \xrightarrow{\varepsilon} \mathbb{1}_{\mathcal{V}}$, with the oplax monoidal counit ε, is a weak equivalence in $\mathcal{V}$.

This is called a ***strong monoidal Quillen adjunction*** *if F is a strong monoidal functor, that is the maps $\delta_{x,y}$ and ε are isomorphisms. In this case the first condition above on F is vacuous, and the second becomes vacuous if the unit object of $\mathcal{W}$ is cofibrant. If a weak monoidal Quillen adjunction is also a Quillen equivalence it is called a* ***weak monoidal Quillen equivalence****.*

Example 2.4.9. *The Dold–Kan correspondence is a weak monoidal Quillen equivalence but not strong since neither the normalization $N : \mathrm{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$ nor the denormalization $\Gamma : \mathrm{Ch}_{\geq 0}(R) \to \mathrm{sMod}_R$ is a strong monoidal functor.*

2.5 Facts about model categories

Here we review some background material on the notion of homotopy category. These are mainly from [Hov99, Chapter 1], [Rie14, Chapter 2] and [MP12, Chapter 14].

Definition 2.5.1. *Let $\mathcal{C}$ be a category with a subcategory of weak equivalences $\mathbf{W}$. The* ***homotopy category*** *of $\mathcal{C}$, if it exists, is a category $\mathrm{Ho}(\mathcal{C})$ together with a functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, which is the identity on objects and takes weak equivalences from $\mathbf{W}$ to isomorphisms and satisfying the following universal property.*

If $F : \mathcal{C} \to \mathcal{D}$ is a functor that sends maps of $\mathbf{W}$ to isomorphisms, then there is a unique functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $\mathrm{Ho}(F) \circ \gamma = F$.

Definition 2.5.2. *Let $\mathcal{C}$ be a model category, and $f, g : B \to X$ be two maps in $\mathcal{C}$.*

1. *A* ***cylinder object*** *for B is a factorization of the fold map $\nabla : B \coprod B \to B$ into a cofibration $B \coprod B \xhookrightarrow{i_0+i_1} B'$ followed by a weak equivalence $B' \underset{\sim}{\xrightarrow{s}} B$.*

2. *A* ***path object*** *for X is a factorization of the diagonal map $X \to X \times X$ into a weak equivalence $X \underset{\sim}{\xrightarrow{r}} X'$ followed by a fibration $X' \xrightarrow{(p_0,p_1)} X \times X$.*

3. *A* ***left homotopy*** *from f to g is a map $H : B' \to X$ for some cylinder object B' for B such that $Hi_0 = f$ and $Hi_1 = g$. We say that f and g are* ***left homotopic****, written $f \overset{l}{\sim} g$, if there is a left homotopy from f to g.*

4. *A* ***right homotopy*** *from f to g is a map $K : B \to X'$ for some path object X' for X such that $p_0K = f$ and $p_1K = g$. We say that f and g are* ***right homotopic****, written $f \overset{r}{\sim} g$, if there is a right homotopy from f to g.*

5. *We say that f and g are* ***homotopic****, written $f \sim g$, if they are both left and right homotopic.*

6. *The map f is a homotopy equivalence if there is a map $h : X \to B$ such that $hf \sim \mathrm{id}_B$ and $fh \sim \mathrm{id}_X$.*

Proposition 2.5.3. *Let $\mathcal{C}$ be a model category, and $f, g : B \to X$ be two maps in $\mathcal{C}$.*

1. *If $f \overset{l}{\sim} g$ and $h : X \to Y$, then $hf \overset{l}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h : A \to B$, then $fh \overset{r}{\sim} gh$.*

2. *If X is fibrant, $f \overset{l}{\sim} g$, and $h : A \to B$, then $fh \overset{l}{\sim} gh$. Dually, if B is cofibrant, $f \overset{r}{\sim} g$, and $h : X \to Y$, then $hf \overset{r}{\sim} hg$.*

3. *If B is cofibrant, then the left homotopy is an equivalence relation on $\mathcal{C}(B, X)$. Dually, if X is fibrant, then the right homotopy is an equivalence relation on $\mathcal{C}(B, X)$.*

4. *If B is cofibrant and $h : X \to Y$ is a trivial fibration or a weak equivalence between fibrant objects, then h induces an isomorphism*

$$\mathcal{C}(B, X)/\overset{l}{\sim} \xrightarrow{\cong} \mathcal{C}(B, Y)/\overset{l}{\sim} .$$

Dually, if X is fibrant and $h : A \to B$ is a trivial cofibration or a weak equivalence between cofibrant objects, then h induces an isomorphism

$$\mathcal{C}(B, X)/\overset{r}{\sim} \xrightarrow{\cong} \mathcal{C}(A, X)/\overset{r}{\sim}\,.$$

5. *If B is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Furthermore, if X' is any path object for X, there is a right homotopy $K : B \to X'$ from f to g. Dually, if X is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$, and there is a left homotopy from f to g using any cylinder object for B.*

Corollary 2.5.4. *Let $\mathcal{C}$ be a model category, B be a cofibrant object of $\mathcal{C}$, and X be a fibrant object of $\mathcal{C}$. Then the left homotopy and right homotopy relations coincide and are equivalence relations on $\mathcal{C}(B, X)$. Furthermore, if $f \sim g : B \to X$, then there is a left homotopy $H : B' \to X$ from f to g using any cylinder object B' for B. Dually, there is a right homotopy $K : B \to X'$ from f to g using any path object X' for X.*

For a model category $\mathcal{C}$, let $\mathcal{C}_c$ (resp. $\mathcal{C}_f$, $\mathcal{C}_{cf}$) denote the full subcategory of cofibrant (resp. fibrant, both cofibrant and fibrant) objects of $\mathcal{C}$.

Corollary 2.5.5. *The homotopy relation on the morphisms of $\mathcal{C}_{cf}$ is an equivalence relation and is compatible with composition. Hence the category $\mathcal{C}_{cf}/\sim$ exists.*

Proposition 2.5.6. *Suppose $\mathcal{C}$ is a model category. Then a map of $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence.*

Corollary 2.5.7. *Suppose $\mathcal{C}$ is a model category. Let $\gamma : \mathcal{C}_{cf} \to \mathrm{Ho}(\mathcal{C}_{cf})$ and $\delta : \mathcal{C}_{cf} \to \mathcal{C}_{cf}/\sim$ be the canonical functors. Then there is a unique isomorphism of categories $\mathcal{C}_{cf}/\sim \underset{\cong}{\xrightarrow{j}} \mathrm{Ho}(\mathcal{C}_{cf})$ such that $j\delta = \gamma$. Furthermore j is the identity on objects.*

Theorem 2.5.8. *Suppose $\mathcal{C}$ is a model category. Let $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ denote the canonical functor.*

1. *The inclusion $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}$ induces an equivalence of categories*

$$\mathcal{C}_{cf}/\sim \xrightarrow{\cong} \mathrm{Ho}(\mathcal{C}_{cf}) \xrightarrow{\cong} \mathrm{Ho}(\mathcal{C}).$$

2. *There are natural isomorphisms*

$$\mathcal{C}(QRX, QRY)/\sim \;\cong \mathrm{Ho}(\mathcal{C})(\gamma X, \gamma Y) \cong \mathcal{C}(RQX, RQY)/\sim$$

where Q denotes a cofibrant replacement and R denotes a fibrant replacement in $\mathcal{C}$. In addition, there is a natural isomorphism

$$\mathrm{Ho}(\mathcal{C})(\gamma X, \gamma Y) \cong \mathcal{C}(QX, RY)/\sim$$

and, if X is cofibrant and Y is fibrant, there is a natural isomorphism $\mathrm{Ho}(\mathcal{C})(\gamma X, \gamma Y) \cong \mathcal{C}(X, Y)/\sim$. In particular, $\mathrm{Ho}(\mathcal{C})$ is a category (with small hom sets).

3. *The functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ identifies left or right homotopic maps.*

4. ***Saturation***: *If $f : A \to B$ is a map in $\mathcal{C}$ such that γf is an isomorphism in* $\mathrm{Ho}(\mathcal{C})$*, then f is a weak equivalence.*

Proposition 2.5.9. *Suppose $\mathcal{C}$ is a model category. Then the inclusion functors induce equivalences of categories*

$$\mathrm{Ho}(\mathcal{C}_{cf}) \to \mathrm{Ho}(\mathcal{C}_c) \to \mathrm{Ho}(\mathcal{C}) \quad \textit{and} \quad \mathrm{Ho}(\mathcal{C}_{cf}) \to \mathrm{Ho}(\mathcal{C}_f) \to \mathrm{Ho}(\mathcal{C}).$$

Definition 2.5.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.*

1. *If $F : \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, define the* ***total left derived functor*** $\mathbf{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ *to be the composite* $\mathrm{Ho}(\mathcal{C}) \xrightarrow{\mathrm{Ho}(Q)} \mathrm{Ho}(\mathcal{C}_c) \xrightarrow{\mathrm{Ho}(F)} \mathrm{Ho}(\mathcal{D})$.

2. *If $G : \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, define the* ***total right derived functor*** $\mathbf{R}G : \mathrm{Ho}(\mathcal{D}) \to \mathrm{Ho}(\mathcal{C})$ *of G to be the composite*

$$\mathrm{Ho}(\mathcal{D}) \xrightarrow{\mathrm{Ho}(R)} \mathrm{Ho}(\mathcal{D}_f) \xrightarrow{\mathrm{Ho}(G)} \mathrm{Ho}(\mathcal{C}).$$

Example 2.5.11. *Let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category tensored over $\mathcal{V}$.*

- *The total left derived tensor product, denoted $- \otimes^{\mathbf{L}} -$ on $\mathrm{Ho}(\mathcal{V})$ is given by $u \otimes^{\mathbf{L}} v := Qu \otimes Qv$, for any $u, v \in \mathcal{V}$.*

- *The total left derived tensoring, denoted $- \otimes^{\mathbf{L}} -$, on $\mathrm{Ho}(\mathcal{C})$ over $\mathrm{Ho}(\mathcal{V})$ is given by $x \otimes^{\mathbf{L}} v := Qx \otimes Qv$, for any $x \in \mathcal{C}$ and $v \in \mathcal{V}$.*

- *The total right derived hom objects, denoted $\mathbf{R}\underline{\mathcal{C}}(-,-)$, on $\mathrm{Ho}(\mathcal{C})$ is given by $\mathbf{R}\underline{\mathcal{C}}(x,y) := \underline{\mathcal{C}}(Qx, Ry)$, for any $x, y \in \mathcal{C}$.*

Remark 2.5.12. *In the case of a $\mathcal{V}$-enriched model category $\mathcal{C}$ that is tensored and cotensored over $\mathcal{V}$, the homotopy category* $\mathrm{Ho}(\mathcal{C})$ *inherits a* $\mathrm{Ho}(\mathcal{V})$*-enrichment, a tensoring and a cotensoring over* $\mathrm{Ho}(\mathcal{V})$*; see [Rie14, Theorem 10.2.12], [MP12, Remark 16.4.13] and [Hov99, Theorem 4.3.4].*

2.6 Projective classes

Here, we review some background material from [CH02].

Definition 2.6.1. *Let $\mathcal{A}$ be an abelian category. For an object $P \in \mathcal{A}$, a map $f : B \to C$ in $\mathcal{A}$ is said to be **P-epic** if the induced map of hom sets* $\mathrm{Hom}_{\mathcal{A}}(P, f) : \mathrm{Hom}_{\mathcal{A}}(P, B) \to \mathrm{Hom}_{\mathcal{A}}(P, C)$ *is a surjection of abelian groups. For a collection of objects $\mathcal{P}$, the map $f : B \to C$ is **$\mathcal{P}$-epic** if it is P-epic for all $P \in \mathcal{P}$.*

Example 2.6.2. *In an abelian category $\mathcal{A}$, any split epimorphism $f : B \to C$ in $\mathcal{A}$ is $\mathcal{A}$-epic. Indeed, $f : B \to C$ is a split epimorphism if and only if the induced map of hom sets* $\mathrm{Hom}_{\mathcal{A}}(P, f) : \mathrm{Hom}_{\mathcal{A}}(P, B) \to \mathrm{Hom}_{\mathcal{A}}(P, C)$ *is a surjection of abelian groups for any object $P \in \mathcal{A}$.*

Definition 2.6.3. *For an abelian category $\mathcal{A}$, a **projective class** on $\mathcal{A}$ is a collection $\mathcal{P}$ of objects of $\mathcal{A}$ and a collection $\mathcal{E}$ of maps in $\mathcal{A}$ such that*

(i) $\mathcal{E}$ is precisely the collection of all $\mathcal{P}$-epic maps;

(ii) $\mathcal{P}$ *is precisely the collection of all objects* P *such that each map in* $\mathcal{E}$ *is* P*-epic;*

(iii) for each object $B \in \mathcal{A}$ *there is a map* $j : P \to B$ *in* $\mathcal{E}$ *with* $P \in \mathcal{P}$.

Example 2.6.4. *Let* $\mathcal{A}$ *be an abelian category. Take* $\mathcal{P}$ *to be the collection of all objects and* $\mathcal{E}$ *to be the collection of all split epimorphisms* $f : B \to C$*. Here* $(\mathcal{P}, \mathcal{E})$ *is a projective class, called the* ***trivial projective class****.*

Example 2.6.5. *Let us consider a functor* $U : \mathcal{A} \to \mathcal{B}$ *of abelian categories, together with a left adjoint* $F : \mathcal{B} \to \mathcal{A}$ *. Then* U *and* F *are additive,* U *is left exact and* F *is right exact. If* $(\mathcal{P}', \mathcal{E}')$ *is a projective class on* $\mathcal{B}$*, we define*

$$\mathcal{P} := \{\textit{retracts of } FP \textit{ for } P \in \mathcal{P}'\} \textit{ and } \mathcal{E} := \{B \xrightarrow{f} C \textit{ such that } UB \xrightarrow{Uf} UC \in \mathcal{E}'\}.$$

Then $(\mathcal{P}, \mathcal{E})$ *is a projective class on* $\mathcal{A}$*, called the* ***pullback*** *of* $(\mathcal{P}', \mathcal{E}')$ *along the right adjoint* U*.*

Chapter 3

The Hurewicz Model Structure on Simplicial R-modules

The chapter is organized as follows. In Section 3.1, we describe the Hurewicz model structure on $\mathrm{Ch}_{\geq 0}(R)$ as an instance of the Christensen–Hovey setup and we give some of its monoidal properties. Now that we have the projective (Quillen) and the Hurewicz model structures on $\mathrm{Ch}_{\geq 0}(R)$, in Section 3.2, we give a description of the resulting mixed model structure together with the enrichment relations between these three model structures. In Section 3.3, we show that Dold–Kan correspondence transfers the Hurewicz model structure from $\mathrm{Ch}_{\geq 0}(R)$ to a monoidal model structure on sMod_R. Finally, in Section 3.4, we explain how the Bousfield model structure on the category $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ of non-negatively graded cochain complexes of objects of $\mathcal{A}$ is recovered from the Christensen–Hovey setup.

3.1 Hurewicz model structure on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$

In this section, we show that the Hurewicz model structure on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is an instance of the Christensen–Hovey model structures, for a bicomplete abelian category $\mathcal{A}$. Also, we prove some related properties.

Proposition 3.1.1. *[CH02, Corollary 6.4] If $\mathcal{A}$ is a bicomplete abelian category with a projective class $\mathcal{P}$, then the category $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ of non-negatively graded chain complexes of objects of $\mathcal{A}$ is endowed with a model structure as follows.*

1. *A map f is a weak equivalence, i.e., a $\mathcal{P}$-**equivalence**, if $\mathrm{Hom}_{\mathcal{A}}(P, f)$ is a quasi-isomorphism for each $P \in \mathcal{P}$.*
2. *A map f is a fibration, i.e., a $\mathcal{P}$-**fibration**, if $\mathrm{Hom}_{\mathcal{A}}(P, f)$ is surjective in positive degrees (but not necessarily in degree 0) for each $P \in \mathcal{P}$.*
3. *A map f is a cofibration, i.e., a $\mathcal{P}$-**cofibration**, if it is a degreewise split monomorphism with degreewise $\mathcal{P}$-projective cokernel.*

Here, $\mathrm{Hom}_{\mathcal{A}}(P, f)$ denotes the map in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ obtained by applying the functor $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{Ab}$ degreewise. In this model structure, every complex is fibrant, and a complex is cofibrant if and only if it is a complex of $\mathcal{P}$-projectives.

The next two lemmas are useful in the description of the Hurewicz model structure on the category $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ of the non-negatively graded chain complexes of objects in an abelian category $\mathcal{A}$ later on.

Lemma 3.1.2. *Let $\mathcal{B}$ be an abelian category and $g : X \to Y$ a map in* $\mathrm{Ch}(\mathcal{B})$*. The map g is a degreewise split epimorphism if and only if for every $M \in \mathcal{B}$, the map $g_* : \mathrm{Hom}_{\mathcal{B}}(M, X) \to \mathrm{Hom}_{\mathcal{B}}(M, Y)$ in $\mathrm{Ch}(\mathbb{Z})$ is a degreewise surjection, where $\mathrm{Hom}_{\mathcal{B}}(M, X) \in \mathrm{Ch}(\mathbb{Z})$ is the chain complex obtained by applying the functor $\mathrm{Hom}_{\mathcal{B}}(M, -) : \mathcal{B} \to \mathrm{Ab}$ degreewise to X.*

Proof. ($\Longrightarrow$) Verified since split epimorphisms are universal, hence the image of the degreewise split epimorphism g by the functor $\mathrm{Hom}_{\mathcal{B}}(M, -) : \mathcal{B} \to \mathrm{Ab}$ is a degreewise split epimorphism.

($\Longleftarrow$) From $M = Y_n$, the map $(g_n)_* : \mathrm{Hom}_{\mathcal{B}}(Y_n, X_n) \to \mathrm{Hom}_{\mathcal{B}}(Y_n, Y_n)$ is surjective. Then take $s_n \in \mathrm{Hom}_{\mathcal{B}}(Y_n, X_n)$ such that $(g_n)_*(s_n) = g_n s_n = \mathrm{id}_{Y_n}$. □

Lemma 3.1.3. *Let $U : \mathcal{A} \to \mathcal{B}$ be a functor of abelian categories, with a left adjoint $F : \mathcal{B} \to \mathcal{A}$. If $\mathcal{P}$ is the projective class on $\mathcal{A}$ given by the pullback of the trivial projective class on $\mathcal{B}$, then we have the following.*

1. *A map $f : X \to Y$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is a $\mathcal{P}$-equivalence if and only if the map $Uf : UX \to UY$ is a chain homotopy equivalence in $\mathrm{Ch}_{\geq 0}(\mathcal{B})$.*

2. *A map $f : X \to Y$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is a $\mathcal{P}$-fibration if and only if the map $Uf : UX \to UY$ in $\mathrm{Ch}_{\geq 0}(\mathcal{B})$ is a degreewise split epimorphism in positive degrees.*

Proof. (1.) The proof of [CH02, Lemma 3.1 (b)] is still valid here since the cofiber C

of Uf is given by $C_n = (UY)_n \oplus (UX)_{n-1}$ and so $C \in \mathrm{Ch}_{\geq 0}(\mathcal{B})$.

(2.) By **Proposition 3.1.1**, the map $f : X \to Y$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is a $\mathcal{P}$-fibration if and only if the following map, induced by Uf given by the adjunction $F \dashv U$,

$$\mathrm{Hom}_{\mathcal{A}}(FM, f) \cong \mathrm{Hom}_{\mathcal{B}}(M, Uf) : \mathrm{Hom}_{\mathcal{B}}(M, UX) \to \mathrm{Hom}_{\mathcal{B}}(M, UY)$$

is degreewise surjective in positive degrees for every $M \in \mathcal{B}$ (trivial projective class). By **Lemma 3.1.2**, this is equivalent to $Uf : UX \to UY$ being degreewise split epimorphism in positive degrees. □

We have the following model structure on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$.

Proposition 3.1.4. *For $\mathcal{A}$ a bicomplete abelian category,* $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ *has a model structure given by the following.*

1. *A map f is a weak equivalence if it is a chain homotopy equivalence.*
2. *A map f is a fibration if it is a degreewise split epimorphism in positive degrees (not necessarily in degree zero).*
3. *A map f is a cofibration if it is a degreewise split monomorphism.*

Moreover, every complex is fibrant and cofibrant.

Proof. Parts (1.) and (2.) are obtained by applying **Lemma 3.1.3** with $U = \mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. In other words, take the trivial projective class on $\mathcal{A}$.

Part (3.) is given by **Proposition 3.1.1** and so a map $f : X \to Y$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is a cofibration if and only if it is a degreewise split monomorphism, and every complex is fibrant and cofibrant, since all objects in $\mathcal{A}$ are $\mathcal{P}$-projectives. $\square$

The model structure on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ defined in **Proposition 3.1.4** is called the **Hurewicz model structure** (or **h-model structure**), in light of the following characterization of the cofibrations and fibrations.

Proposition 3.1.5.

1. *A map $i : A \to X$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is an h-cofibration if and only if it satisfies the homotopy extension property (HEP).*
2. *A map $p : E \to B$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is an h-fibration if and only if it satisfies the homotopy lifting property (HLP).*

Proof. (1.) Recall from [MP12, Proposition 18.3.6] that an r-cofibration and an r-fibration are respectively a degreewise split monomorphism and a degreewise split epimorphism of unbounded complexes. The proof of [MP12, Proposition 18.3.6] applies here, since any degreewise split monomorphism $i : A \to X$ of non-negatively graded chain complexes is an r-cofibration when viewed as a map of unbounded chain complexes and its mapping cylinder $Mi = X \cup_i A \otimes I$, computed in $\mathrm{Ch}(\mathcal{A})$, is a non-negatively graded chain complex. Here, the interval complex I is given by

$$\mathbb{Z} \xrightarrow{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z}$$

in degrees 1 and 0, and zero in all other degrees.

(2.) ($\Longrightarrow$) Assume that a map $p : E \to B$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is a h-fibration. Consider the inclusion $i_0 : A \to A \otimes I$ given in degree $n \geq 0$ by

$$(i_0)_n : A_n \to (A \otimes I)_n \cong A_{n-1} \oplus A_n \oplus A_n$$
$$a \mapsto (0, a, 0).$$

As in [MP12, Proposition 18.3.6], since i_0 is an acyclic cofibration, by **Proposition 3.1.4** the lifting property of the model structure gives that p satisfies the HLP.

($\Longleftarrow$) Assume that a map $p : E \to B$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ satisfies the HLP. We want to show that p is a h-fibration, i.e., a degreewise split epimorphism in positive degree. The mapping cocylinder $\tau_{\geq 0}(Np) = \tau_{\geq 0}\left(E \times_B B^I\right)$ of p is given in degree n by

$$\left(\tau_{\geq 0}\left(E \times_B B^I\right)\right)_n \cong \begin{cases} E_n \oplus B_n \oplus B_{n+1}, & \text{if } n \geq 1 \\ E_0 \oplus B_1, & \text{if } n = 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau_{\geq 0} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is the good truncation functor. Consider the map (ev_0, p_*) given at any degree $n > 0$ by

$$\begin{tikzcd}
\left(E^I\right)_n \arrow[r, "(\mathrm{ev}_0,p_*)_n"] \arrow[d, "\cong"'] & \left(\tau_{\geq 0}\left(E \times_B B^I\right)\right)_n \arrow[d, "\cong"] \\
E_n \oplus E_n \oplus E_{n+1} \arrow[r, "(\mathrm{id},p_n,p_{n+1})"'] & E_n \oplus B_n \oplus B_{n+1}.
\end{tikzcd}$$

Since p satisfies the HLP, the map (ev_0, p_*) has a section $\sigma : \tau_{\geq 0}(Np) \to E^I$. For $n > 0$, the map $s_n : B_n \to E_n$ given by the composition

$$\begin{tikzcd}
B_n \arrow[r, hook] \arrow[rrr, bend left, "s_n"] & \left(E \times_B B^I\right)_n \arrow[r, "\sigma_n"] \arrow[d, "\cong"] & \left(E^I\right)_n \arrow[r, "(\mathrm{ev}_1)_n"] \arrow[d, "\cong"] & E_n \\
B_n \arrow[r, hook] & E_n \oplus B_n \oplus B_{n+1} \arrow[r, "\sigma_n"] & E_n \oplus E_n \oplus E_{n+1} \arrow[r, "\mathrm{proj}_2"] & E_n \\
b \arrow[r, mapsto] & (0,b,0) \arrow[r, mapsto] & (e_0,e_1,e_2) \arrow[r, mapsto] & e_1
\end{tikzcd} \tag{3.1.1}$$

is a section of $p_n : E_n \to B_n$. Thus $p : E \to B$ is a degreewise split epimorphism in positive degree. □

For background on the HEP and HLP, see [May99, §6.1, 7.1].

Now we provide an example of a map of non-negatively graded chain complexes which is not an h-fibration when viewed as a map in $\mathrm{Ch}(\mathcal{A})$ but is an h-fibration in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, where $\mathcal{A}$ is some abelian category.

Example 3.1.6. *For C in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, take the chain complex C' with $C'_n = C_n$ for $n \neq 0$ and $C'_0 = B_0(C)$ the 0-boundaries of C. The map $p : C' \to C$ given by the identity in positive degrees and the inclusion of 0-boundaries $B_0(C) \hookrightarrow C_0$ in* *degree* 0 *is an h-fibration. By* ***Proposition 3.1.5****, it satisfies the HLP. Let us check this fact directly. Considering the left-hand side diagram below with* $f \sim g$ *with the homotopy* H *and given a lift* $\widetilde{f}$*, we want to lift the homotopy* H *into* $\widetilde{H}$ *such that* $\partial\widetilde{H} + \widetilde{H}\partial = \widetilde{g} - \widetilde{f}$*, for some lift* $\tilde{g}$ *of* g*.*

$$\begin{tikzcd}
 & C' \arrow[dd, "p"] \\
 & \\
A \arrow[uur, "\widetilde{f}"] \arrow[r, phantom, ""] \arrow[rr, bend left, "f", ""{name=U, below}] \arrow[rr, bend right, "g"', ""{name=D}] & & C
\arrow[Rightarrow, from=U, to=D, "H"]
\end{tikzcd}$$

$$\Longrightarrow$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
A_2 \arrow[ur, "\widetilde{H}_2"] \arrow[r, "\widetilde{f}_2{,}\widetilde{g}_2"] \arrow[d] & C_2 \arrow[r, "p_2"] \arrow[d, "\partial"] & C_2 \arrow[d] \\
A_1 \arrow[ur, "\widetilde{H}_1"] \arrow[r, "\widetilde{f}_1{,}\widetilde{g}_1"] \arrow[d] & C_1 \arrow[r, "p_1"] \arrow[d, "\partial"] & C_1 \arrow[d] \\
A_0 \arrow[ur, "\widetilde{H}_0"] \arrow[r, "\widetilde{f}_0{,}\widetilde{g}_0"] \arrow[d] & B_0(C) \arrow[r, hook, "p_0"] \arrow[d] & C_0 \arrow[d] \\
0 \arrow[ur, "\widetilde{H}_{-1}"] & 0 & 0
\end{tikzcd}$$

For $n \geq 1$*, we take* $\widetilde{H}_n = H_n$ *and* $\widetilde{g}_n = g_n$*. For* $n = 0$*, we have* $\partial\widetilde{H}_0 + \widetilde{H}_{-1}\partial = \widetilde{g}_0 - \widetilde{f}_0$*. So* $\widetilde{g}_0 = \widetilde{f}_0 + \partial H_0$*, with* $\widetilde{H}_0 = H_0$ *since* $\widetilde{H}_{-1} = 0$*.*

The Hurewicz model structure on the category $\mathrm{Ch}_{\geq 0}(R)$ of the non-negative graded chain complexes of (left) R-modules satisfies the following monoidal property.

Proposition 3.1.7.

1. *For a commutative ring* R*, the Hurewicz model structure on* $\mathrm{Ch}_{\geq 0}(R)$ *is monoidal.*

2. *For an arbitrary ring* R*, the Hurewicz model structure on* $\mathrm{Ch}_{\geq 0}(R)$ *is enriched over* $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$*.*

Proof. The proof follows from its analogue on $\mathrm{Ch}(R)$ [MP12, End of §18.3]. If we take maps $i : A \to X$ in $\mathrm{Ch}_{\geq 0}(R)$ and $j : Y \to Z$ in $\mathrm{Ch}_{\geq 0}(R)$ (or in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$), then all objects in the diagrams in that proof are non-negatively graded, i.e., live in $\mathrm{Ch}_{\geq 0}(R)$ (or in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$). $\square$

3.2 Relation with other model structures on $\mathrm{Ch}_{\geq 0}(R)$

Given the Hurewicz model structure described above and the well-known Quillen model structure on the non-negatively graded chain complexes of left R-modules $\mathrm{Ch}_{\geq 0}(R)$, by [Col06, Theorem 2.1] there is a new model structure on $\mathrm{Ch}_{\geq 0}(R)$ given as follows.

Proposition 3.2.1. *There is a* ***mixed model structure*** *on* $\mathrm{Ch}_{\geq 0}(R)$ *given by the following classes of maps.*

1. *The m-weak equivalences are the quasi-isomorphisms.*

2. *The m-fibrations are the degreewise split epimorphisms in positive degrees (not necessarily in degree zero).*

3. *The m-cofibrations are the maps satisfying the left lifting property (LLP) with respect to acyclic fibrations (*$\boldsymbol{F} \cap \boldsymbol{W}$*).*

Remark 3.2.2. *Since the Quillen and Hurewicz model categories, respectively denoted by* $\mathrm{Ch}_{\geq 0}(R)_q$ *[Jar03, Remark 1.6] and* $\mathrm{Ch}_{\geq 0}(R)_h$ *are monoidal, then the above mentioned mixed model category, denoted by* $\mathrm{Ch}_{\geq 0}(R)_m$ *is also monoidal, by [Col06, Proposition 6.6].*

The next two lemmas are useful in the description of the relationships between the three monoidal model category mentioned in **Remark 3.2.2**.

Lemma 3.2.3. *In* $\mathrm{Ch}_{\geq 0}(R)$ *(or in* $\mathrm{Ch}(R)$*), if a map f is a q-cofibration and a q-weak equivalence, i.e., a quasi-isomorphism, then f is a h-weak equivalence, i.e., a chain homotopy equivalence.*

Proof. Let $f : A \overset{\sim}{\hookrightarrow} X$ be an acyclic q-cofibration. We have the following split short exact sequence of graded R-modules

$$0 \longrightarrow A \xhookrightarrow[\sim]{f} X \longrightarrow X/A \longrightarrow 0$$

since $\mathrm{coker}(f) = X/A$ is degreewise projective. Since f is a quasi-isomorphism, we have $H_*(X/A) = 0$ by the long exact sequence, i.e., X/A is weakly contractible. Also, X/A is a q-fibrant and q-cofibrant complex, and so it is contractible by Whitehead's Theorem in model categories [Hov99, Proposition 1.2.8]. Therefore, by [MP12, Lemma 18.2.8], f is a chain homotopy equivalence. □

Lemma 3.2.4. *In* $\mathrm{Ch}_{\geq 0}(R)$ *(or in* $\mathrm{Ch}(R)$*), if a map j is an m-cofibration and an m-weak equivalence, i.e., a quasi-isomorphism, then j is an h-weak equivalence, i.e., a chain homotopy equivalence.*

Proof. Let $j : A \overset{\sim}{\hookrightarrow} X$ be an acyclic m-cofibration, i.e., an m-cofibration and a quasi-isomorphism. As an m-cofibration, by [MP12, Theorem 17.3.5], j factors as follows

$$\begin{tikzcd}
A \arrow[rr, hook, "j", "\sim"'] \arrow[dr, hook, "i"'] & & X \ , \\
& X' \arrow[ur, "f"', "\sim"] &
\end{tikzcd}$$

where i is a q-cofibration and f is a chain homotopy equivalence. Since j is also a quasi-isomorphism, then by the 2-out-of-3 property, i is a quasi-isomorphism too. Hence by **Lemma 3.2.3**, i is a chain homotopy equivalence and so by the 2-out-of-3 property, j is a chain homotopy equivalence too. □

We have the following relations between three of the model structures on $\mathrm{Ch}_{\geq 0}(R)$.

Proposition 3.2.5. *For a commutative ring R, we have the following.*

1. $\mathrm{Ch}_{\geq 0}(R)_h$ *is enriched over* $\mathrm{Ch}_{\geq 0}(R)_q$.

2. $\mathrm{Ch}_{\geq 0}(R)_h$ *is enriched over* $\mathrm{Ch}_{\geq 0}(R)_m$.

3. $\mathrm{Ch}_{\geq 0}(R)_m$ *is enriched over* $\mathrm{Ch}_{\geq 0}(R)_q$.

Proof. Before we prove each of the enrichments, let us recall that we have the following comparison between the different classes of maps defining the three model structures. $W_h \subsetneq W_m = W_q$, $C_q \subsetneq C_m \subsetneq C_h$ and $F_h = F_m \subsetneq F_q$.

For (1.), let $i : K \hookrightarrow L$ and $j : A \hookrightarrow X$ be respectively a h-cofibration and a q-cofibration. $i \Box j$ is an h-cofibration since $C_q \subsetneq C_h$ and $\mathrm{Ch}_{\geq 0}(R)_h$ is monoidal. A similar argument works for the case where i is an acyclic h-cofibration. If instead

j is an acyclic q-cofibration, then by **Lemma 3.2.3**, j is an acyclic h-cofibration. Hence, the above argument applies again.

For (2.), the argument here is the same as in (1.) above, after replacing **Lemma 3.2.3** by **Lemma 3.2.4**.

For (3.), $\mathrm{Ch}_{\geq 0}(R)_m$ is enriched over $\mathrm{Ch}_{\geq 0}(R)_q$, since we have $C_q \subsetneq C_m$ and $W_m = W_q$, and also $\mathrm{Ch}_{\geq 0}(R)_m$ is monoidal by **Remark 3.2.2**. □

Lemma 3.2.6. *In* $\mathrm{Ch}_{\geq 0}(R)$*, the chain complex* $M[0]$ *given by the* R*-module* M *concentrated in degree* 0*, is m-cofibrant if and only if* M *is projective.*

Proof. By [Col06, Corollary 3.7], $M[0]$ is m-cofibrant if and only if it is h-cofibrant and homotopy equivalent to a q-cofibrant complex. But all complexes are h-cofibrant, and q-cofibrant complexes are the degreewise projective ones. Hence, $M[0]$ is m-cofibrant if and only if it is homotopy equivalent to a degreewise projective complex P, in which case M is a retract of the projective R-module P_0, hence M is projective. Conversely, if M is projective, then $M[0]$ is q-cofibrant. □

Proposition 3.2.7. *For a commutative ring* R *that admits a non-projective module, we have the following.*

1. $\mathrm{Ch}_{\geq 0}(R)_q$ *is not enriched over* $\mathrm{Ch}_{\geq 0}(R)_h$.
2. $\mathrm{Ch}_{\geq 0}(R)_m$ *is not enriched over* $\mathrm{Ch}_{\geq 0}(R)_h$.
3. $\mathrm{Ch}_{\geq 0}(R)_q$ *is not enriched over* $\mathrm{Ch}_{\geq 0}(R)_m$.

Proof. For (1.), consider the q-cofibration $i : 0 \to R[0]$ and the h-cofibration $j : 0 \to L$ that is not a q-cofibration, i.e., where L is not degreewise projective. We have that the map $i \square j = j : 0 \to L$ is not a q-cofibration.

For (2.), similarly, consider the h-cofibration $j : 0 \to M[0]$, with M non-projective. The map j is not an m-cofibration since by **Lemma 3.2.6**, $M[0]$ is not m-cofibrant.

For (3.), similarly, consider the complex L given by $M \xrightarrow{1} M$ in degrees 1 and 0, and zero in all other degrees, with M non-projective. The map $j : 0 \to L$ is an m-cofibration since L is a contractible complex. But j is not a q-cofibration. $\square$

3.3 Induced Hurewicz model structure on sMod_R

The following well-known fact tells us that having an equivalence of categories, one can transport a model structure from one category to the other.[1]

Proposition 3.3.1. *Let $\mathcal{C}$ be a model category given by the three classes of maps $\mathbf{W}_\mathcal{C}$, $\mathbf{C}_\mathcal{C}$ and $\mathbf{F}_\mathcal{C}$, and $U : \mathcal{D} \to \mathcal{C}$ an equivalence of categories. Then the classes*

$$\mathbf{W}_\mathcal{D} := \{f : d \to d' \mid Uf \in \mathbf{W}_\mathcal{C}\}$$

$$\mathbf{C}_\mathcal{D} := \{f : d \to d' \mid Uf \in \mathbf{C}_\mathcal{C}\}$$

$$\mathbf{F}_\mathcal{D} := \{f : d \to d' \mid Uf \in \mathbf{F}_\mathcal{C}\}$$

form a model structure on $\mathcal{D}$.

---

[1] For more details, see for instance: `https://math.stackexchange.com/questions/2171572/transferring-model-structures-along-an-equivalence-of-categories`

Proof. Observe that the category $\mathcal{C}$ is complete and cocomplete if and only if the category $\mathcal{D}$ is complete and cocomplete. The class $\mathbf{W}_{\mathcal{D}}$ satisfies 2-out-of-3 since the class $\mathbf{W}_{\mathcal{C}}$ does. The three classes in $\mathcal{D}$ are closed under retracts since the three classes in $\mathcal{C}$ are. Also, the three classes $\mathbf{W}_{\mathcal{D}}$, $\mathbf{C}_{\mathcal{D}}$ and $\mathbf{F}_{\mathcal{D}}$ are closed under composition and contain isomorphisms, since $\mathbf{W}_{\mathcal{C}}$, $\mathbf{C}_{\mathcal{C}}$ and $\mathbf{F}_{\mathcal{C}}$ do. In particular, the three classes are closed under isomorphisms in the arrow category. Therefore, the factorization and lifting axioms are invariant under equivalence of category. □

Hence, the Dold–Kan correspondence induces a model structure on the category of simplicial R-modules sMod_R, also called the Hurewicz model structure, and given by the following classes of maps.

1. A map f is a weak equivalence if its normalization $N(f)$ is an h-weak equivalence in $\mathrm{Ch}_{\geq 0}(R)$.
2. A map p is a fibration if its normalization $N(f)$ is an h-fibration in $\mathrm{Ch}_{\geq 0}(R)$.
3. A map i is a cofibration if its normalization $N(f)$ is an h-cofibration in $\mathrm{Ch}_{\geq 0}(R)$.

Moreover, every object is fibrant and cofibrant.

Proposition 2.3.18 induces the following.

Proposition 3.3.2. *Let R be a ring (not necessary commutative). For $A \in \mathrm{sAb}$ and $B \in \mathrm{sMod}_R$, the chain complex $N(B)^{N(A)}$ is a deformation retract of $N\left(B^A\right)$ in* $\mathrm{Ch}_{\geq 0}(R)$ *and this is given by the following two diagrams*

$$\begin{tikzcd}
N(B)^{N(A)} \arrow[r, "\mathrm{AW}^*"] \arrow[rr, bend right=50, "\mathrm{id}"'] & N\left(B^A\right) \arrow[r, "\mathrm{EZ}^*"] \arrow[d, phantom, "\Vert"] & N(B)^{N(A)}, & N\left(B^A\right) \arrow[r, "\mathrm{EZ}^*"] \arrow[rr, bend right=50, "\mathrm{id}"'] & N(B)^{N(A)} \arrow[r, "\mathrm{AW}^*"] \arrow[d, phantom, "\Downarrow"] & N\left(B^A\right). \\
 & {} & & & {} &
\end{tikzcd}$$

That is, the composite $\mathrm{EZ}^* \circ \mathrm{AW}^*$ *is the identity, and the composite* $\mathrm{AW}^* \circ \mathrm{EZ}^*$ *is naturally chain homotopic to the identity.*

Proof. Here we consider the categories $\mathrm{Ch}_{\geq 0}(R)$ and sMod_R enriched respectively over the closed symmetric monoidal categories $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ and sAb as stated in **Lemmas 2.3.6** and **2.3.8**.

Consider D^n the n-disk chain complex, i.e., the chain complex with $\mathbb{Z} \xrightarrow{1} \mathbb{Z}$ in degrees n and $n-1$, and zero in all other degrees. In degree n, we have:

$$\begin{aligned}
N\left(B^A\right)_n &\cong \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(D^n, N\left(B^A\right)) \\
&\cong \mathrm{Hom}_{\mathrm{sAb}}\left(\Gamma(D^n), B^A\right), \text{ by the Dold–Kan correspondence} \\
&\cong \mathrm{Hom}_{\mathrm{sAb}}(A \otimes \Gamma(D^n), B), \text{ by the unenriched tensor-hom adjunction} \\
&\cong \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(N(A \otimes \Gamma(D^n)), N(B)), \text{ by the Dold–Kan correspondence}
\end{aligned}$$

and

$$\begin{aligned}
\left(N(B)^{N(A)}\right)_n &\cong \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(D^n, N(B)^{N(A)}\right) \\
&\cong \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(N(A) \otimes D^n, N(B)), \text{ by the unenriched tensor-hom} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{adjunction.}
\end{aligned}$$

Hence, we have the following comparison of n-chains

$$\begin{tikzcd}
N\left(B^A\right)_n \arrow[r, shift left, "\mathrm{EZ}^*"] \arrow[d, "\cong"] & \left(N(B)^{N(A)}\right)_n \arrow[l, shift left, "\mathrm{AW}^*"] \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(N(A \otimes \Gamma(D^n)), N(B)) & \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(N(A) \otimes D^n, N(B))
\end{tikzcd}$$

given by:

$$\begin{tikzcd}
N(B) & N(A \otimes \Gamma(D^n)) \arrow[l, "f"'] \arrow[d, shift left, "\mathrm{AW}"] \arrow[dr, "\mathrm{AW}^*(g)"] & \\
 & N(A) \otimes D^n \arrow[u, shift left, "\mathrm{EZ}"] \arrow[ul, "\mathrm{EZ}^*(f)"] \arrow[r, "g"'] & N(B).
\end{tikzcd}$$

From **Proposition 2.3.18** and by functoriality, we have

$$\mathrm{EZ}^* \circ \mathrm{AW}^* = (\mathrm{AW} \circ \mathrm{EZ})^* = \mathrm{id}.$$

We now show that the map $\mathrm{AW}^* \circ \mathrm{EZ}^*$ is chain homotopic to the identity, by adapting a proof due to Opadotun of an analogous statement [Opa21, Theorem 6.3.1].

The n-disk D^n induces, as n varies, in the category $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ the diagram $D^\bullet$ given by

$$0 \longrightarrow D^0 \xrightarrow{\delta^0} D^1 \xrightarrow{\delta^1} D^2 \xrightarrow{\delta^2} D^3 \longrightarrow \cdots$$

where the coboundary map δ^n is the identity in degree n and 0 elsewhere. Hence, $D^\bullet$ forms a cochain complex in the category $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$. Applying the denormalization Γ degreewise, we obtain $\Gamma(D^\bullet)$

$$0 \longrightarrow \Gamma(D^0) \xrightarrow{\Gamma(\delta^0)} \Gamma\left(D^1\right) \xrightarrow{\Gamma\left(\delta^1\right)} \Gamma\left(D^2\right) \xrightarrow{\Gamma\left(\delta^2\right)} \Gamma\left(D^3\right) \longrightarrow \cdots$$

as a cochain complex in the category sAb. From **Proposition 2.3.18**, there is a chain homotopy $H^n : EZ \circ AW \sim \mathrm{id}$ for each n. By naturality of H^n, as n varies, we obtain a diagram of cochain complexes of chain complexes

$$\begin{tikzcd}
N(A \otimes \Gamma(D^\bullet)) \arrow[d, hook, "\iota_0"'] \arrow[dr, bend left, "EZ \circ AW"] & \\
N(A \otimes \Gamma(D^\bullet)) \otimes N\left(\mathbb{Z}\Delta^1\right) \arrow[r, "H"] & N\left(A \otimes \Gamma\left(D^\bullet\right)\right) \\
N\left(A \otimes \Gamma\left(D^\bullet\right)\right). \arrow[u, hook', "\iota_1"] \arrow[ur, bend right, "\mathrm{id}"'] &
\end{tikzcd}$$

Applying $\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(-, N(B))$ degreewise, we obtain a diagram in $\mathrm{Ch}_{\geq 0}(R)$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}(N(A \otimes \Gamma(D^\bullet)), N(B)) & \\
\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N(A \otimes \Gamma(D^\bullet)) \otimes N\left(\mathbb{Z}\Delta^1\right), N(B)\right) \arrow[u, "\iota_0^*"] \arrow[d, "\iota_1^*"'] & \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N\left(A \otimes \Gamma\left(D^\bullet\right)\right), N(B)\right) \arrow[l, "H^*"'] \arrow[ul, bend right, "AW^* \circ EZ^*"'] \arrow[dl, bend left, "\mathrm{id}"] \\
\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N\left(A \otimes \Gamma\left(D^\bullet\right)\right), N(B)\right). &
\end{tikzcd}$$

To show that $AW^* \circ EZ^* \sim \mathrm{id}$ it suffices to show that

$$\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N\left(A \otimes \Gamma\left(D^\bullet\right)\right) \otimes N\left(\mathbb{Z}\Delta^1\right), N(B)\right)$$

is a path object for $\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N\left(A \otimes \Gamma\left(D^\bullet\right)\right), N(B)\right)$, i.e., the restriction map in $\mathrm{Ch}_{\geq 0}(R)$

$$\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N\left(A \otimes \Gamma\left(D^\bullet\right)\right), N(B)\right) \to \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N\left(A \otimes \Gamma\left(D^\bullet\right)\right) \otimes N\left(\mathbb{Z}\Delta^1\right), N(B)\right) \tag{3.3.1}$$

induced by $\Delta^1 \to \Delta^0$ is a chain homotopy equivalence. By **Proposition 2.3.18**, the map (3.3.1) is a retract of the map

$$\mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N\left(A \otimes \Gamma\left(D^\bullet\right)\right), N(B)\right) \to \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathbb{Z})}\left(N\left(A \otimes \Gamma\left(D^\bullet\right) \otimes \mathbb{Z}\Delta^1\right), N(B)\right)$$

which a calculation identifies as the map

$$N\left(B^{A}\right) \to N\left(B^{A\otimes\mathbb{Z}\Delta^{1}}\right). \tag{3.3.2}$$

The map $A \otimes \mathbb{Z}\Delta^1 \to A$ collapsing the cylinder is a homotopy equivalence in sAb. Hence the induced map on cotensoring $B^A \to B^{A\otimes\mathbb{Z}\Delta^1}$ is a homotopy equivalence in sMod_R. Applying normalization yields a chain homotopy equivalence in $\mathrm{Ch}_{\geq 0}(R)$, namely the map (3.3.2). □

The following proposition gives us some homotopy properties satisfies by the Hurewicz model structure on sMod_R given above.

Proposition 3.3.3. *In the Hurewicz model structure on* sMod_R*, the three classes of maps are characterized as follows.*

1. *A map is a weak equivalence if and only if it is a homotopy equivalence.*
2. *A map is a fibration if and only if it satisfies the HLP.*
3. *A map is a cofibration if and only if it satisfies the HEP.*

Proof. Here we consider the categories $\mathrm{Ch}_{\geq 0}(R)$ and sMod_R enriched, tensored and cotensored over $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ and sAb respectively, as describe in **Lemmas 2.3.6** and **2.3.8**.

(1.) Two maps $f, f' : A \to B$ in sMod_R are homotopic if and only if their normalizations $N(f), N(f') : N(A) \to N(B)$ are chain homotopic [Wei94, Theorem 8.4.1].

Therefore f is a homotopy equivalence if and only if $N(f)$ is a chain homotopy equivalence, i.e., f is a weak equivalence in sMod_R.

(2.) ($\Longrightarrow$) Let $p : E \to B$ be a fibration in sMod_R. Consider the lifting problem in sMod_R given by

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "\iota_0"', "\sim"] & E \arrow[d, two heads, "p"] \\
A \otimes \mathbb{Z}\Delta^1 \arrow[r, "g"'] \arrow[ur, dotted, "?"] & B.
\end{tikzcd}$$

Applying the normalization N,

$$\begin{tikzcd}
N(A) \arrow[rr, "N(f)"] \arrow[d, hook, "\iota_0"', "\sim"] \arrow[dr, hook, "N(\iota_0)", "\sim"'] & & N(E) \arrow[d, two heads, "N(p)"] \\
N(A) \otimes N\left(\mathbb{Z}\Delta^1\right) \arrow[r, hook, "\sim", "\mathrm{EZ}"'] \arrow[urr, dotted, "\exists h"] & N\left(A \otimes \mathbb{Z}\Delta^1\right) \arrow[r, "N(g)"'] \arrow[ur, dotted, "\exists H"'] & N(B)
\end{tikzcd}$$

there exists a lift h by assumption. Also, there exists a lift H in the square given by the identity $N(g) \circ \mathrm{EZ} = N(p) \circ h$, thanks to the fact that the map EZ is a split monomorphism and a chain homotopy equivalence by **Proposition 2.3.18**, i.e., EZ is a trivial Hurewicz cofibration and so satisfies the Left Lifting Property (LLP) with respect to the Hurewicz fibration $N(p)$ in $\mathrm{Ch}_{\geq 0}(R)$. Applying the denormalization Γ,

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, hook, "\Gamma(\iota_0)"', "\sim"] \arrow[dr, hook, "\iota_0", "\sim"'] & & E \arrow[d, two heads, "p"] \\
\Gamma\left(N(A) \otimes N\left(\mathbb{Z}\Delta^1\right)\right) \arrow[r, hook, "\sim", "\Gamma(\mathrm{EZ})"'] \arrow[urr, dotted, "\Gamma(h)"] & A \otimes \mathbb{Z}\Delta^1 \arrow[r, "g"'] \arrow[ur, dotted, "\Gamma(H)"'] & B.
\end{tikzcd}$$

So the homotopy $\Gamma(H)$ gives a solution to the lifting problem. Therefore p satisfies the HLP in sMod_R.

($\Longleftarrow$) Let $p : E \to B$ be a map satisfying the HLP in sMod_R. Thanks to **Proposition 3.1.5**, it suffices to solve the following lifting problem in $\text{Ch}_{\geq 0}(R)$ given by

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "\iota_0"', "\sim"] & N(E) \arrow[d, "N(p)"] \\
A \otimes N\left(\mathbb{Z}\Delta^1\right) \arrow[r, "g"'] \arrow[ur, dotted, "?"] & N(B).
\end{tikzcd}$$

Applying the denormalization Γ,

$$\begin{tikzcd}
\Gamma(A) \arrow[rr, "\Gamma(f)"] \arrow[d, hook, "\iota_0"', "\sim"] & & E \arrow[d, "p"] \\
\Gamma(A) \otimes \mathbb{Z}\Delta^1 \arrow[r, two heads, "\sim", "\Gamma(AW)"'] \arrow[urr, dotted, "\exists h"] & \Gamma\left(A \otimes N\left(\mathbb{Z}\Delta^1\right)\right) \arrow[r, "\Gamma(g)"'] & B
\end{tikzcd}$$

there exists a lift h by assumption. Applying the normalization N,

$$\begin{tikzcd}
 & A \arrow[rr, "f"] \arrow[d, hook, "N(\iota_0)"', "\sim"] \arrow[ddl, hook', bend right, "\iota_0"', "\sim"] & & N(E) \arrow[d, two heads, "N(p)"] \\
 & N\left(\Gamma(A) \otimes \mathbb{Z}\Delta^1\right) \arrow[r, two heads, "\mathrm{AW}", "\sim"'] \arrow[urr, dotted, "N(h)"] & A \otimes N\left(\mathbb{Z}\Delta^1\right) \arrow[r, "g"'] & N(B) \\
A \otimes N\left(\mathbb{Z}\Delta^1\right) \arrow[ur, hook, "\mathrm{EZ}", "\sim"'] \arrow[urr, equal] & & &
\end{tikzcd}$$

we have $\text{AW} \circ \text{EZ} = \text{id}$ by **Proposition 2.3.18**. So the homotopy $H = N(h) \circ \text{EZ}$ gives a solution to the lifting problem. Therefore p is a fibration.

(3.) The proof here is quite similar to the case of fibrations in (2.) above.

($\Longrightarrow$) Let $i : A \to X$ be a cofibration in sMod_R. Consider the lifting problem in

sMod_R given by

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & B^{\mathbb{Z}\Delta^1} \arrow[d, two heads, "ev_0", "\sim"'] \\
X \arrow[r, "g"'] \arrow[ur, dotted, "?"] & B.
\end{tikzcd}$$

Applying the normalization N,

$$\begin{tikzcd}
N(A) \arrow[r, "N(f)"] \arrow[d, hook, "N(i)"'] & N\left(B^{\mathbb{Z}\Delta^1}\right) \arrow[r, two heads, "\mathrm{EZ}^*", "\sim"'] \arrow[dr, two heads, "\sim", "N(ev_0)"'] & N(B)^{N\left(\mathbb{Z}\Delta^1\right)} \arrow[d, two heads, "ev_0", "\sim"'] \\
N(X) \arrow[rr, "N(g)"'] \arrow[ur, dotted, "\exists H"] \arrow[urr, dotted, "\exists h"] & & N(B)
\end{tikzcd}$$

there exists a lift h by assumption. Also, there exists a lift H in the square given by the identity $\mathrm{EZ}^* \circ N(f) = h \circ N(i)$, thanks to the fact that the map EZ^* is a split epimorphism and a chain homotopy equivalence by **Proposition 3.3.2**, i.e., EZ^* is a trivial Hurewicz fibration and so satisfies the Right Lifting Property (RLP) with respect to the Hurewicz cofibration $N(i)$ in $\mathrm{Ch}_{\geq 0}(R)$. Applying the denormalization Γ,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & B^{\mathbb{Z}\Delta^1} \arrow[r, two heads, "\Gamma(\mathrm{EZ}^*)", "\sim"'] \arrow[dr, two heads, "ev_0"'] & \Gamma\left(N(B)^{N(\mathbb{Z}\Delta^1)}\right) \arrow[d, two heads, "\Gamma(ev_0)", "\sim"'] \\
X \arrow[rr, "g"'] \arrow[ur, dotted, "\Gamma(H)"] \arrow[urr, dotted, "\Gamma(h)", "\sim"'] & & B.
\end{tikzcd}$$

So the homotopy $\Gamma(H)$ gives a solution to the lifting problem. Therefore i satisfies the HEP in sMod_R.

($\Longleftarrow$) Let $i : A \to X$ be a map satisfying the HEP in sMod_R. Thanks to **Proposition 3.1.5**, it suffices to solve the following lifting problem in $\mathrm{Ch}_{\geq 0}(R)$ given by

$$\begin{tikzcd}
N(A) \arrow[r, "f"] \arrow[d, "N(i)"'] & B^{N\left(\mathbb{Z}\Delta^1\right)} \arrow[d, two heads, "\sim"', "ev_0"] \\
N(X) \arrow[r, "g"'] \arrow[ur, dotted, "?"] & B.
\end{tikzcd}$$

Applying the denormalization Γ,

$$\begin{tikzcd}
A \arrow[r, "\Gamma(f)"] \arrow[d, "i"'] & \Gamma\left(B^{N\left(\mathbb{Z}\Delta^1\right)}\right) \arrow[r, hook, "\Gamma(\mathrm{AW}^*)", "\sim"'] \arrow[dr, two heads, "\sim", "\Gamma(ev_0)"'] & \Gamma(B)^{\mathbb{Z}\Delta^1} \arrow[d, two heads, "\sim"', "ev_0"] \\
X \arrow[rr, "\Gamma(g)"'] \arrow[urr, dotted, "\exists h"] & & \Gamma(B)
\end{tikzcd}$$

where $\Gamma(\mathrm{AW}^*)$ is the denormalization of the map AW^* given in **Proposition 3.3.2**, there exists a lift h by assumption. Applying the normalization N,

$$\begin{tikzcd}
 & & & B^{N\left(\mathbb{Z}\Delta^1\right)} \arrow[dl, two heads, bend left, "\sim"', "ev_0"] \\
N(A) \arrow[r, "f"] \arrow[d, hook, "N(i)"'] & B^{N\left(\mathbb{Z}\Delta^1\right)} \arrow[r, hook, "\sim", "\mathrm{AW}^*"'] \arrow[urr, Rightarrow, no head] & N\left(\Gamma(B)^{\mathbb{Z}\Delta^1}\right) \arrow[d, two heads, "N(ev_0)"', "\sim"] \arrow[ur, two heads, "\sim", "\mathrm{EZ}^*"'] & \\
N(X) \arrow[rr, "g"'] \arrow[urr, dotted, "N(h)"] & & B &
\end{tikzcd}$$

we have $\mathrm{EZ}^* \circ \mathrm{AW}^* = \mathrm{id}$ by **Proposition 3.3.2**. So the homotopy $H = \mathrm{EZ}^* \circ N(h)$ gives a solution to the lifting problem. Therefore i is a cofibration. □

The Hurewicz model structure on the category sMod_R of the simplicial R-modules satisfies the following monoidal property.

Proposition 3.3.4. *1. For a commutative ring R, the category sMod_R with the Hurewicz model structure is a monoidal model category.*

2. *For an arbitrary ring R, the Hurewicz model structure on* sMod_R *is an enriched model category over the category* sAb *of simplicial abelian groups, with the Hurewicz model structure.*

Proof. For item (1.), we consider the category sMod_R endowed with the closed symmetric monoidal structure as described in **Lemma 2.3.8**.

Let $i : X \to Y$ and $k : V \to W$ be cofibrations in sMod_R. Let us show that their pushout product $i \square k : X \otimes W \amalg_{X \otimes V} Y \otimes V \to Y \otimes W$ is a cofibration too, i.e., for any acyclic fibration $j : A \to B$, the following lifting problem admits a solution by **Proposition 3.3.3**,

$$\begin{tikzcd}
X \otimes W \amalg_{X \otimes V} Y \otimes V \arrow[r, "\widetilde{f}"] \arrow[d, "i \square k"'] & A \arrow[d, two heads, "\sim"', "j"] \\
Y \otimes W \arrow[r, "\widetilde{g}"'] \arrow[ur, dotted, "?"] & B.
\end{tikzcd}$$

Applying the normalization N,

$$\begin{tikzcd}
N(X) \otimes N(W) \amalg_{N(X) \otimes N(V)} N(Y) \otimes N(V) \arrow[r, hook, "\mathrm{EZ} \amalg_{\mathrm{EZ}} \mathrm{EZ}", "\sim"'] \arrow[d, hook, "N(i) \square N(k)"'] & N(X \otimes W \amalg_{X \otimes V} Y \otimes V) \arrow[r, "N(\widetilde{f})"] \arrow[ddl, "N(i \square k)"'] & N(A) \arrow[dd, two heads, "\sim"', "N(j)"] \\
N(Y) \otimes N(W) \arrow[d, hook, "\mathrm{EZ}"', "\sim"] & & \\
N(Y \otimes W) \arrow[rr, "N(\widetilde{g})"'] \arrow[uurr, dotted, "\exists h"] & & N(B)
\end{tikzcd} \tag{3.3.3}$$

there exists a lift h in the outer square thanks to the fact that $\mathrm{EZ} \circ (N(i) \square N(k))$ is the composition of cofibrations by **Proposition 2.3.18** and so satisfies the LLP with respect to the acyclic fibration $N(j)$ in $\mathrm{Ch}_{\geq 0}(R)$, since the Hurewicz model structure turns $\mathrm{Ch}_{\geq 0}(R)$ into a monoidal model category as we have shown in **Proposition 3.1.7**. Applying the denormalization Γ, we have

$$\begin{tikzcd}
\Gamma\big(N(X)\otimes N(W) \amalg_{N(X)\otimes N(V)} N(Y)\otimes N(V)\big) \arrow[r, hook, "\Gamma(\mathrm{EZ}\amalg_{\mathrm{EZ}}\mathrm{EZ})", "\sim"'] \arrow[d, hook, "\Gamma(N(i)\square N(k))"'] & X\otimes W \amalg_{X\otimes V} Y\otimes V \arrow[r, "\widetilde{f}"] \arrow[ddl, hook', "i\square k"'] & A \arrow[dd, two heads, "\sim"', "j"] \\
\Gamma(N(Y)\otimes N(W)) \arrow[d, hook, "\Gamma(\mathrm{EZ})"', "\sim"] & & \\
Y\otimes W \arrow[rr, "\widetilde{g}"'] \arrow[uurr, dotted, "\Gamma(h)"'] & & B.
\end{tikzcd}$$

Now assume moreover that $i : X \to Y$ and $k : V \to W$ are cofibrations (with either of the two being additionally a weak equivalence) in sMod_R. We have to show that their pushout product $i\square k : X \otimes W \amalg_{X\otimes V} Y \otimes V \to Y \otimes W$ is an acyclic cofibration. We have that $N(i)\square N(k)$ is an acyclic cofibration in the monoidal model category $\mathrm{Ch}_{\geq 0}(R)_h$. Hence, considering the upper-left triangle of the diagram (3.3.3), we have that $N(i)\square N(k)$ and EZ are both chain homotopy equivalences in $\mathrm{Ch}_{\geq 0}(R)_h$ by **Proposition 2.3.18**. Moreover, because the two pushouts in

$$N(X)\otimes N(W) \amalg_{N(X)\otimes N(V)} N(Y)\otimes N(V) \xhookrightarrow[\sim]{\mathrm{EZ}\amalg_{\mathrm{EZ}}\mathrm{EZ}} N(X\otimes W) \amalg_{N(X\otimes V)} N(Y\otimes V)$$

are homotopy pushouts in the Hurewicz model category $\mathrm{Ch}_{\geq 0}(R)_h$, the map $\mathrm{EZ}\amalg_{\mathrm{EZ}}\mathrm{EZ}$ is a chain homotopy equivalence. So, by the 2-out-of-3 property, $N(i\square k)$ is a chain homotopy equivalence. Therefore, $i\square k$ is an acyclic cofibration in sMod_R.

For item (2.), consider the proof of item (1.) where we take the map k from sAb and apply item (2.) from **Proposition 3.1.7**. □

3.4 Bousfield model structure on $\mathrm{Ch}^{\geq 0}(\mathcal{A})$

For a bicomplete abelian category $\mathcal{A}$ with an injective class $\mathcal{I}$, i.e., the dual of a projective class [Chr98, §2], one can recover the Bousfield model structure [Bou03, §4.4] on the category $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ of non-negatively graded cochain complexes over $\mathcal{A}$. For a bicomplete pointed category $\mathcal{A}$ (not necessarily abelian) with an injective class $\mathcal{I}$, since the opposite category $(c\mathcal{A})^{\mathrm{op}}$ of cosimplicial objects in $\mathcal{A}$ satisfies $(c\mathcal{A})^{\mathrm{op}} \cong s(\mathcal{A}^{\mathrm{op}})$, applying [CH02, Theorem 6.3] to the opposite category $\mathcal{A}^{\mathrm{op}}$ yields the following model structure on $c\mathcal{A}$.

Proposition 3.4.1. *If for each X in $c\mathcal{A}$ and each I in $\mathcal{I}$, $\mathcal{A}(X, I)$ is a fibrant simplicial set, then the following classes of maps form a model structure on $c\mathcal{A}$.*

1. *A map f is a weak equivalence if it is an $\mathcal{I}$-equivalence, i.e., $\mathcal{A}(f, I)$ is a weak equivalence of simplicial sets, for each $I \in \mathcal{I}$.*
2. *A map f is a cofibration if $\mathcal{A}(f, I)$ is a fibration of simplicial sets, for each $I \in \mathcal{I}$.*
3. *A map f is a fibration if it has the RLP with respect to all $\mathcal{I}$-acyclic cofibrations.*

Now, assuming $\mathcal{A}$ to be abelian, the hypothesis of **Proposition 3.4.1** holds. The

Dold–Kan correspondence yields the diagram

$$\begin{tikzcd}
s(\mathcal{A}^{\mathrm{op}}) \arrow[r, shift left, "N"] \arrow[r, phantom, "\cong"] \arrow[d, "\cong"] & \mathrm{Ch}_{\geq 0}(\mathcal{A}^{\mathrm{op}}) \arrow[l, shift left, "\Gamma"] \arrow[d, "\cong"] \\
(c\mathcal{A})^{\mathrm{op}} \arrow[r, "\cong"] & (\mathrm{Ch}^{\geq 0}(\mathcal{A}))^{\mathrm{op}}
\end{tikzcd}$$

and so induces a model structure on the category $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ of non-negatively graded cochain complexes of objects in $\mathcal{A}$ given as follows by dualizing [CH02, Corollary 6.4], also found in [Bou03, §4.4].

Corollary 3.4.2 (Bousfield model structure)**.** *Let $\mathcal{A}$ be a bicomplete abelian category with an injective class $\mathcal{I}$. The category $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ of non-negatively graded cochain complexes of objects in $\mathcal{A}$ has a model structure given as follows.*

1. *A map f is a weak equivalence if $\mathcal{A}(f, I)$ is a quasi-isomorphism in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$, for each $I \in \mathcal{I}$.*
2. *A map f is a cofibration if it is $\mathcal{I}$-monic in positive degrees (but not necessarily in degree 0) in $\mathcal{A}$, i.e., $\mathcal{A}(f, I)$ is surjective in positive degrees, for each $I \in \mathcal{I}$.*
3. *A map f is a fibration if it is a degreewise split epimorphism with $\mathcal{I}$-injective kernel.*

Moreover, every complex is cofibrant, and a complex is fibrant if and only if it is a complex of $\mathcal{I}$-injectives.

Taking $\mathcal{I}$ to be the trivial injective class given by all objects of $\mathcal{A}$ yields the following example.

Example 3.4.3. *For $\mathcal{A}$ a bicomplete abelian category,* $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ *has a model structure given by:*

1. *A map is a weak equivalence if it is a cochain homotopy equivalence.*
2. *A map is a cofibration if it is a degreewise split monomorphism in positive degrees.*
3. *A map is a fibration if it is a degreewise split epimorphism.*

Moreover, every complex is fibrant and cofibrant.

Chapter 4

Change of Enrichment along a Weak Monoidal Quillen Pair

The chapter is organized as follows. In Section 4.1, we give the conditions under which the tensoring and cotensoring structures are preserved. In Section 4.2, we introduce the notion of weak enriched adjunction and we show that any weak monoidal Quillen adjunction lifts to a weak enriched adjunction. In Section 4.3, we introduce the notions of weak tensoring and weak cotensoring, and we show that they are preserved by any weak monoidal Quillen adjunction. In Section 4.4, we introduce equivalent formulations of the unit axiom and we prove that some implications between them hold, whenever the tensoring is replaced by its weak version. In Section 4.5, we introduce the notion of weak $\mathcal{V}$-model category and we show that this is preserved by a change of enrichment along a weak monoidal Quillen adjunction. Finally, in Section 4.6, we apply some of the above mentioned results in the case of the Dold–Kan correspondence.

4.1 Preservation of tensoring and cotensoring

Given an adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$, the following lemma gives us a necessary and sufficient condition for which the change of enrichment along the right adjoint G preserves the underlying category, that is, $(G_*\mathcal{C})_0 \cong \mathcal{C}_0$ for any $\mathcal{V}$-category $\mathcal{C}$. In this case, any potential model structure on $\mathcal{C}_0$ is preserved.

Lemma 4.1.1. *Let* $(F \dashv G)$ *be an adjunction given by* $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$ *, where the right adjoint* G *is lax monoidal. The following are equivalent.*

1. $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$.

2. G *commutes with the underlying set functors, that is, the following diagram commutes (up to natural isomorphism)*

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "G"] \arrow[dr, "\mathcal{V}(\mathbb{1}_{\mathcal{V}}, -)=U"'] & \mathcal{W} \arrow[d, "\mathcal{W}(\mathbb{1}_{\mathcal{W}}, -)=U"] \\
& \mathbf{Set},
\end{tikzcd} \tag{4.1.1}$$

where U *is the underlying set functor.*

3. *Change of enrichment along* $G : \mathcal{V} \to \mathcal{W}$ *preserves the underlying category for every* $\mathcal{V}$*-enriched category* $\mathcal{C}$*, that is, the following diagram commutes (up to*

natural isomorphism)

$$\begin{tikzcd}
\mathcal{V}\text{−}\mathbf{Cat} \arrow[r, "G_*"] \arrow[dr, "U_*"'] & \mathcal{W}\text{−}\mathbf{Cat} \arrow[d, "U_*"] \\
& \mathbf{Cat},
\end{tikzcd} \tag{4.1.2}$$

where U_* is the forgetful functor.

Proof. (1) $\iff$ (2) The adjunction $(F \dashv G)$ gives us the following natural bijection

$$\mathcal{V}(F(\mathbb{1}_{\mathcal{W}}), v) \cong \mathcal{W}(\mathbb{1}_{\mathcal{W}}, Gv), \forall v \in \mathcal{V}, \tag{4.1.3}$$

and the commutativity of (4.1.1) gives us the following natural bijection

$$\mathcal{V}(\mathbb{1}_{\mathcal{V}}, v) \cong \mathcal{W}(\mathbb{1}_{\mathcal{W}}, Gv), \forall v \in \mathcal{V}. \tag{4.1.4}$$

Hence (4.1.3) and (4.1.4) gives us the following natural bijection

$$\mathcal{V}(F(\mathbb{1}_{\mathcal{W}}), v) \cong \mathcal{V}(\mathbb{1}_{\mathcal{V}}, v) \iff F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}, \text{ by the Yoneda Lemma.}$$

(2) $\implies$ (3) Since (2) is equivalent to (1), $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$ holds. Then the underlying categories are preserved and so the commutativity of (4.1.2) holds on objects. On hom objects, we have

$$\begin{aligned}\underline{U_* G_* \mathcal{C}}(x, y) &:= \underline{(UG)_* \mathcal{C}}(x, y), \text{ by } \textbf{Proposition 2.1.7} \\ &\cong \underline{U_* \mathcal{C}}(x, y), \ \forall x, y \in \mathcal{C},\end{aligned}$$

natural since (4.1.1) is.

(3) $\Longrightarrow$ (2) The category $\mathcal{V}$ is $\mathcal{V}$-enriched as a monoidal category and we have

$$\begin{tikzcd}
& \underline{U_* G_* \mathcal{V}}(\mathbb{1}_{\mathcal{V}}, v) \arrow[r, "\cong"] \arrow[d, ":="'] & \underline{U_* \mathcal{V}}(\mathbb{1}_{\mathcal{V}}, v) \arrow[d, ":="] & \\
& UG\underline{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, v) \arrow[d, "\cong"'] & U\underline{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, v) \arrow[d, "\cong"] & \\
\mathcal{W}(\mathbb{1}_{\mathcal{W}}, Gv) \arrow[r, equal] & :UGv \arrow[r, dashed, "\therefore \cong"'] & Uv: \arrow[r, equal] & \mathcal{V}(\mathbb{1}_{\mathcal{V}}, v),
\end{tikzcd} \tag{4.1.5}$$

for any $v \in \mathcal{V}$, that is, the solid diagram induces an isomorphism $UGv \cong Uv$ given by the dashed map. All the maps in diagram (4.1.5) are natural in $\mathcal{W}$, so we are done. □

Corollary 4.1.2. *For any adjunction $(F \dashv G)$ with F strong monoidal, G preserves the underlying sets.*

Proof. This follows from **Lemma 4.1.1**, since F strong monoidal implies $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$ and G lax monoidal, as the right adjoint of the strong (and so oplax) monoidal functor F. □

The following example illustrates **Lemma 4.1.1**.

Example 4.1.3. *1. The normalization N and the denormalization Γ defining the Dold–Kan correspondence* $\mathrm{sMod}_R \underset{\Gamma}{\overset{N}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} \mathrm{Ch}_{\geq 0}(R)$ *are both not strong monoidal but preserve underlying sets.*

- *The underlying set of a simplicial left R-module $A_\bullet$ is given by its 0-simplices, i.e., $U(A_\bullet) = A_0$, and the underlying set of a chain complex* *is given by its* 0*-cycles, in particular,* $U(N(A_\bullet)) = Z_0(N(A_\bullet))$*. But since* $N(A_\bullet)$ *is a non-negatively graded chain complex, we have* $Z_0(N(A_\bullet)) = N(A_\bullet)_0 = A_0$ *by definition.*

- *The underlying set of a chain complex* C *is given by its* 0*-cycles, i.e.,* $U(C) = Z_0(C) = C_0$ *since* C *is non-negatively graded. But since the underlying set of a simplicial* R*-module is given by its* 0*-simplices, then* $U(\Gamma(C)) = \Gamma(C)_0 = C_0$.

2. *The geometric realization* $\mathbf{sSet} \xrightarrow{|\cdot|} \mathbf{Top}$ *is strong monoidal but* $U(|X_\bullet|) \ncong U(X_\bullet) = X_0$*, i.e.,* $|\cdot|$ *does not preserve the underlying set. Note that the geometric realization* $|\cdot|$ *does not admit a left adjoint, since it doesn't preserve infinite products.*

The following proposition gives us a necessary and sufficient condition, that is, F strong monoidal, under which the tensoring or the cotensoring is preserved by a change of enrichment along the right adjoint of the lax monoidal adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$.

Proposition 4.1.4. *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be two closed symmetric monoidal categories, and* G *be a lax monoidal functor such that the following is an adjunction* $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$.

1. *If* F *is strong monoidal, then for a* $\mathcal{V}$*-category* $\mathcal{C}$*, the* $\mathcal{W}$*-category* $G_*\mathcal{C}$ *admits a tensoring or a cotensoring over* $\mathcal{W}$ *given respectively by:*

$$x \otimes w := x \otimes Fw \text{ and } x^w := x^{Fw}, \text{ for any } x \in \mathcal{C} \text{ and } w \in \mathcal{W}.$$

2. *If $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$ and the $\mathcal{W}$-category $G_*\mathcal{V}$ admits a tensoring or a cotensoring over $\mathcal{W}$, then F is strong monoidal.*

Proof. (1.) See [Rie14, Proposition 3.7.11].

(2.) Assume that $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$ and the $\mathcal{W}$-category $G_*\mathcal{V}$ admits a tensoring over $\mathcal{W}$. We have the following bijection for any $w \in \mathcal{W}$ and $v_1, v_2 \in \mathcal{V}$.

$$\begin{aligned}
\mathrm{Hom}_{G_*\mathcal{V}}(v_1 \otimes w, v_2) &\cong \mathrm{Hom}_{\mathcal{W}}(w, \underline{G_*\mathcal{V}}(v_1, v_2)), \text{ by definition of the tensoring} \\
\mathrm{Hom}_{\mathcal{V}}(v_1 \otimes w, v_2) &\cong \mathrm{Hom}_{\mathcal{W}}(w, G\underline{\mathcal{V}}(v_1, v_2)), \ (G_*\mathcal{V})_0 \cong \mathcal{V}_0 \text{ since } F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}} \\
&\cong \mathrm{Hom}_{\mathcal{V}}(Fw, \underline{\mathcal{V}}(v_1, v_2)), \text{ by the unenriched adjunction } F \dashv G \\
&\cong \mathrm{Hom}_{\mathcal{V}}(v_1 \otimes Fw, v_2), \text{ by the unenriched tensor-hom adjunction} \\
\implies v_1 \otimes w &\cong v_1 \otimes Fw, \text{ by the unenriched Yoneda Lemma applied to } \mathcal{V}.
\end{aligned}$$

In particular for $v_1 = \mathbb{1}_{\mathcal{V}}, \mathbb{1}_{\mathcal{V}} \otimes w \cong \mathbb{1}_{\mathcal{V}} \otimes F(w) \cong F(w)$. Hence, applying the associativity of the tensoring we have the following for any $w_1, w_2 \in \mathcal{W}$:

$$\begin{tikzcd}
(\mathbb{1}_{\mathcal{V}} \otimes w_1) \otimes w_2 \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathbb{1}_{\mathcal{V}} \otimes (w_1 \otimes w_2) \arrow[d, "\cong"] & \\
F(w_1) \otimes w_2 \arrow[d, "\cong"'] & F(w_1 \otimes w_2) \arrow[r, Rightarrow] & F(w_1 \otimes w_2) \cong F(w_1) \otimes F(w_2). \\
F(w_1) \otimes F(w_2) \arrow[ur, dashed, "\therefore \cong"'] & &
\end{tikzcd}$$

Now, assume that $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$ and the $\mathcal{W}$-enriched category $G_*\mathcal{V}$ admits a cotensoring over $\mathcal{W}$. We have the following bijection for any $w \in \mathcal{W}$ and $v_1, v_2 \in \mathcal{V}$.

$$\begin{aligned}
\mathrm{Hom}_{G_*\mathcal{V}}(v_1, v_2^w) &\cong \mathrm{Hom}_{\mathcal{W}}(w, \underline{G_*\mathcal{V}}(v_1, v_2)), \text{ by definition of the cotensoring} \\
\mathrm{Hom}_{\mathcal{V}}(v_1, v_2^w) &\cong \mathrm{Hom}_{\mathcal{W}}(w, G\underline{\mathcal{V}}(v_1, v_2)), \text{ since } F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}} \\
&\cong \mathrm{Hom}_{\mathcal{V}}(Fw, \underline{\mathcal{V}}(v_1, v_2)), \text{ by the unenriched adjunction } F \dashv G \\
&\cong \mathrm{Hom}_{\mathcal{V}}(v_1 \otimes Fw, v_2), \text{ by the unenriched tensor-hom adjunction} \\
&\cong \mathrm{Hom}_{\mathcal{V}}(v_1, \underline{\mathcal{V}}(Fw, v_2)), \text{ by the unenriched tensor-hom adjunction} \\
\Longrightarrow\ v_2^w &\cong \underline{\mathcal{V}}(Fw, v_2) =: v_2^{Fw}, \text{ by the unenriched Yoneda Lemma applied in } \mathcal{V}.
\end{aligned}$$

By the exponential law and the Yoneda Lemma, we have the following for any $w_1, w_2 \in \mathcal{W}$ and $v \in \mathcal{V}$:

$$\begin{tikzcd}
(v^{w_2})^{w_1} \arrow[r, "\cong"] \arrow[d, "\cong"'] & v^{w_1 \otimes w_2} \arrow[d, "\cong"] & \\
\underline{\mathcal{V}}(F(w_1), \underline{\mathcal{V}}(F(w_2), v)) \arrow[d, "\cong"'] & \underline{\mathcal{V}}(F(w_1 \otimes w_2), v) \arrow[r, Rightarrow] & F(w_1 \otimes w_2) \cong F(w_1) \otimes F(w_2). \\
\underline{\mathcal{V}}(F(w_1) \otimes F(w_2), v) \arrow[ur, dashed, "\therefore \cong"'] & &
\end{tikzcd}$$

This completes the proof. $\square$

4.2 Weak enriched adjunction

The proposition bellow gives the necessary and sufficient condition under which the lax monoidal adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$ lifts to a $\mathcal{W}$-adjunction $F : \mathcal{W} \rightleftarrows G_*\mathcal{V} : G$.

Proposition 4.2.1. *The lax monoidal adjunction* $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$ *lifts to a* $\mathcal{W}$*-adjunction* $F : \mathcal{W} \rightleftarrows G_*\mathcal{V} : G$ *if and only if the left adjoint* F *is strong monoidal.*

Proof. ($\implies$) Let us show that the functor F satisfies the natural isomorphisms $F(w \otimes w') \cong Fw \otimes Fw'$, for any $w, w' \in \mathcal{W}$.

$$\begin{aligned}
\underline{G_*\mathcal{V}}(F(w \otimes w'), v) &\cong \underline{\mathcal{W}}(w \otimes w', Gv) \text{ by enriched } F \dashv G \\
&\cong \underline{\mathcal{W}}(w, \underline{\mathcal{W}}(w', Gv)) \text{ by the tensor-hom adjunction on } \mathcal{W} \\
&\cong \underline{\mathcal{W}}(w, \underline{G_*\mathcal{V}}(Fw', v)) \text{ by enriched } F \dashv G \\
&:= \underline{\mathcal{W}}(w, G\underline{\mathcal{V}}(Fw', v)) \text{ by definition} \\
&\cong \underline{G_*\mathcal{V}}(Fw, \underline{\mathcal{V}}(Fw', v)) \text{ by enriched } F \dashv G \\
&\cong \underline{G_*\mathcal{V}}(Fw \otimes Fw', v) \text{ by the tensor-hom adjunction on } \mathcal{V}.
\end{aligned}$$

Therefore, $F(w \otimes w') \cong Fw \otimes Fw'$ by the enriched Yoneda Lemma and the naturality is given by the naturality of the unenriched adjunction $F \dashv G$.

($\impliedby$) The converse is found in [Rie14, Corollary 3.7.12]. This relied on [Rie14, Proposition 3.7.10], which we spelled out in **Proposition 2.2.4**. □

Below we provide more details about the converse of the above proof. Let us start by lifting both functors F and G to $\mathcal{W}$-functors and produce some natural isomorphisms $\underline{G_*\mathcal{V}}(Fw, v) \cong \underline{\mathcal{W}}(w, Gv)$ in $\mathcal{W}$. This is done under the assumption that the left adjoint F is strong monoidal.

1. Lifting F to a $\mathcal{W}$-functor.

- On objects, the $\mathcal{W}$-functor is given by the original unenriched functor F as follows

$$\mathrm{Ob}(\mathcal{W}) \xrightarrow{F} \mathrm{Ob}(G_*\mathcal{V}) := \mathrm{Ob}(\mathcal{V}).$$

- On hom objects, the $\mathcal{W}$-functor

$$\underline{\mathcal{W}}(x,y) \xrightarrow{F_{x,y}} \underline{G_*\mathcal{V}}(Fx,Fy) := G\underline{\mathcal{V}}(Fx,Fy)$$

  is defined by its adjunct $\overline{\widetilde{F_{x,y}}}$ obtained successively through the adjunction $F \dashv G$ and the tensor-hom adjunction, by the composite

$$F\underline{\mathcal{W}}(x,y) \otimes Fx \underset{\longrightarrow}{\xleftarrow[\cong]{\delta}} F(\underline{\mathcal{W}}(x,y) \otimes x) \xrightarrow{F(\mathrm{ev})} Fy.$$

- The $\mathcal{W}$-functor F preserves compositions as shown by the following diagram chase obtained successively through the adjunction $F \dashv G$ and the tensor-hom adjunction.

$$\begin{tikzcd}
 & F(\underline{\mathcal{W}}(y,z) \otimes \underline{\mathcal{W}}(x,y)) \otimes Fx \arrow[dl, shift right, "\delta"'] \arrow[dr, "F(\circ)\otimes Fx"] & \\
F\underline{\mathcal{W}}(y,z) \otimes F\underline{\mathcal{W}}(x,y) \otimes Fx \arrow[ur, shift right, "\cong"'] \arrow[rr, "\circ\otimes Fx"] \arrow[d, "F(F_{y,z})\otimes F(F_{x,y})\otimes Fx"'] & & F\underline{\mathcal{W}}(x,z) \otimes Fx \arrow[dd, "\overline{\widetilde{F_{x,z}}}"'] \\
FG\underline{\mathcal{V}}(Fy,Fz) \otimes FG\underline{\mathcal{V}}(Fx,Fy) \otimes Fx \arrow[d, "(\epsilon\otimes\epsilon)\otimes Fx"'] & \underline{\mathcal{V}}(Fy,Fz) \otimes Fy \arrow[dr, "\mathrm{ev}"] & \\
\underline{\mathcal{V}}(Fy,Fz) \otimes \underline{\mathcal{V}}(Fx,Fy) \otimes Fx \arrow[ur, "\mathrm{id}\otimes\mathrm{ev}"] \arrow[rr, "\bar{\circ}"] \arrow[dr, "\circ\otimes Fx"'] & & Fz \\
 & \underline{\mathcal{V}}(Fx,Fz) \otimes Fx. \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

(4.2.1)

- The $\mathcal{W}$-functor F preserves identities as shown by the following diagram chase obtained successively through the adjunction $F \dashv G$ and the tensor-hom adjunction,

$$\begin{tikzcd}
F\mathbb{1}_{\mathcal{W}} \otimes Fx \arrow[r, "F1_x \otimes Fx"] \arrow[d, shift right, "\epsilon\otimes Fx"'] & F\underline{\mathcal{W}}(x,x) \otimes Fx \arrow[d, "\widetilde{F_{x,x}}"] \\
\mathbb{1}_{\mathcal{V}} \otimes Fx \arrow[r, "\lambda_{Fx}"'] \arrow[u, shift right, "\cong"'] & Fx.
\end{tikzcd} \tag{4.2.2}$$

- The underlying functor of the $\mathcal{W}$-functor $F : \underline{\mathcal{W}} \to \underline{G_*\mathcal{V}}$ is the original functor $F : \mathcal{W} \to \mathcal{V}$. Indeed, for any map $f : x \to y$ in $\mathcal{W}$, we have

$$\left(Fx \xrightarrow{Ff} Fy\right) = \!=\!= \mathbb{1}_{\mathcal{W}} \xrightarrow{\quad Ff \quad} \underline{G_*\mathcal{V}}(Fx,Fy) = G\underline{\mathcal{V}}(Fx,Fy)$$

$$\Big\Vert$$

$$\mathbb{1}_{\mathcal{W}} \xrightarrow{\quad f \quad} \underline{\mathcal{W}}(x,y) \xrightarrow{F_{xy}} \underline{G_*\mathcal{V}}(Fx,Fy) = G\underline{\mathcal{V}}(Fx,Fy)$$

since the following diagram, formed by the adjuncts of the maps above, is commutative:

$$\begin{tikzcd}
F\mathbb{1}_{\mathcal{W}} \otimes Fx \arrow[r, "\cong"] \arrow[d, shift left, "\cong"] & F(\mathbb{1}_{\mathcal{W}} \otimes x) \arrow[d, "\cong"] & \\
\mathbb{1}_{\mathcal{V}} \otimes Fx \arrow[u, shift left, "\eta\otimes Fx"] \arrow[r, shift left, "\lambda_x"] \arrow[d, "1\otimes Fx"'] & Fx \arrow[l, shift left, "\cong"] \arrow[r, "Ff"] & Fy \arrow[d, shift left, "\cong"] \\
F\underline{\mathcal{W}}(x,y) \otimes Fx \arrow[rr, shift left, "\cong"] & & F(\underline{\mathcal{W}}(x,y) \otimes x). \arrow[ll, shift left, "\delta"] \arrow[u, shift left, "F(ev)"]
\end{tikzcd} \tag{4.2.3}$$

2. Lifting G to a $\mathcal{W}$-functor.

- On objects,

$$\mathrm{Ob}(\mathcal{V}) =: \mathrm{Ob}(G_*\mathcal{V}) \xrightarrow{G} \mathrm{Ob}(\mathcal{W})$$

  given by the original unenriched functor G.

- On hom objects,

$$G\underline{\mathcal{V}}(x,y) =: \underline{G_*\mathcal{V}}(x,y) \xrightarrow{G_{x,y}} \underline{\mathcal{W}}(Gx,Gy)$$

  is given by its adjunct $\widetilde{G_{x,y}}$ obtained, through the tensor-hom adjunction by the composite

$$G\underline{\mathcal{V}}(x,y) \otimes Gx \xrightarrow{\mu} G(\underline{\mathcal{V}}(x,y) \otimes x) \xrightarrow{G(ev)} Gy.$$

- The $\mathcal{W}$-functor G preserves compositions as shown by the following diagram chase obtained through the tensor-hom adjunction.

$$\begin{tikzcd}
G\underline{\mathcal{V}}(y,z) \otimes G\underline{\mathcal{V}}(x,y) \otimes Gx \arrow[rr, "\circ\otimes Gx"] \arrow[dr, "G\underline{\mathcal{V}}(y,z)\otimes\widetilde{G_{x,y}}"] \arrow[dd, "G_{y,z}\otimes G_{x,y}\otimes Gx"'] & & G\underline{\mathcal{V}}(x,z) \otimes Gx \arrow[dd, "\widetilde{G_{x,z}}"] \\
& G\underline{\mathcal{V}}(y,z) \otimes Gy \arrow[dr, "\widetilde{G_{y,z}}"] & \\
\underline{\mathcal{W}}(Gy,Gz) \otimes \underline{\mathcal{W}}(Gx,Gy) \otimes Gx \arrow[rr, "\widetilde{\circ}"] \arrow[dr, "\circ\otimes Gx"'] & & Gz \\
& \underline{\mathcal{W}}(Gx,Gz) \otimes Gx \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd} \tag{4.2.4}$$

- The $\mathcal{W}$-functor G preserves identities as shown by the following diagram chase obtained through the tensor-hom adjunction.

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{W}} \otimes Gx \arrow[r, "1_x \otimes Gx"] \arrow[dr, "\lambda_{Gx}"'] & G\underline{\mathcal{V}}(x,x) \otimes Gx \arrow[d, "\widetilde{G_{x,x}}"] \\
 & Gx
\end{tikzcd} \tag{4.2.5}$$

- The underlying functor of the $\mathcal{W}$-functor $G : \underline{G_*\mathcal{V}} \to \underline{\mathcal{W}}$ is $G : \mathcal{V} \to \mathcal{W}$. Indeed, for any map $f : x \to y$ in $\mathcal{V}$, we have

  $$\begin{tikzcd}
  \left(Gx \xrightarrow{Gf} Gy\right) \arrow[r, equal] & \mathbb{1}_{\mathcal{W}} \arrow[rr, "Gf"] & {} \arrow[d, equal] & \underline{\mathcal{W}}(Gx, Gy) \\
   & \mathbb{1}_{\mathcal{W}} \arrow[r, "f"] & \underline{G_*\mathcal{V}}(x,y) = G\underline{\mathcal{V}}(x,y) \arrow[r, "G_{xy}"] & \underline{\mathcal{W}}(Gx, Gy)
  \end{tikzcd}$$

  since the following diagram, formed by the adjuncts of the maps above, is commutative:

  $$\begin{tikzcd}
  \mathbb{1}_{\mathcal{W}} \otimes Gx \arrow[r, "\cong"] \arrow[dr, "1 \otimes Gx"'] & Gx \arrow[r, "Gf"] & Gy \\
   & G\underline{\mathcal{V}}(x,y) \otimes Gx \arrow[r, "\mu"'] & G(\underline{\mathcal{V}}(x,y) \otimes x) \arrow[u, "G(\mathrm{ev})"']
  \end{tikzcd} \tag{4.2.6}$$

3. The isomorphisms $\underline{G_*\mathcal{V}}(Fw, v) \cong \underline{\mathcal{W}}(w, Gv)$ in $\mathcal{W}$ is given by the Yoneda

Lemma. Indeed,

$$\begin{aligned}
\mathcal{W}(w', \underline{G_*\mathcal{V}}(Fw, v)) &:= \mathcal{W}(w', G\underline{\mathcal{V}}(Fw, v)) \text{ by definition} \\
&\cong \mathcal{V}(Fw', \underline{\mathcal{V}}(Fw, v)) \text{ by the unenriched adjunction } F \dashv U \\
&\cong \mathcal{V}(Fw \otimes Fw', v) \text{ by the tensor-hom adjunction} \\
&\cong \mathcal{V}(F(w \otimes w'), v) \text{ since } F \text{ is strong monoidal} \\
&\cong \mathcal{W}(w \otimes w', Gv) \text{ by the unenriched adjunction } F \dashv U \\
&\cong \mathcal{W}(w', \underline{\mathcal{W}}(w, Gv)) \text{ by the tensor-hom adjunction.}
\end{aligned}$$

The naturality is given by the one of the different adjunctions.

In the case of a weak Quillen adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$, the left adjoint F is not necessary strong monoidal and so, the adjunct $F \dashv G$ lift to a weak version of the $\mathcal{W}$-adjunction $F : \mathcal{W} \rightleftarrows G_*\mathcal{V} : G$ defined as follows.

Definition 4.2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-model categories. A* ***weak $\mathcal{V}$-adjunction*** $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *consists of natural maps $\underline{\mathcal{D}}(Fc, d) \to \underline{\mathcal{C}}(c, Gd)$ that are weak equivalences in $\mathcal{V}$, for any cofibrant $c \in \mathcal{C}$ and fibrant $d \in \mathcal{D}$.*

Proposition 4.2.3. *Let $\mathcal{V}$ and $\mathcal{W}$ be monoidal model categories. Any weak monoidal Quillen adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$ lifts to a weak $\mathcal{W}$-adjunction $F : \mathcal{W} \rightleftarrows G_*\mathcal{V} : G$.*

Proof. For any $w_1 \in \mathcal{W}$, any cofibrant $w_2 \in \mathcal{W}$ and any fibrant $v \in \mathcal{V}$, we have the following.

$$\begin{aligned}
\mathcal{W}(w_1, \underline{G_*\mathcal{V}}(Fw_2, v)) &:= \mathcal{W}(w_1, G\underline{\mathcal{V}}(Fw_2, v)), \text{ by definition} \\
&\cong \mathcal{V}(Fw_1, \underline{\mathcal{V}}(Fw_2, v)), \text{ by the adjunction } F \dashv G \\
&\cong \mathcal{V}(Fw_1 \otimes Fw_2, v), \text{ by the tensor-hom adjunction} \\
&\xrightarrow{\delta^*} \mathcal{V}(F(w_1 \otimes w_2), v), \ \delta \text{ the comultiplication map of } F \\
&\cong \mathcal{W}(w_1 \otimes w_2, Gv), \text{ by the adjunction } F \dashv G \\
&\cong \mathcal{W}(w_1, \underline{\mathcal{W}}(w_2, Gv)), \text{ by the tensor-hom adjunction.} \quad (4.2.7)
\end{aligned}$$

For $w_1 = \underline{G_*\mathcal{V}}(Fw_2, v)$, the image of the identity gives us the comparison map

$$\underline{G_*\mathcal{V}}(Fw_2, v) \xrightarrow{\phi_{w_2,v}} \underline{\mathcal{W}}(w_2, Gv).$$

For w_1, w_2 cofibrant in $\mathcal{W}$, $F(w_1 \otimes w_2) \underset{\sim}{\xrightarrow{\delta}} Fw_1 \otimes Fw_2$ is a weak equivalence in $\mathcal{V}$. Then applying $\underline{\mathcal{V}}(-, v)$, with fibrant $v \in \mathcal{V}$, we obtain the following weak equivalence

$$\underline{\mathcal{V}}(Fw_1 \otimes Fw_2, v) \underset{\sim}{\xrightarrow{\delta^*}} \underline{\mathcal{V}}(F(w_1 \otimes w_2), v),$$

since $\underline{\mathcal{V}}(-, v)$, with v fibrant, preserves weak equivalences between cofibrant objects in $\mathcal{V}$ by [Rie14, Lemma 9.2.3].

We want to show that the map $\phi_{w_2,v}$ constructed above is a weak equivalence when $w_2 \in \mathcal{W}$ is cofibrant and $v \in \mathcal{V}$ is fibrant. By saturation [Hov99, Theorem 1.2.10], it suffices to show that the induced map $[w_1, \underline{G_*\mathcal{V}}(Fw_2, v)] \xrightarrow{(\phi_{w_2,v})_*} [w_1, \underline{\mathcal{W}}(w_2, Gv)]$ of hom sets in Ho($\mathcal{W}$), is a bijection for any cofibrant $w_1 \in \mathcal{W}$. By **Remark 2.5.12**,

$\mathrm{Ho}(\mathcal{V})$ and $\mathrm{Ho}(\mathcal{W})$ are closed symmetric monoidal categories, and this fact gives us the following.

$$\begin{aligned}
[w_1, \underline{G_*\mathcal{V}}(Fw_2, v)] &:= [w_1, G\underline{\mathcal{V}}(Fw_2, v)], \text{ by definition} \\
&\cong [w_1, \mathbf{R}G\left(\underline{\mathcal{V}}(Fw_2, v)\right)], \text{ since } \underline{\mathcal{V}}(Fw_2, v) \text{ is fibrant} \\
&\cong [\mathbf{L}F(w_1), \underline{\mathcal{V}}(Fw_2, v)], \text{ by the derived adjunction} \\
&\cong [\mathbf{L}F(w_1) \otimes^{\mathbf{L}} F(w_2), v], \text{ by the derived tensor-hom in } \mathrm{Ho}(\mathcal{V}) \\
&\cong [F(w_1) \otimes F(w_2), v], \text{ since } w_1 \text{ and } w_2 \text{ are cofibrant} \\
&\underset{\cong}{\xrightarrow{\delta^*}} [F(w_1 \otimes w_2), v], \text{ since } \delta \text{ is a weak equivalence} \\
&\cong [\mathbf{L}F(w_1 \otimes w_2), v], \text{ since } w_1 \otimes w_2 \text{ is cofibrant} \\
&\cong [w_1 \otimes w_2, \mathbf{R}G(v)], \text{ by the derived adjunction} \\
&\cong [w_1, \underline{\mathcal{W}}(w_2, \mathbf{R}G(v))], \text{ by the derived tensor-hom in } \mathrm{Ho}(\mathcal{W}) \\
&\cong [w_1, \underline{\mathcal{W}}(w_2, Gv)], \text{ since } v \text{ is fibrant,}
\end{aligned}$$

where $\mathbf{R}G$ and $\mathbf{L}F$ are respectively the right and the left derived functor of the adjunction $F \dashv G$. $\square$

The following lemma is used in the proof of **Proposition 4.3.3**.

Lemma 4.2.4. *Consider the map* $\mathcal{W}(w_1, \underline{G_*\mathcal{V}}(Fw_2, v)) \to \mathcal{W}(w_1, \underline{\mathcal{W}}(w_2, Gv))$ *constructed in Equation (4.2.7). In the special case* $w_1 = Gv$ *and* $w_2 = \mathbb{1}_{\mathcal{W}}$*, we have*

$$\mathcal{W}(Gv, \underline{G_*\mathcal{V}}(F\mathbb{1}_{\mathcal{W}}, v)) \to \mathcal{W}(Gv, \underline{\mathcal{W}}(\mathbb{1}_{\mathcal{W}}, Gv))$$

$$G(\varepsilon^* \circ \eta) \mapsto \tilde{\eta},$$

with $v \xrightarrow[\cong]{\eta} \underline{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, v) \xrightarrow{\varepsilon^*} \underline{\mathcal{V}}(F\mathbb{1}_{\mathcal{W}}, v)$.

Proof. From the series of maps in (4.2.7), diagram chasing gives the following:

$$\begin{tikzcd}
\mathcal{W}(Gv, G\underline{\mathcal{V}}(F(\mathbb{1}_{\mathcal{W}}), v)) & & \ni & G(\varepsilon^* \circ \eta) \arrow[d, mapsto] \\
\xrightarrow{\cong} \mathcal{V}(FGv, \underline{\mathcal{V}}(F(\mathbb{1}_{\mathcal{W}}), v)) & & \ni & \varepsilon^* \circ \eta \circ \epsilon_v \arrow[d, mapsto] \\
\xrightarrow{\cong} \mathcal{V}(FGv \otimes F(\mathbb{1}_{\mathcal{W}}), v) & & \ni & \mathrm{ev} \circ (\eta \otimes \mathrm{id}) \circ (\epsilon \otimes \varepsilon) \arrow[d, mapsto] \\
\xrightarrow{\delta^*} \mathcal{V}(F(Gv \otimes \mathbb{1}_{\mathcal{W}}), v) \arrow[d, "\cong"'] & \mathcal{V}(FGv, v) \arrow[l, "\cong", "F(\tilde{\rho})^*"'] \arrow[d, "\cong"'] & \ni & \epsilon \arrow[d, mapsto] \\
\xrightarrow{\cong} \mathcal{W}(Gv \otimes \mathbb{1}_{\mathcal{W}}, Gv) & \mathcal{W}(Gv, Gv) \arrow[l, "\cong", "\tilde{\rho}^*"'] & \ni & \mathrm{id} \arrow[d, mapsto] \\
\xrightarrow{\cong} \mathcal{W}(Gv, \underline{\mathcal{W}}(\mathbb{1}_{\mathcal{W}}, Gv)) & & \ni & \tilde{\eta}.
\end{tikzcd}$$

The step with δ^* relied on the counitality equation of the oplax monoidal functor F. □

The change of enrichment along a weak monoidal Quillen adjunction preserved the SM7 axiom, whenever the left adjoint preserved the tensor unit as stated by the following lemma.

Lemma 4.2.5. *Let $\mathcal{V}$ and $\mathcal{W}$ be monoidal model categories. For any weak monoidal Quillen adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$ such that $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$, the change of enrichment along G preserves SM7.*

Proof. Consider a $\mathcal{V}$-enriched model category $\mathcal{C}$ satisfying SM7. The category $G_*\mathcal{C}$ is a $\mathcal{W}$-enriched category since G is a lax monoidal functor. Also, $G_*\mathcal{C}$ has the same underlying category as $\mathcal{C}$ by **Lemma 4.1.1**, i.e., $(G_*\mathcal{C})_0 = \mathcal{C}_0$, since $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$. In particular, the underlying category $(G_*\mathcal{C})_0$ of G_*C has the same model structure as $\mathcal{C}_0$. Let us show that the model category $G_*\mathcal{C}$ satisfies SM7. Let $i : a \hookrightarrow b$ be a cofibration and $p : x \twoheadrightarrow y$ be a fibration in $(G_*\mathcal{C})_0 = \mathcal{C}_0$. We have

$$\begin{tikzcd}
\underline{G_*\mathcal{C}}(b,x) \arrow[r, "(i^*,p_*)"] \arrow[dd, equal, "①"'] & \underline{G_*\mathcal{C}}(a,x) \times_{\underline{G_*\mathcal{C}}(a,y)} \underline{G_*\mathcal{C}}(b,y) \arrow[d, equal, "③"] \\
 & G\underline{\mathcal{C}}(a,x) \times_{G\underline{\mathcal{C}}(a,y)} G\underline{\mathcal{C}}(b,y) \\
G\underline{\mathcal{C}}(b,x) \arrow[r, "G(i^*,p_*)"'] & G\left(\underline{\mathcal{C}}(a,x) \times_{\underline{\mathcal{C}}(a,y)} \underline{\mathcal{C}}(b,y)\right) \arrow[u, "\cong"', "②"]
\end{tikzcd}$$

where the ① and ③ are given by the definition of the change of enrichment. The isomorphism ② holds since G preserves limits as a right adjoint functor. Since the $\mathcal{V}$-enriched model category $\mathcal{C}$ satisfies SM7, the map

$$(i^*, p_*) : \mathcal{C}(b,x) \to \mathcal{C}(a,x) \times_{\mathcal{C}(a,y)} \mathcal{C}(b,y)$$

is a fibration. Hence $G(i^*, p_*)$ is also a fibration since $G : \mathcal{V} \to \mathcal{W}$ is a right Quillen functor. Same for acyclic fibrations. $\square$

4.3 Weak tensoring and weak cotensoring

We define the weak tensoring and the weak cotensoring as follows.

Definition 4.3.1. *Let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category satisfying SM7.*

- *A **weak tensoring** of $\mathcal{C}$ over $\mathcal{V}$ is a bifunctor $- \otimes - : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$, satisfying the external pushout-product axiom, together with a natural map*

$$\varphi_{v,x,y} : \underline{\mathcal{C}}(x \otimes v, y) \to \underline{\mathcal{V}}(v, \underline{\mathcal{C}}(x,y)) \tag{4.3.1}$$

which is a weak equivalence in $\mathcal{V}$ for cofibrant $v \in \mathcal{V}$, cofibrant $x \in \mathcal{C}$, and fibrant $y \in \mathcal{C}$, and a natural map $\rho_x : x \otimes \mathbb{1} \to x$, for any $x \in \mathcal{C}$, satisfying the following commutative diagram

$$\begin{tikzcd}
\underline{\mathcal{C}}(x \otimes \mathbb{1}, y) \arrow[d, "\varphi_{\mathbb{1},x,y}"'] & \underline{\mathcal{C}}(x,y) \arrow[l, "\rho_x^*"'] \arrow[dl, "\cong"] \\
\underline{\mathcal{V}}(\mathbb{1}, \underline{\mathcal{C}}(x,y)). &
\end{tikzcd}$$

- *A **weak cotensoring** of $\mathcal{C}$ over $\mathcal{V}$ is a bifunctor $(-)^- : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$, satisfying the external pullback-power axiom, together with natural map*

$$\psi_{v,x,y} : \underline{\mathcal{C}}(x, y^v) \to \underline{\mathcal{V}}(v, \underline{\mathcal{C}}(x,y)) \tag{4.3.2}$$

which is a weak equivalence in $\mathcal{V}$ for cofibrant $v \in \mathcal{V}$, cofibrant $x \in \mathcal{C}$ and fibrant $y \in \mathcal{C}$, and a natural map $\eta_y : y \to y^{\mathbb{1}}$, for any $y \in \mathcal{C}$, satisfying the following *commutative diagram*

$$\begin{tikzcd}
\underline{\mathcal{C}}(x, y^{\mathbb{1}}) \arrow[d, "\psi_{\mathbb{1},x,y}"'] & \underline{\mathcal{C}}(x, y) \arrow[l, "(\eta_y)_*"'] \arrow[dl, "\cong"] \\
\underline{\mathcal{V}}(\mathbb{1}, \underline{\mathcal{C}}(x, y)). &
\end{tikzcd}$$

Example 4.3.2. *Tensoring and cotensoring are a special kind of weak tensoring and weak cotensoring, respectively.*

The change of enrichment along a weak monoidal Quillen adjunction preserved the weak tensoring and the weak cotensoring, whenever the left adjoint preserved the tensor unit as stated by the following result.

Proposition 4.3.3. *Let $\mathcal{V}$ and $\mathcal{W}$ be monoidal model categories. For any weak monoidal Quillen adjunction $F : \mathcal{W} \rightleftarrows \mathcal{V} : G$ such that $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$, the change of enrichment G_* along the right adjoint $G : \mathcal{V} \to \mathcal{W}$ preserves the weak tensoring and the weak cotensoring.*

Proof. **SM7:** Note that $G_*\mathcal{C}$ satisfies SM7, by **Lemma 4.2.5**.

Weak tensoring: Assume that $\mathcal{C}$ is weakly tensored over $\mathcal{V}$. Let us show that the bifunctor $- \otimes - : G_*\mathcal{C} \times \mathcal{W} \to G_*\mathcal{C}$, given by $x \otimes w := x \otimes Fw$, satisfies the external pushout-product axiom. Let $i : a \to b$ be a (acyclic) cofibration in $\mathcal{W}$ and $k : x \to y$ be a (acyclic) cofibration in $(G_*\mathcal{C})_0 = \mathcal{C}_0$. The pushout-product $i \square k$ of i and k given by

$$\begin{tikzcd}
x \otimes b \amalg_{x\otimes a} y \otimes a \arrow[r, "k\Box i"] \arrow[d, equal] & y \otimes b \arrow[d, equal] \\
x \otimes Fb \amalg_{x\otimes Fa} y \otimes Fa \arrow[r, "k\Box Fi"'] & y \otimes Fb
\end{tikzcd}$$

is a (acyclic) cofibration in $(G_*\mathcal{C})_0 = \mathcal{C}_0$ since $Fi : Fa \to Fb$ is a (acyclic) cofibration in $\mathcal{V}$ as F is a left Quillen functor and the bifunctor $- \otimes - : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$ satisfies the external pushout-product axiom.

Define the natural map $\tilde{\varphi}_{w,x,y} : \underline{G_*\mathcal{C}}(x \otimes w, y) \to \underline{\mathcal{W}}(w, G\underline{\mathcal{C}}(x, y))$ to be the composite

$$\underline{G_*\mathcal{C}}(x \otimes w, y) := \underline{G_*\mathcal{C}}(x \otimes Fw, y) \xrightarrow{G\varphi_{Fw,x,y}} \underline{G_*\mathcal{V}}(Fw, \underline{\mathcal{C}}(x, y)) \to \underline{\mathcal{W}}(w, G\underline{\mathcal{C}}(x, y)), \quad (4.3.3)$$

where the last arrow is the natural weak equivalence thanks to the existing weak $\mathcal{W}$-adjunction by **Proposition 4.2.3**, given by the weak monoidal Quillen adjunction $F \dashv G$. For a cofibrant object $w \in \mathcal{W}$, $Fw \in \mathcal{V}$ is also cofibrant and so the map $\tilde{\varphi}_{w,x,y}$ is a weak equivalence, for cofibrant objects $x \in \mathcal{C}$ and $w \in \mathcal{W}$, and the fibrant object $y \in \mathcal{C}$, since the map $G\varphi_{Fw,x,y}$ is a weak equivalence as the right Quillen functor G preserves weak equivalences between fibrant objects. Consider the natural map $\tilde{\rho}_x : x \otimes \mathbb{1}_{\mathcal{W}} \to x$ given by the composite

$$\begin{tikzcd}
x \otimes \mathbb{1}_{\mathcal{W}} : \arrow[r, equal] \arrow[rrr, bend right, "\tilde{\rho}_x"'] & x \otimes F(\mathbb{1}_{\mathcal{W}}) \arrow[r, "x\otimes\varepsilon"] & x \otimes \mathbb{1}_{\mathcal{V}} \arrow[r, "\rho_x"] & x
\end{tikzcd} , \quad (4.3.4)$$

where $F\mathbb{1}_{\mathcal{W}} \xrightarrow{\varepsilon} \mathbb{1}_{\mathcal{V}}$ is the counit map of the oplax monoidal functor F. The unitality condition follows from the following commutative diagram

$$\begin{tikzcd}
G\underline{\mathcal{C}}(x\otimes \mathbb{1}_{\mathcal{W}},y) : \arrow[r, equal] \arrow[ddr, "\tilde{\varphi}"'] & G\underline{\mathcal{C}}(x\otimes F(\mathbb{1}_{\mathcal{W}}),y) \arrow[d, "G\varphi"'] \arrow[dr, phantom, "①"] & G\underline{\mathcal{C}}(x\otimes \mathbb{1}_{\mathcal{V}},y) \arrow[l, "G(\mathrm{id}\otimes\varepsilon)^*"'] \arrow[d, "G\varphi"] & G\underline{\mathcal{C}}(x,y) \arrow[l, "G\rho^*"'] \arrow[dl, "\cong"', "G\eta"] \\
& G\underline{\mathcal{V}}(F(\mathbb{1}_{\mathcal{W}}),\underline{\mathcal{C}}(x,y)) \arrow[d, "\psi"'] \arrow[dr, phantom, "②"] & G\underline{\mathcal{V}}(\mathbb{1}_{\mathcal{V}},\underline{\mathcal{C}}(x,y)) \arrow[l, "G\varepsilon^*"] & \\
& \underline{\mathcal{W}}(\mathbb{1}_{\mathcal{W}},G\underline{\mathcal{C}}(x,y)) & G\underline{\mathcal{C}}(x,y), \arrow[l, "\tilde{\eta}"', "\cong"] \arrow[u, "\cong"', "G\eta"] &
\end{tikzcd}$$

where the square ① is commutative by the naturality of φ and the square ② is commutative by **Lemma 4.2.4**, where $v = \underline{\mathcal{C}}(x,y)$.

Weak cotensoring: Now, assume that $\mathcal{C}$ is weakly cotensored over $\mathcal{V}$. Let us show that this bifunctor $(-)^{-} : G_*\mathcal{C} \times \mathcal{W}^{\mathrm{op}} \to G_*\mathcal{C}$, given by $y^w := y^{Fw}$, satisfies the external pullback-power axiom. Let $i : a \to b$ be a (acyclic) cofibration in $\mathcal{W}$ and $p : x \to y$ be a (acyclic) fibration in $(G_*\mathcal{C})_0 = \mathcal{C}_0$. The pullback-power (i^*, p_*) of i and p given by

$$\begin{tikzcd}
x^b \arrow[r, "(i^*,p_*)"] \arrow[d, equal] & x^a \times_{y^a} y^b \arrow[d, equal] \\
x^{Fb} \arrow[r, "((Fi)^*,p_*)"'] & x^{Fa} \times_{y^{Fa}} y^{Fb}
\end{tikzcd}$$

is a (acyclic) fibration in $(G_*\mathcal{C})_0 = \mathcal{C}_0$ since $Fi : Fa \to Fb$ is a (acyclic) cofibration in $\mathcal{V}$ as F is a left Quillen functor and the bifunctor $(-)^{-} : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$ satisfies the external pullback-power axiom.

Define the natural map $\tilde{\psi}_{w,x,y} : \underline{G_*\mathcal{C}}(x, y^w) \to \underline{\mathcal{W}}(w, G\underline{\mathcal{C}}(x, y))$ to be the composite

$$\underline{G_*\mathcal{C}}(x, y^w) := G\underline{\mathcal{C}}(x, y^{Fw}) \xrightarrow{G\psi_{Fw,x,y}} G\underline{\mathcal{V}}(Fw, \underline{\mathcal{C}}(x, y)) \to \underline{\mathcal{W}}(w, G\underline{\mathcal{C}}(x, y)), \qquad (4.3.5)$$

where the last arrow is the natural weak equivalence thanks to the existing weak $\mathcal{W}$-adjunction, by **Proposition 4.2.3**, given by the weak monoidal Quillen adjunction $F \dashv G$. For a cofibrant object $w \in \mathcal{W}$, $Fw \in \mathcal{V}$ is also cofibrant and so the map $\tilde{\psi}_{w,x,y}$ is a weak equivalence, for cofibrant objects $x \in \mathcal{C}$ and $w \in \mathcal{W}$, and the fibrant object $y \in \mathcal{C}$, since the map $G\psi_{Fw,x,y}$ is a weak equivalence as the right Quillen functor G preserves weak equivalences between fibrant objects. Consider the natural map $\tilde{\eta}_x : x \to x^{\mathbb{1}_{\mathcal{W}}}$ given by the composite

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[rrr, bend right, "\tilde{\eta}_x"'] & x^{\mathbb{1}_{\mathcal{V}}} \arrow[r, "x^{\varepsilon}"] & x^{F(\mathbb{1}_{\mathcal{W}})} \arrow[r, equal, ":"] & x^{\mathbb{1}_{\mathcal{W}}} \end{tikzcd} . \qquad (4.3.6)$$

The unitality condition holds as the dual of the weak tensoring case proved above. □

Remark 4.3.4. *Since an isomorphism is in particular a weak equivalence, the change of enrichment G_* of a $\mathcal{V}$-enriched model category $\mathcal{C}$ satisfying SM7 and tensored and cotensored over $\mathcal{V}$ yields a $\mathcal{W}$-enriched model category $G_*\mathcal{C}$ endowed with a weak tensoring and a weak cotensoring over $\mathcal{W}$.*

An analogous result to **Remark 2.5.12** for weak tensoring and weak cotensoring is given by the following proposition.

Proposition 4.3.5. *Let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category satisfying SM7.*

1. *The homotopy category* $\mathrm{Ho}(\mathcal{C})$ *inherits a* $\mathrm{Ho}(\mathcal{V})$*-enrichment [Rie14, Theorem 10.2.12].*

2. *If the category* $\mathcal{C}$ *is weakly tensored over* $\mathcal{V}$*, then* $\mathrm{Ho}(\mathcal{C})$ *inherits a tensoring over* $\mathrm{Ho}(\mathcal{V})$.

3. *If the category* $\mathcal{C}$ *is weakly cotensored over* $\mathcal{V}$*, then* $\mathrm{Ho}(\mathcal{C})$ *inherits a cotensoring over* $\mathrm{Ho}(\mathcal{V})$.

Proof. (1.) Let us just observe that the homotopy category $\mathrm{Ho}(\mathcal{C})$ is enriched over the homotopy category $\mathrm{Ho}(\mathcal{V})$, with the enrichment given by the total derived functor

$$\mathbf{R}\underline{\mathcal{C}} : \mathrm{Ho}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{V})$$

$$\mathbf{R}\underline{\mathcal{C}}(x, y) = \gamma\underline{\mathcal{C}}(Qx, Ry), \tag{4.3.7}$$

where $\gamma : \mathcal{V} \to \mathrm{Ho}(\mathcal{V})$ is the localization functor, that is, the functor that inverts the weak equivalences in $\mathcal{V}$. For the details of the proof, see [Rie14, Theorem 10.2.12].

(2.) The weak tensoring on $\mathcal{C}$ over $\mathcal{V}$ is given by a map

$$\varphi_{v,x,y} : \underline{\mathcal{C}}(x \otimes v, y) \to \underline{\mathcal{V}}(v, \underline{\mathcal{C}}(x, y))$$

which is a weak equivalence for cofibrant $v \in \mathcal{V}$, cofibrant $x \in \mathcal{C}$, and fibrant $y \in \mathcal{C}$. Applying the functor $\gamma : \mathcal{V} \to \mathrm{Ho}(\mathcal{V})$, we obtain the map

$$\gamma\varphi_{v,x,y} : \gamma\underline{\mathcal{C}}(x \otimes v, y) \to \gamma\underline{\mathcal{V}}(v, \underline{\mathcal{C}}(x, y))$$

which is an isomorphism for cofibrant $v \in \mathcal{V}$, cofibrant $x \in \mathcal{C}$, and fibrant $y \in \mathcal{C}$. Hence,

$$\mathbf{R}\underline{\mathcal{C}}(x \otimes v, y) \cong \mathbf{R}\underline{\mathcal{V}}(v, \mathbf{R}\underline{\mathcal{C}}(x, y)) \tag{4.3.8}$$

for cofibrant $v \in \mathcal{V}$, cofibrant $x \in \mathcal{C}$, and fibrant $y \in \mathcal{C}$ since the tensoring of cofibrant objects is cofibrant and the hom object from cofibrant to fibrant objects is fibrant by **Lemma 2.4.6**. Now, let us show that the total derived functor $\otimes^{\mathbf{L}}$ of the weak tensoring on $\mathcal{C}$ over $\mathcal{V}$ defines a tensoring on $\mathrm{Ho}(\mathcal{C})$ over $\mathrm{Ho}(\mathcal{V})$. Let $v \in \mathcal{V}$, $x, y \in \mathcal{C}$. We have,

$$\begin{aligned}
\mathbf{R}\underline{\mathcal{C}}(x \otimes^{\mathbf{L}} v, y) &:= \mathbf{R}\underline{\mathcal{C}}(Qx \otimes Qv, y) \\
&\cong \mathbf{R}\underline{\mathcal{C}}(Qx \otimes Qv, Ry) \\
&\cong \mathbf{R}\underline{\mathcal{V}}(Qv, \mathbf{R}\underline{\mathcal{C}}(Qx, Ry)) \text{ by (4.3.8)} \\
&\cong \mathbf{R}\underline{\mathcal{V}}(v, \mathbf{R}\underline{\mathcal{C}}(x, y)).
\end{aligned}$$

The naturality here is given by the naturality of $\mathbf{R}\underline{\mathcal{V}}$ and $\mathbf{R}\underline{\mathcal{C}}$.

(3.) The proof for the cotensoring over $\mathrm{Ho}(\mathcal{V})$ is similar to the one of tensoring. □

4.4 Unit Axiom

In this section, we present some alternate forms of the external unit axiom (see item (ii) in **Definition 2.4.4**) not necessarily involving the tensoring. We start with the following statement that generalizes the one on simplicial model categories from

[GJ99, Proposition II.3.10].

Proposition 4.4.1. *Let $\mathcal{C}$ be a $\mathcal{V}$-model category. Let x and y be respectively cofibrant and fibrant objects of $\mathcal{C}$, and v be a cofibrant object of $\mathcal{V}$. Then there are natural bijections of hom sets in* $\mathrm{Ho}(\mathcal{V})$ *given by*

$$[v, \underline{\mathcal{C}}(x,y)] \cong [x \otimes v, y] \text{ and } [v, \underline{\mathcal{C}}(x,y)] \cong [x, y^v].$$

Proof. By definition of the tensoring, we have the following:

$$\begin{tikzcd}
\mathcal{C}(x \otimes v, y) \arrow[r, "\cong"] \arrow[d, "\text{quotient}"'] & \mathcal{V}(v, \underline{\mathcal{C}}(x,y)) \arrow[d, "\text{quotient}"] \\
\mathcal{C}(x \otimes v, y)/\sim \arrow[r, dashed, "\overset{?}{\cong}"] \arrow[d, "\cong"'] & \mathcal{V}(v, \underline{\mathcal{C}}(x,y))/\sim \arrow[d, "\cong"] \\
{[x \otimes v, y]} & {[v, \underline{\mathcal{C}}(x,y)],}
\end{tikzcd}$$

where the two downward maps under the quotient maps are bijections since $x \otimes v$ is cofibrant and $\underline{\mathcal{C}}(x,y)$ is fibrant by **Lemma 2.4.6**. Hence, it suffices to show that for any $f, g : x \otimes v \to y$ in $\mathcal{C}$ and their corresponding maps $f', g' : v \to \underline{\mathcal{C}}(x,y)$ in $\mathcal{V}$,

$$f \sim g \text{ in } \mathcal{C} \iff f' \sim g' \text{ in } \mathcal{V}.$$

($\Longrightarrow$) Assume $f \sim g$ in $\mathcal{C}$. Take a path object for y

$$\begin{tikzcd}
y \arrow[r, hook, "\sim", "c"'] \arrow[rr, bend right, "\Delta"'] & \mathrm{Path}(y) \arrow[r, two heads, "{(ev_0, ev_1)}"] & y \times y.
\end{tikzcd}$$

By [Hov99, Proposition 1.2.5], since $x \otimes v$ is cofibrant, there is a right homotopy

$$\begin{tikzcd}
 & & y \\
x \otimes v \arrow[urr, bend left, "f"] \arrow[r, "H"] \arrow[drr, bend right, "g"'] & \mathrm{Path}(y) \arrow[ur, "ev_0"'] \arrow[dr, "ev_1"] & \\
 & & y.
\end{tikzcd}$$

By the tensor-hom adjunction, we have

$$\begin{tikzcd}
 & & \underline{\mathcal{C}}(x,y) \\
v \arrow[urr, bend left, "f'"] \arrow[r, "H'"] \arrow[drr, bend right, "g'"'] & \underline{\mathcal{C}}(x, \mathrm{Path}(y)) \arrow[ur, "(ev_0)_*"'] \arrow[dr, "(ev_1)_*"] & \\
 & & \underline{\mathcal{C}}(x,y).
\end{tikzcd}$$

It suffices to show that

$$\begin{tikzcd}
\underline{\mathcal{C}}(x,y) \arrow[rr, bend left, "\underline{\mathcal{C}}(x,\Delta)"] \arrow[r, "\underline{\mathcal{C}}(x,c)", "=c_*"'] \arrow[drr, bend right, "\Delta"'] & \underline{\mathcal{C}}(x, \mathrm{Path}(y)) \arrow[r, "\underline{\mathcal{C}}(x,(ev_0,ev_1))", "=(ev_0,ev_1)_*"'] & \underline{\mathcal{C}}(x, y \times y) \arrow[d, "\cong"] \\
 & & \underline{\mathcal{C}}(x,y) \times \underline{\mathcal{C}}(x,y)
\end{tikzcd}$$

is a path object. Since x is cofibrant, then by [Rie14, Lemma 9.2.3] the functor $\underline{\mathcal{C}}(x,-) : \mathcal{C} \to \mathcal{V}$ preserves weak equivalences between fibrant objects and any fibration, as a right Quillen adjoint of $x \otimes -$. But $\mathrm{Path}(y)$ is fibrant, since $y \times y$ is, and (ev_0, ev_1) is a fibration. Hence c_* is a weak equivalence and $(ev_0, ev_1)_*$ is a fibration.

$(\Longleftarrow)$ Assume $f' \sim g'$ in $\mathcal{V}$. Take a cylinder object for v

$$\begin{tikzcd}
v \coprod v \arrow[r, hook, "i_0+i_1"] \arrow[rr, bend right, "\nabla"'] & \mathrm{Cyl}(v) \arrow[r, two heads, "\mathrm{coll}_v", "\sim"'] & v.
\end{tikzcd}$$

By [Hov99, Proposition 1.2.5], since $\underline{\mathcal{C}}(x, y)$ is fibrant, there is a left homotopy

$$\begin{tikzcd}
v \arrow[dr, "i_0"'] \arrow[drr, bend left, "f'"] & & \\
& \mathrm{Cyl}(v) \arrow[r, "H"] & \underline{\mathcal{C}}(x, y). \\
v \arrow[ur, "i_1"] \arrow[urr, bend right, "g'"'] & &
\end{tikzcd}$$

By the tensor-hom adjunction, we have

$$\begin{tikzcd}
x \otimes v \arrow[dr, "x\otimes i_0"'] \arrow[drr, bend left, "f"] & & \\
& x \otimes \mathrm{Cyl}(v) \arrow[r, "\widetilde{H}"] & y. \\
x \otimes v \arrow[ur, "x\otimes i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

It suffices to show that

$$\begin{tikzcd}
x \otimes (v \coprod v) \arrow[r, "x\otimes(i_0+i_1)"] \arrow[rr, bend left, "x\otimes\nabla"] & x \otimes \mathrm{Cyl}(v) \arrow[r, "x\otimes\mathrm{coll}_v"] & x \otimes v \\
(x \otimes v) \coprod (x \otimes v) \arrow[u, "\cong"] \arrow[urr, "\nabla"'] & &
\end{tikzcd}$$

is a cylinder object. Since x is cofibrant, then by [Rie14, Lemma 9.2.3] the functor $x \otimes - : \mathcal{V} \to \mathcal{C}$ preserves weak equivalences between cofibrant objects and any cofibration, as a left Quillen adjoint of $\underline{\mathcal{C}}(x, -)$. But $\mathrm{Cyl}(v)$ is cofibrant, since $v \coprod v$ is, and $i_0 + i_1$ is a cofibration. Hence $x \otimes \mathrm{coll}_v$ is a weak equivalence and $x \otimes (i_0 + i_1)$ is a cofibration. □

A similar argument also works for the case of weak tensoring and weak cotensoring.

The following corollary gives us an equivalent statement of the external unit axiom (see item (ii) in **Definition 2.4.4**).

Corollary 4.4.2. *Let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category tensored over $\mathcal{V}$. The following statements are equivalent.*

1. *The category $\mathcal{C}$ satisfies the external unit axiom.*
2. *If $x \in \mathcal{C}$ is cofibrant and $y \in \mathcal{C}$ is fibrant, then $[x, y] \cong [\mathbb{1}, \underline{\mathcal{C}}(x, y)]$.*

Proof. Let $q : Q\mathbb{1} \xrightarrow{\sim} \mathbb{1}$ be a cofibrant replacement of the tensor unit $\mathbb{1}$. Let $x \in \mathcal{C}$ be cofibrant and $y \in \mathcal{C}$ be fibrant.

(1.) $\Longrightarrow$ (2.) By the external unit axiom, the map $x \otimes Q\mathbb{1} \xrightarrow[x\otimes q]{\sim} x$ is a weak equivalence and so, by **Proposition 4.4.1**, it induces the following bijection in homotopy

$$\begin{tikzcd}
{[x \otimes Q\mathbb{1}, y]} \arrow[r, "\cong"] & {[Q\mathbb{1}, \underline{\mathcal{C}}(x, y)]} \\
{[x, y]} \arrow[u, "(x\otimes q)^*", "\cong"'] \arrow[r, dashed, "\therefore\ \cong"'] & {[\mathbb{1}, \underline{\mathcal{C}}(x, y)],} \arrow[u, "\cong", "q^*"']
\end{tikzcd}$$

that is, the solid diagram induces an isomorphism $[x, y] \cong [\mathbb{1}, \underline{\mathcal{C}}(x, y)]$ given by the dash map. (2.) $\Longrightarrow$ (1.) We have the following

$$\begin{aligned}
[x, y] &\cong [\mathbb{1}, \underline{\mathcal{C}}(x, y)] \\
&\xrightarrow[\cong]{q^*} [Q\mathbb{1}, \underline{\mathcal{C}}(x, y)] \\
&\cong [x \otimes Q\mathbb{1}, y], \text{ by } \textbf{Proposition 4.4.1}.
\end{aligned}$$

Therefore, the map $x \otimes Q\mathbb{1} \underset{\sim}{\xrightarrow{x \otimes q}} x$ is a weak equivalence, by saturation. Hence the external unit axiom holds in $\mathcal{C}$. $\square$

The following statement generalizes the one on simplicial model categories from [GJ99, Lemma II.4.2]. This gives us a second equivalent statement of the external unit axiom.

Proposition 4.4.3. *Let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category tensored and cotensored over $\mathcal{V}$, and satisfying SM7. Let $f : a \to b$ be a map in $\mathcal{C}$ between cofibrant objects.*

1. *If f is a weak equivalence, then for any fibrant object $z \in \mathcal{C}$, the induced map on hom objects $f^* : \underline{\mathcal{C}}(b, z) \to \underline{\mathcal{C}}(a, z)$ is a weak equivalence in $\mathcal{V}$.*

2. *The converse implication holds if and only if $\mathcal{C}$ satisfies the external unit axiom.*

Proof. By [GJ99, Lemma II.9.4], the map $f : a \to b$ between cofibrant objects in $\mathcal{C}$ has a factorization

$$\begin{tikzcd}
 & x \arrow[d, "q"] \\
a \arrow[ur, "j"] \arrow[r, "f"'] & b
\end{tikzcd}$$

such that j is a cofibration and q is a left inverse to a trivial cofibration $i : b \to x$.

1. If $f : a \to b$ is a weak equivalence, then the map $j : a \to x$ is a trivial cofibration, and hence induces a trivial fibration $j^* : \underline{\mathcal{C}}(x, z) \to \underline{\mathcal{C}}(a, z)$ for any fibrant object z. Similarly, the trivial cofibration i induces a trivial fibration i^*, so that

the map $q^* : \underline{\mathcal{C}}(b, z) \to \underline{\mathcal{C}}(x, z)$ is a weak equivalence. Therefore, $f^* = j^* q^*$ is a weak equivalence.

2. ($\Longrightarrow$) By assumption, $\mathcal{V}$ satisfies the unit axiom. Hence, for a cofibrant replacement $q : Q\mathbb{1} \xrightarrow{\sim} \mathbb{1}$ of the tensor unit $\mathbb{1}$, we have

$$\begin{tikzcd}
\underline{\mathcal{V}}(\mathbb{1}, \underline{\mathcal{C}}(x, z)) \arrow[r, "q^*", "\sim"'] & \underline{\mathcal{V}}(Q\mathbb{1}, \underline{\mathcal{C}}(x, z)) \\
\underline{\mathcal{C}}(x, z) \arrow[u, "\cong"] \arrow[r, "\therefore \sim", "(x \otimes q)^*"'] & \underline{\mathcal{C}}(x \otimes Q\mathbb{1}, z), \arrow[u, "\cong"']
\end{tikzcd}$$

for $x \in \mathcal{C}$ cofibrant and $z \in \mathcal{C}$ fibrant. Here, q^* is a weak equivalence since the category $\mathcal{V}$ satisfies the unit axiom as the hom object $\underline{\mathcal{C}}(x, z)$ is a fibrant object. Therefore, the map $x \otimes q : x \otimes Q\mathbb{1} \xrightarrow{\sim} x$ is a weak equivalence since $\mathcal{C}$ satisfies the **detection property** (see item (3.) below) by assumption.

($\Longleftarrow$) Suppose that

$$f^* : \underline{\mathcal{C}}(b, z) \to \underline{\mathcal{C}}(a, z) \tag{4.4.1}$$

is a weak equivalence for all fibrant object $z \in \mathcal{C}$. It suffices to show that

$$[b, z] \xrightarrow{f^*} [a, z]$$

is a bijection for any fibrant object $z \in \mathcal{C}$. Applying $[\mathbb{1}, -]$ to (4.4.1), we obtain the following commutative diagram by **Corollary 4.4.2**

$$\begin{tikzcd}
{[\mathbb{1}, \underline{\mathcal{C}}(b, z)]} \arrow[r, "(f^*)_*", "\cong"'] & {[\mathbb{1}, \underline{\mathcal{C}}(a, z)]} \\
{[b, z]} \arrow[u, "\cong"] \arrow[r, "\therefore \cong", "f^*"'] & {[a, z].} \arrow[u, "\cong"']
\end{tikzcd}$$

Hence, the map $f : a \to b$ is a weak equivalence. □

Let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category $\mathcal{C}$ weakly tensored and weakly cotensored over $\mathcal{V}$. Consider the following axioms.

1. **Homotopy tensor unit**: For any cofibrant replacement $q : Q\mathbb{1} \xrightarrow{\sim} \mathbb{1}$ and cofibrant object $x \in \mathcal{C}$, the map $x \otimes q : x \otimes Q\mathbb{1} \xrightarrow{\sim} x$ is a weak equivalence in $\mathcal{C}$.
2. π_0 **of mapping space**: For any $x, y \in \mathcal{C}$ respectively cofibrant and fibrant objects, we have $[x, y] \cong [\mathbb{1}, \underline{\mathcal{C}}(x, y)]$.
3. **Detection property**: If a map $f : x \to y$ between cofibrant objects in $\mathcal{C}$ is such that its restriction $f^* : \underline{\mathcal{C}}(y, z) \xrightarrow{\sim} \underline{\mathcal{C}}(x, z)$ is a weak equivalence in $\mathcal{V}$, for all fibrant object $z \in \mathcal{C}$, then f is also a weak equivalence in $\mathcal{C}$.

Remark 4.4.4. *Observe that the homotopy tensor unit is a generalization of the external unit axiom, where the tensoring is replaced by the weak tensoring.*

By **Corollary 4.4.2** and **Proposition 4.4.3**, when $\mathcal{C}$ is tensored and cotensored over $\mathcal{V}$, those three axioms are equivalent formulations of the unit axiom. Below, we show a few implications between them in the weak case.

Proposition 4.4.5. *Let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category weakly tensored over $\mathcal{V}$. We have the following implications.*

1. ***Homotopy tensor unit*** $\Longrightarrow$ π_0 ***of mapping space****.*
2. π_0 ***of mapping space*** $\Longrightarrow$ ***Detection property****.*

Proof. By assumption, for any objects $x, y \in \mathcal{C}$ and $v \in \mathcal{V}$ there is a map

$$\varphi_{x,y,v} : \underline{\mathcal{C}}(x \otimes v, y) \to \underline{\mathcal{V}}(v, \underline{\mathcal{C}}(x, y))$$

which is a weak equivalence whenever x and v are cofibrant, and y is fibrant.

(1.) Let $q : Q\mathbb{1} \overset{\sim}{\to} \mathbb{1}$ be a cofibrant replacement of the tensor unit $\mathbb{1}$ in $\mathcal{V}$. Let $x, y \in \mathcal{C}$ respectively cofibrant and fibrant objects. Assume that $\mathcal{C}$ satisfies the homotopy tensor unit, so the map $x \otimes Q\mathbb{1} \xrightarrow[\sim]{x \otimes q} x$ is a weak equivalence. Hence, we have

$$\begin{aligned} [x, y] &\xrightarrow[\cong]{(x\otimes q)^*} [x \otimes Q\mathbb{1}, y] \\ &\xrightarrow[\cong]{\gamma(\varphi_{x,y,Q\mathbb{1}})_*} [Q\mathbb{1}, \underline{\mathcal{C}}(x, y)] \\ &\xrightarrow{\cong} [\mathbb{1}, \underline{\mathcal{C}}(x, y)]. \end{aligned}$$

(2.) It suffices to show that the map $[y, z] \xrightarrow[\cong]{f^*} [x, z]$ is a bijection for any fibrant object $z \in \mathcal{C}$. Applying $[\mathbb{1}, -]$ to the map $\underline{\mathcal{C}}(y, z) \xrightarrow[\sim]{f^*} \underline{\mathcal{C}}(x, z)$, we have the commutative diagram

$$\begin{tikzcd}
{[\mathbb{1}, \underline{\mathcal{C}}(y, z)]} \arrow[r, "(f^*)_*", "\cong"'] & {[\mathbb{1}, \underline{\mathcal{C}}(x, z)]} \\
{[y, z]} \arrow[u, "\cong"] \arrow[r, "\therefore \cong", "f^*"'] & {[x, z].} \arrow[u, "\cong"']
\end{tikzcd}$$

Hence, the map $f : x \to y$ is a weak equivalence. $\square$

Proposition 4.4.6. *Let* $\mathcal{W} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{V}$ *be a weak monoidal Quillen adjunction such that* $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$.

1. *The change of enrichment along* G *preserves the* ***homotopy tensor unit****.*

2. *The change of enrichment along G preserves the π_0* ***of mapping space axiom****.*

3. *If G reflects weak equivalences between fibrant objects, then the change of enrichment along G preserves the* ***detection property****.*

Proof. (1.) Assume that the weak $\mathcal{V}$-tensoring on $\mathcal{C}$ satisfies the homotopy tensor unit, that is, the composite $\begin{tikzcd} x \otimes Q\mathbb{1}_{\mathcal{V}} \arrow[r, "x\otimes q"] \arrow[rr, bend right, "\sim"'] & x \otimes \mathbb{1}_{\mathcal{V}} \arrow[r, "\rho_x"] & x \end{tikzcd}$ is a weak equivalence for x cofibrant. For the weak $\mathcal{W}$-tensoring on $G_*\mathcal{C}$, the following

$$\begin{tikzcd} x \otimes Q\mathbb{1}_{\mathcal{W}} := x \otimes F(Q\mathbb{1}_{\mathcal{W}}) \arrow[r, "x\otimes\tilde{q}"] \arrow[rr, bend right, "\sim"'] & x \otimes \mathbb{1}_{\mathcal{V}} \arrow[r, "\rho_x"] & x \end{tikzcd}$$

is a weak equivalence since $\tilde{q}$ is the weak equivalence $\begin{tikzcd} F(Q\mathbb{1}_{\mathcal{W}}) \arrow[r, "F(q)"] \arrow[rr, bend right, "\sim", "\tilde{q}"'] & F(\mathbb{1}_{\mathcal{W}}) \arrow[r, "\varepsilon"] & \mathbb{1}_{\mathcal{V}} \end{tikzcd}$, by **Definition 2.4.8**, making $F(Q\mathbb{1}_{\mathcal{W}})$ be a cofibrant replacement of $\mathbb{1}_{\mathcal{V}}$.

(2.) Let $x, y \in \mathcal{C}$, with x cofibrant and y fibrant. We have

$$\begin{aligned} [x, y] &\cong [\mathbb{1}_{\mathcal{V}}, \underline{\mathcal{C}}(x, y)] \\ &\xrightarrow[\cong]{\tilde{q}^*} [F(Q\mathbb{1}_{\mathcal{W}}), \underline{\mathcal{C}}(x, y)], \text{ since } \tilde{q} \text{ is a weak equivalence} \\ &\cong [\mathbf{L}F(\mathbb{1}_{\mathcal{W}}), \underline{\mathcal{C}}(x, y)] \\ &\cong [\mathbb{1}_{\mathcal{W}}, \mathbf{R}G\underline{\mathcal{C}}(x, y)], \text{ by the derived adjunction} \\ &\cong [\mathbb{1}_{\mathcal{W}}, G\underline{\mathcal{C}}(x, y)], \text{ since } \underline{\mathcal{C}}(x, y) \text{ is fibrant} \\ &\cong [\mathbb{1}_{\mathcal{W}}, \underline{G_*\mathcal{C}}(x, y)]. \end{aligned}$$

(3.) Assume G reflects weak equivalences between fibrant objects and that $\mathcal{C}$ satisfies the detection property. Let $f : a \to b$ be in $(G_*\mathcal{C})_0 = \mathcal{C}_0$, with cofibrant objects

$a, b \in \mathcal{C}$. Assume that $Gf^* : \underline{G_*\mathcal{C}}(b, z) \overset{\sim}{\to} \underline{G_*\mathcal{C}}(a, z)$, with $z \in \mathcal{C}$ fibrant, is a weak equivalence. The weak equivalence Gf^* equals

$$\underline{G_*\mathcal{C}}(b, z) := G\underline{\mathcal{C}}(b, z) \xrightarrow[\sim]{G(f^*)} G\underline{\mathcal{C}}(a, z) =: \underline{G_*\mathcal{C}}(a, z), \text{ i.e.,}$$

$$Gf^* = G\left(\underline{\mathcal{C}}(b, z) \xrightarrow{f^*} \underline{\mathcal{C}}(a, z)\right) \text{ is a weak equivalence.}$$

Hence, the restriction $\underline{\mathcal{C}}(b, z) \xrightarrow[\sim]{f^*} \underline{\mathcal{C}}(a, z)$ is a weak equivalence since G reflects weak equivalences between fibrant objects and so the map f is a weak equivalence as the $\mathcal{V}$-enriched model category $\mathcal{C}$ satisfies the detection property. Therefore, the category $G_*\mathcal{C}$ satisfies the detection property also. □

Example 4.4.7. *The following are examples of right Quillen functors $G : \mathcal{D} \to \mathcal{C}$ that reflect weak equivalences between fibrant objects.*

1. *Quillen equivalences [Hov99, Corollary 1.3.16], for instance:*
    (a) *Dold–Kan correspondence* $N : \mathrm{sMod}_R \underset{\longleftarrow}{\overset{\cong}{\longrightarrow}} \mathrm{Ch}_{\geq 0}(R) : \Gamma$,
    (b) *Geometric realization and the singular set functor* $|\cdot| : \mathrm{sSet} \underset{\longleftarrow}{\longrightarrow} \mathrm{Top} : \mathrm{Sing}$.
2. *Let $\mathcal{C}$ be a model category and $G : \mathcal{D} \to \mathcal{C}$ a right adjoint. If $\mathcal{D}$ admits the right-induced model structure along G, then G reflects weak equivalences.*

4.5 Preservation of the enriched model structure

The following is the definition of a weak $\mathcal{V}$-model category, which is basically a $\mathcal{V}$-model category where the tensoring is replaced by a weak tensoring.

Definition 4.5.1. *Let $\mathcal{V}$ be a closed symmetric monoidal model category. A* ***weak $\mathcal{V}$-model category*** *is a $\mathcal{V}$-enriched model category $\mathcal{C}$ weakly tensored and weakly cotensored over $\mathcal{V}$, satisfying SM7 and the homotopy tensor unit.*

Remark 4.5.2. *A $\mathcal{V}$-model category $\mathcal{C}$ is in particular a weak $\mathcal{V}$-model category.*

Theorem 4.5.3. *Let $\mathcal{W} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{V}$ be a weak monoidal Quillen adjunction such that $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$. If $\mathcal{C}$ is a weak $\mathcal{V}$-model category, then $G_*\mathcal{C}$ is a weak $\mathcal{W}$-model category.*

Proof. The category $G_*\mathcal{C}$ is a $\mathcal{W}$-enriched category since G is a lax monoidal functor. Also, $G_*\mathcal{C}$ has the same underlying category as $\mathcal{C}$ by **Lemma 4.1.1**, since $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$. In particular, the underlying category of G_*C still has a model structure. The category $G_*\mathcal{C}$ satisfies SM7 by **Lemma 4.2.5**, and inherits a weak tensoring and weak cotensoring over $\mathcal{W}$ by **Proposition 4.3.3**. Finally, the category $G_*\mathcal{C}$ satisfies the homotopy tensor unit by item (1.) from **Proposition 4.4.6**. □

Corollary 4.5.4. *Let $\mathcal{W} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{V}$ be a weak monoidal Quillen adjunction such that $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$. If $\mathcal{C}$ is a $\mathcal{V}$-model category, then $G_*\mathcal{C}$ is a weak $\mathcal{W}$-model category.*

Proof. This is a special case of **Proposition 4.5.3**, by **Remark 4.5.2**. □

Proposition 4.5.5. *Let $\mathcal{W} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{V}$ be a weak monoidal Quillen adjunction such that $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$. If $\mathcal{C}$ is a $\mathcal{V}$-enriched model category satisfying SM7 and the π_0 of mapping space axiom, then $G_*\mathcal{C}$ is a $\mathcal{W}$-enriched model category satisfying SM7 and the π_0 of mapping space axiom.*

Proof. Apply **Proposition 4.5.3** and item (2) from **Proposition 4.4.6**. ☐

Proposition 4.5.6. *Let* $\mathcal{W} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{V}$ *be a weak monoidal Quillen adjunction such that* $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$ *and* G *reflects weak equivalences between fibrant objects. If* $\mathcal{C}$ *is a* $\mathcal{V}$*-enriched model category satisfying SM7 and the detection property, then* $G_*\mathcal{C}$ *is a* $\mathcal{W}$*-enriched model category satisfying SM7 and the detection property.*

Proof. Apply **Proposition 4.5.3** and item (3) from **Proposition 4.4.6**. ☐

4.6 Application: Comparing two enrichments

Via the Dold–Kan correspondence, we can produce two enrichments of $\mathrm{Ch}_{\geq 0}(R)$ over sMod_R, for a commutative ring R: one by applying Dold–Kan globally to the whole category, the other by applying Dold–Kan locally to each hom complex.

Definition 4.6.1. *(a) The* ***"global enrichment"*** *of* $\mathrm{Ch}_{\geq 0}(R)$ *over* sMod_R *is defined by*

$$\underline{\mathrm{Hom}}^{\mathrm{global}}_{\mathrm{Ch}_{\geq 0}(R)}(C, D) := \underline{\mathrm{Hom}}_{\mathrm{sMod}_R}(\Gamma(C), \Gamma(D)), \text{ for all } C, D \in \mathrm{Ch}_{\geq 0}(R).$$

(b) The ***"local enrichment"*** *of* $\mathrm{Ch}_{\geq 0}(R)$ *over* sMod_R *is defined by*

$$\underline{\mathrm{Hom}}^{\mathrm{local}}_{\mathrm{Ch}_{\geq 0}(R)}(C, D) := \Gamma\tau_{\geq 0}\underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}(C, D), \text{ for all } C, D \in \mathrm{Ch}_{\geq 0}(R).$$

Here, the bifunctors $\underline{\mathrm{Hom}}_{\mathrm{sMod}_R}$ *and* $\underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}$ *are the respective internal hom's.*

Michael Opadotun [Opa21, Theorem 6.3.1] proved that "global enrichment" and "local enrichment" of $\mathrm{Ch}_{\geq 0}(R)$ over sMod_R defined above, are homotopy equivalent and his result is stated as follows.

Theorem 4.6.2. *For a commutative ring R, the natural map of simplicial R-modules*

$$AW^* : \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(R)}(C \otimes N(R\Delta^\bullet), D) \to \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(R)}(N(\Gamma C \otimes R\Delta^\bullet), D)$$

is a homotopy equivalence, with homotopy inverse

$$EZ^* : \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(R)}(N(\Gamma C \otimes R\Delta^\bullet), D) \to \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(R)}(C \otimes N(R\Delta^\bullet), D),$$

for all chain complexes C and D.

In **Proposition 3.3.2**, we proved an analogous statement to the above one from Opadotun.

By applying **Proposition 2.2.7** to the "global enrichment" defined above, we obtain the following corollary.

Corollary 4.6.3. *Considering the "global enrichment"*[1] *of $\mathrm{Ch}_{\geq 0}(R)$ over sMod_R.*

(i) $C \otimes A := N(\Gamma(C) \otimes A)$ defines a tensoring on $\mathrm{Ch}_{\geq 0}(R)$ over sMod_R, for any $C \in \mathrm{Ch}_{\geq 0}(R)$ and $A \in \mathrm{sMod}_R$.

(ii) $C^A := N\underline{\mathrm{Hom}}_{\mathrm{sMod}_R}(A, \Gamma(C))$ defines a cotensoring on $\mathrm{Ch}_{\geq 0}(R)$ over sMod_R, for any $C \in \mathrm{Ch}_{\geq 0}(R)$ and $A \in \mathrm{sMod}_R$.

---

[1] *The "global enrichment", the tensoring and cotensoring defined in (i) and (ii) from **Corollary 4.6.3** respectively are the one referred to on page 3 of Background material for TALBOT 2012: https://math.mit.edu/events/talbot/2012/2012TalbotExercises.pdf.*

The following example illustrates the result of item (2.) from **Proposition 4.1.4**.

Example 4.6.4. *There is neither a tensoring nor a cotensoring on* $\mathrm{Ch}_{\geq 0}(R)$ *over* sMod_R *associated to the "local enrichment" since the normalization* $N : \mathrm{sMod}_R \to \mathrm{Ch}_{\geq 0}(R)$ *is not strong monoidal. Similarly, since the denormalization* $\Gamma : \mathrm{Ch}_{\geq 0}(R) \to \mathrm{sMod}_R$ *is not strong monoidal, then there is neither a tensoring nor a cotensoring on* sMod_R *over* $\mathrm{Ch}_{\geq 0}(R)$ *associated to the "local enrichment" given by* $\underline{Hom}(A, B) := N\underline{Hom}_{\mathrm{sMod}_R}(A, B)$, *for all* $A, B \in \mathrm{sMod}_R$.

As an application of **Remark 4.3.4**, we have the following result.

Proposition 4.6.5. *For all* $C, D \in \mathrm{Ch}_{\geq 0}(R)$ *and* $A, B \in \mathrm{sMod_R}$, *there is a natural homotopy equivalence of simplicial R-modules* $\Gamma(D^C) \simeq \Gamma(D)^{\Gamma(C)}$ *and a natural chain homotopy equivalence* $N(B^A) \simeq N(B)^{N(A)}$ *for the Dold–Kan correspondence* $N : \mathrm{sMod}_R \overset{\cong}{\underset{\longleftarrow}{\longrightarrow}} \mathrm{Ch}_{\geq 0}(R) : \Gamma$.

The above homotopy equivalences are given by weak equivalences in the respective Hurewicz model categories [NN23].

Proof. By **Definition 4.3.2** and **Remark 4.3.4**, the change of enrichment of the self-enriched model category $\mathrm{Ch}_{\geq 0}(R)$ along Γ gives the sMod_R-enriched model category with weak tensoring and weak cotensoring. There is a map

$$\Gamma\underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}(E, D^C) \to \underline{\mathrm{Hom}}_{\mathrm{sMod}_R}(\Gamma C, \Gamma\underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}(E, D)) \tag{4.6.1}$$

in sMod_R which is a weak equivalence when $C, E \in \mathrm{Ch}_{\geq 0}(R)$ are cofibrant and $D \in \mathrm{Ch}_{\geq 0}(R)$ is fibrant. But in the Hurewicz model structure, every object is cofibrant and fibrant. Then the equation (4.6.1) is a homotopy equivalence for all C, D, E. Specializing to $E = \mathbb{1}_{\mathrm{Ch}_{\geq 0}(R)} = R[0]$, (4.6.1) becomes

$$\begin{aligned} \Gamma \underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}\left(R[0], D^C\right) &\simeq \underline{\mathrm{Hom}}_{\mathrm{sMod}_R}\left(\Gamma C, \Gamma \underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}\left(R[0], D\right)\right) \\ &\cong \underline{\mathrm{Hom}}_{\mathrm{sMod}_R}\left(\Gamma C, \Gamma D\right), \end{aligned}$$

i.e, $\Gamma(D^C) \simeq \Gamma(D)^{\Gamma(C)}$.

Similarly by **Proposition 4.3.3**, the change of enrichment of the self-enriched model category sMod_R along N gives the $\mathrm{Ch}_{\geq 0}(R)$-enriched model category with weak tensoring and weak cotensoring. There is a map

$$N\underline{\mathrm{Hom}}_{\mathrm{sMod}_R}(X, B^A) \to \underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}(N(A), N\underline{\mathrm{Hom}}_{\mathrm{sMod}_R}(X, B)) \tag{4.6.2}$$

in $\mathrm{Ch}_{\geq 0}(R)$ which is a weak equivalence when $A, X \in \mathrm{sMod}_R$ are cofibrant and $B \in \mathrm{sMod}_R$ is fibrant. But in the Hurewicz model structure, every object is cofibrant and fibrant. Then the morphism (4.6.2) is a homotopy equivalence for all A, B, X. Specializing to $X = \mathbb{1}_{\mathrm{sMod}_R} = c(R)$, the arrow (4.6.2) becomes

$$\begin{aligned} N\underline{\mathrm{Hom}}_{\mathrm{sMod}_R}\left(c(R), B^A\right) &\xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}\left(N(A), N\underline{\mathrm{Hom}}_{\mathrm{sMod}_R}\left(c(R), B\right)\right) \\ &\xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathrm{Ch}_{\geq 0}(R)}(N(A), N(B)), \end{aligned}$$

i.e, $N(B^A) \simeq N(B)^{N(A)}$. □

Remark 4.6.6. *For a non-commutative ring R, the statements in this section are analogous, using the enrichment, tensoring, and cotensoring of* sMod_R *over* sAb *and of* $\mathrm{Ch}_{\geq 0}(R)$ *over* $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$*, as in* ***Lemmas 2.3.6*** *and* ***2.3.8****.*

Chapter 5

Future Research Possibilities

There are a couple of ways the work presented in this dissertation could be extended. Hence, as future work we have the following.

(i) The Dold–Kan correspondence is a weak Quillen equivalence and we have shown in Chapter 3 that the homotopy and the monoidal properties of the Hurewicz model structure are both preserved after being transferred along the Dold–Kan correspondence. One can check if this is a general property of weak Quillen equivalences, that is, if similar homotopy and monoidal properties of a given model structure are preserved after being transferred along any convenient weak Quillen equivalence.

(ii) One might look into some potential applications of the results or examples of the new concepts, mostly theoretical, developed in Chapter 4. For instance, it could be interesting to produce some proper examples of weak $\mathcal{V}$-adjunction, weak tensoring, weak cotensoring and weak $\mathcal{V}$-model category.

(iii) Also, one could investigate the discussion from Chapter 4 in the case of monoids. Let $\mathcal{C}$ be a weakly $\mathcal{V}$-tensored category and R be a monoid in $\mathcal{V}$, a left R-module in $\mathcal{C}$ is an object X in $\mathcal{C}$ equipped with an action map

$$\mu : R \otimes X \to X \text{ in } \mathcal{C},$$

where the left-hand side denotes the weak tensoring of $\mathcal{C}$ over $\mathcal{V}$ as defined in **Definition 4.3.1**. We have the following interesting questions.

- Is there a well-behaved homotopy theory of R-modules in this weak setting?
- Does the change of enrichment behave well with respect to modules, inducing a nice Quillen functor

$$G_* : \{R\text{-modules in } \mathcal{C}\} \to \{G(R)\text{-modules in } G_*\mathcal{C}\}?$$

Here, one can require strict associativity and unitality if necessary.

(iv) Finally, another question that comes to mind is whether the notions of weak tensoring and weak cotensoring, given by **Definition 4.3.1**, are model category presentations of the ∞-categorical analogues. In other words, given a weak $\mathcal{V}$-model category $\mathcal{C}$, defined in **Definition 4.5.1**, is the underlying ∞-category of $\mathcal{C}$ tensored and cotensored over the underlying symmetric monoidal ∞-category of $\mathcal{V}$? As references on this topic, see [Lur17, § 4.2], [GH15] and [Hei23].

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