Math 527 - Homotopy Theory
Additional notes

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March 4, 2013

1 Fiber sequences

Definition 1.1. Let (X, x_0) be a pointed space. The **path space** of X is the space

$$PX = \{\gamma \in X^I \mid \gamma(0) = x_0\}$$

of paths in X starting at the basepoint. This can be expressed as the pullback

$$\begin{array}{ccc} PX & \longrightarrow & X^I \\ \big\downarrow & \lrcorner & \big\downarrow \mathrm{ev}_0 \\ \{x_0\} & \underset{\iota}{\hookrightarrow} & X \end{array}$$

Definition 1.2. The **path-loop fibration** on X is the evaluation map $\mathrm{ev}_1 \colon PX \to X$.

This is indeed a fibration, in fact the fibration $P(\iota)$ obtained when replacing the map $\iota \colon \{x_0\} \hookrightarrow X$ by a fibration (via the path space construction).

The (strict) fiber of ev_1 is the based loop space

$$\Omega X = \{\gamma \in X^I \mid \gamma(0) = \gamma(1) = x_0\},$$

hence the name of the fibration. We often write $\Omega X \to PX \to X$.

Definition 1.3. Let $f \colon (X, x_0) \to (Y, y_0)$ be a pointed map between pointed spaces. The **homotopy fiber** of f is

$$F(f) := \{(x, \gamma) \in X \times Y^I \mid \gamma(0) = y_0, \gamma(1) = f(x)\}.$$

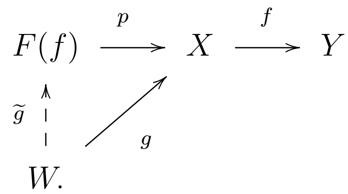
This can be expressed as the pullback

$$\begin{array}{ccc} F(f) & \longrightarrow & PY \\ \big\downarrow & \lrcorner & \big\downarrow \mathrm{ev}_1 \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

so that the projection $p\colon F(f) \to X$ given by $p(x,\gamma) = x$ is automatically a fibration.

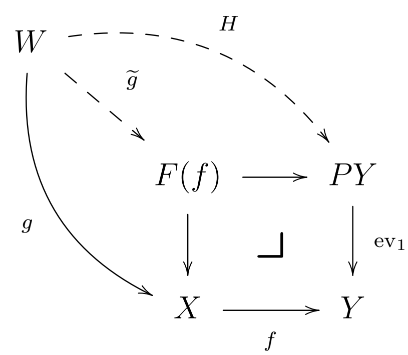
The sequence $F(f) \xrightarrow{p} X \xrightarrow{f} Y$ is called a **fiber sequence**

Proposition 1.4. *Let W be a pointed space and $g\colon W \to X$ a pointed map. Consider the lifting problem in* $\mathbf{Top}_*$



Then lifts $\widetilde{g}\colon W \to F(f)$ of g correspond bijectively to pointed null-homotopies of the composite $fg\colon W \to Y$. In particular, a lift exists if and only if fg is pointed null-homotopic.

Proof. Note that limits (but not colimits) in $\mathbf{Top}$ and $\mathbf{Top}_*$ agree, so that the pullback diagram defining $F(f)$ can be viewed in either category. Lifts of g



correspond to (pointed) maps $H\colon W \to PY$ making the diagram above commute, i.e. satisfying $\mathrm{ev}_1 \circ H = fg$. These are precisely pointed null-homotopies of fg. □

The particular case of the statement can be reinterpreted as follows.

Corollary 1.5. *For any pointed space W, applying the functor $[W,-]_*\colon \mathbf{Top}_* \to \mathbf{Set}_*$ to the fiber sequence $F(f) \xrightarrow{p} X \xrightarrow{f} Y$ yields*

$$[W,F(f)]_* \xrightarrow{p_*} [W,X]_* \xrightarrow{f_*} [W,Y]_*$$

which is an exact sequence of pointed sets.

There is a canonical "inclusion of the strict fiber into the homotopy fiber" $\iota\colon f^{-1}(y_0) \to F(f)$ defined by

$$\iota(x) = (x, c_{y_0})$$

where $c_{y_0}\colon I \to Y$ is the constant path at $y_0 \in Y$.

Proposition 1.6. *If $f\colon X \to Y$ is a fibration, then the canonical map $\iota\colon f^{-1}(y_0) \to F(f)$ is a homotopy equivalence.*

Proof. Homework 6 Problem 2. ☐

Remark 1.7. The homotopy fiber $F(f)$ is rarely a kernel of f in the homotopy category $\mathrm{Ho}(\mathbf{Top}_*)$.

Exercise 1.8. Let (X, x_0) and (Y, y_0) be pointed spaces. Consider the sequence

$$X \xrightarrow{\iota_X} X \times Y \xrightarrow{p_Y} Y$$

where ι_X is the "slice inclusion" defined by $\iota_X(x) = (x, y_0)$ and p_Y is the projection onto the second factor. Show that $\iota_X\colon X \to X \times Y$ is the kernel of p_Y in $\mathrm{Ho}(\mathbf{Top}_*)$.

Exercise 1.9. Consider the real axis inside the complex plane $\mathbb{R} \subset \mathbb{C}$, and the corresponding inclusion $\mathbb{R}^{n+1} \setminus \{0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$. This descends to a map $\mathbb{R}P^n \to \mathbb{C}P^n$ on the quotients by the actions of $O(1) = \{-1, 1\} \subset \mathbb{R}^\times$ and $U(1) = S^1 \subset \mathbb{C}^\times$ respectively. As n goes to infinity, this defines a map $f\colon \mathbb{R}P^\infty \to \mathbb{C}P^\infty$. Alternately, f can be thought of as the map classifying the complexification of the tautological real line bundle over $\mathbb{R}P^\infty$.

Show that the map f does *not* admit a kernel in $\mathrm{Ho}(\mathbf{Top}_*)$.

Remark 1.10. This shows in particular that $\mathrm{Ho}(\mathbf{Top}_*)$ is not complete, though it does have all small products, which are given by the Cartesian product as in $\mathbf{Top}_*$.

2 Homotopy invariance

Note that taking the homotopy fiber is functorial in the input $f\colon X \to Y$, i.e. is a functor $\mathrm{Arr}(\mathbf{Top}_*) \to \mathbf{Top}_*$ from the arrow category of $\mathbf{Top}_*$, and such that the map $F(f) \to X$ is a natural transformation. Thus, a map of diagrams

$$\varphi\colon \left(X \xrightarrow{f} Y\right) \to \left(X' \xrightarrow{f'} Y'\right),$$

which is a (strictly) commutative diagram in $\mathbf{Top}_*$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi_X}\downarrow & & \downarrow{\scriptstyle\varphi_Y} \\ X' & \xrightarrow[f']{} & Y' \end{array} \tag{1}$$

induces a map between homotopy fibers $\varphi_F\colon F(f) \to F(f')$ making the diagram

$$\begin{array}{ccccc} F(f) & \xrightarrow{p} & X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi_F}\downarrow & & {\scriptstyle\varphi_X}\downarrow & & \downarrow{\scriptstyle\varphi_Y} \\ F(f') & \xrightarrow[p']{} & X' & \xrightarrow[f']{} & Y' \end{array} \tag{2}$$

in $\mathbf{Top}_*$ commute. Moreover, this assignment preserves compositions (as in "stacking another square" below the right-hand square).

Let us study to what extent the homotopy fiber is a homotopy invariant construction.

Proposition 2.1. *If a map of diagrams*

$$\varphi\colon \left(X \xrightarrow{f} Y\right) \to \left(X' \xrightarrow{f'} Y'\right)$$

is an objectwise pointed homotopy equivalence, i.e. both maps $\varphi_X\colon X \xrightarrow{\simeq} X'$ *and* $\varphi_Y\colon Y \xrightarrow{\simeq} Y'$ *are pointed homotopy equivalences, then the induced map on homotopy fibers* $\varphi_F\colon F(f) \to F(f')$ *is also a pointed homotopy equivalence.*

Proof. Let $\psi_X\colon X' \to X$ and $\psi_Y\colon Y' \to Y$ be homotopy inverses of φ_X and φ_Y respectively. In the diagram (2), the composite $f\psi_X p'\colon F(f') \to Y$ is (pointed) null-homotopic, in fact by the (pointed) null-homotopy

$$\begin{aligned} f\psi_X p' &\simeq \psi_Y\varphi_Y f\psi_X p' \\ &= \psi_Y f'\varphi_X\psi_X p' \\ &\simeq \psi_Y f' p' \\ &\simeq \psi_Y \circ * \quad \text{via the homotopy } \psi_Y(H') \\ &= * \end{aligned}$$

where $H'\colon f'p' \Rightarrow *$ is the canonical null-homotopy of $f'p'\colon F(f') \to Y'$.

This choice of null-homotopy of $f\psi_X p'\colon F(f') \to Y$ defines a (pointed) map

$$\psi_F\colon F(f') \to F(f)$$

which we claim is (pointed) homotopy inverse to φ_F.

$\varphi_F\psi_F \simeq \mathrm{id}_{F(f')}$. The map

$$\mathrm{id}\colon F(f') \to F(f')$$

corresponds to $p'\colon F(f') \to X'$ and the canonical null-homotopy $H'\colon f'p' \Rightarrow *$.

The map

$$\varphi_F\psi_F\colon F(f') \to F(f')$$

corresponds to $\varphi_X\psi_X p'\colon F(f') \to X'$ and the null-homotopy

$$\begin{aligned} f'\varphi_X\psi_X p' &= \varphi_Y f\psi_X p' \\ &\simeq \varphi_Y\psi_Y\varphi_Y f\psi_X p' \\ &= \varphi_Y\psi_Y f'\varphi_X\psi_X p' \\ &\simeq \varphi_Y\psi_Y f'p' \\ &\simeq \varphi_Y\psi_Y \circ * \ \text{ via the homotopy } \varphi_Y\psi_Y(H'). \end{aligned}$$

The two maps are (pointed) homotopic, since they are related by operations of the form: replacing $\varphi_X\psi_X$ by $\mathrm{id}_{X'}$.

$\psi_F\varphi_F \simeq \mathrm{id}_{F(f)}$. Similar argument. □

Corollary 2.2. *The pointed homotopy type of the homotopy fiber $F(f)$ only depends on the pointed homotopy class of f.*

Proof. Let $f, g\colon X \to Y$ be pointed-homotopic maps, and let $H\colon X \wedge I_+ \to Y$ be a pointed homotopy form f to g. Proposition 2.1 ensures that the two induced maps on homotopy fibers in the (strictly) commutative diagram

$$\begin{array}{ccccc} F(f) & \longrightarrow & X & \xrightarrow{f} & Y \\ \simeq\downarrow & & \iota_0 \downarrow \simeq & & \| \\ F(H) & \longrightarrow & X \wedge I_+ & \xrightarrow{H} & Y \\ \simeq\uparrow & & \iota_1 \uparrow \simeq & & \| \\ F(g) & \longrightarrow & X & \xrightarrow{g} & Y \end{array}$$

are (pointed) homotopy equivalences. □

Remark 2.3. If the spaces involved X, Y , X', and Y' are well-pointed, then any pointed map which is an unpointed homotopy equivalence is automatically a pointed homotopy equivalence. In that case, the condition on φ is that it be an objectwise homotopy equivalence.

Remark 2.4. The map of diagrams $\varphi\colon \left(X \xrightarrow{f} Y\right) \to \left(X' \xrightarrow{f'} Y'\right)$ being an objectwise homotopy equivalence does *not* guarantee that there is a choice of homotopy inverses $\psi_X\colon X' \to X$ and $\psi_Y\colon Y' \to Y$ making the diagram in the reverse direction commute, i.e. satisfying $f \circ \psi_X = \psi_Y \circ f'$.

Example 2.5. Consider the (strictly) commutative diagram in $\mathbf{Top}_*$

$$\begin{array}{ccc} S^n & \hookrightarrow & D^{n+1} \\ \| & & \downarrow \simeq \\ S^n & \longrightarrow & * \end{array}$$

where both downward arrows are (pointed) homotopy equivalences. Then there is no choice of "upward" homotopy inverses $S^n \to S^n$ and $* \to D^{n+1}$ making the "upward" diagram

$$\begin{array}{ccc} S^n & \hookrightarrow & D^{n+1} \\ \uparrow & & \uparrow \\ S^n & \longrightarrow & * \end{array}$$

commute strictly. Indeed, in any such strictly commutative diagram, the map $S^n \to S^n$ on the left is constant, hence not a homotopy equivalence.

Exercise 2.6. (Hatcher § 4.1 Exercise 9) Assume that a map of pairs $f\colon (X, A) \to (X', A')$ induces isomorphisms as in the diagram

$$\begin{array}{ccccccccc} \pi_1(A) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X,A) & \xrightarrow{\partial} & \pi_0(A) & \longrightarrow & \pi_0(X) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow f_* & & \downarrow \simeq & & \downarrow \simeq \\ \pi_1(A') & \longrightarrow & \pi_1(X') & \longrightarrow & \pi_1(X',A') & \xrightarrow{\partial} & \pi_0(A') & \longrightarrow & \pi_0(X') \end{array}$$

for any basepoint $a_0 \in A$. Show that the 5-lemma holds in this situation, i.e. that the middle map $f_*\colon \pi_1(X, A) \to \pi_1(X', A')$ is also an isomorphism.

Recall that $\pi_1(X)$ naturally acts on $\pi_1(X, A)$, in such a way that $\partial(\alpha) = \partial(\beta)$ holds if and only if the elements $\alpha, \beta \in \pi_1(X, A)$ differ by the action, i.e. $\alpha = \gamma \cdot \beta$ for some $\gamma \in \pi_1(X)$.

Proposition 2.7. *If a map of diagrams*

$$\varphi\colon \left(X \xrightarrow{f} Y\right) \to \left(X' \xrightarrow{f'} Y'\right)$$

is an objectwise weak homotopy equivalence, i.e. both maps $\varphi_X\colon X \xrightarrow{\sim} X'$ *and* $\varphi_Y\colon Y \xrightarrow{\sim} Y'$ *are weak homotopy equivalences, then the induced map on homotopy fibers* $\varphi_F\colon F(f) \to F(f')$ *is also a weak homotopy equivalence.*

Proof. Consider the map of long exact sequences of homotopy groups induced by the map of pairs $\varphi\colon (Y, X) \to (Y', X')$. The result follows from the natural isomorphism $\pi_n(Y, X) \cong \pi_{n-1}(F(f))$ and the generalized 5-lemma 2.6. $\square$

3 Iterated fiber sequence

Proposition 3.1. *Consider the fiber sequence* $F(f) \xrightarrow{p} X \xrightarrow{f} Y$*. Then the inclusion of the strict fiber of* p *into its homotopy fiber* $F(p)$ *is a homotopy equivalence making the following diagram commute:*

$$\begin{array}{ccccccc} \Omega Y & \xrightarrow{\iota} & F(f) & \xrightarrow{p} & X & \xrightarrow{f} & Y \\ {\scriptstyle\simeq}\big\downarrow{\scriptstyle\varphi} & \nearrow & & & & & \\ F(p) & & & & & & \end{array}$$

where $\iota\colon \Omega Y \to F(f)$ *is defined by*

$$\iota(\gamma) = (x_0, \gamma) \in F(f) = X \times_Y PY.$$

Proof. The strict fiber of p is the subset of $F(f)$

$$p^{-1}(x_0) = \{(x_0, \gamma) \mid \gamma(0) = y_0, \gamma(1) = f(x_0) = y_0\} \cong \Omega Y$$

and the composite $\Omega Y \to F(p) \to F(f)$ is

$$\gamma \mapsto (\iota(\gamma), \text{ constant path at the basepoint of } F(f)) \mapsto \iota(\gamma).$$

The inclusion of the strict fiber $\varphi\colon \Omega Y \xrightarrow{\simeq} F(p)$ is a homotopy equivalence, since p is a fibration (and using 1.6). □

In light of the proposition, the sequence $\Omega Y \to F(f) \to X$ is sometimes also called a fiber sequence.

Proposition 3.2. *The following triangle*

$$\begin{array}{ccc} \Omega X & \xrightarrow{-\Omega f} & \Omega Y \\ & {\scriptstyle\iota'}\searrow & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\varphi} \\ & & F(p) \end{array}$$

commutes up to homotopy. Here, the map $\iota'\colon \Omega X \to F(p)$ *is defined by*

$$\iota'(\gamma) = (*_{F(f)}, \gamma) \in F(p) = F(f) \times_X PX.$$

See May § 8.6 for more details.

Definition 3.3. The (long) **fiber sequence** generated by a pointed map $f\colon X \to Y$ is the sequence

$$\ldots \to \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega F(f) \xrightarrow{-\Omega p} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} F(f) \xrightarrow{p} X \xrightarrow{f} Y \qquad (3)$$

where $p\colon F(f) \to X$ and $\iota\colon \Omega Y \to F(f)$ are defined above.

Such a sequence is sometimes called a **Puppe sequence**.

Proposition 3.4. *Let $f\colon X \to Y$ be a pointed map, and W any pointed space. Then applying the functor $[W, -]_*\colon \mathbf{Top}_* \to \mathbf{Set}_*$ to the fiber sequence generated by f yields*

$$\ldots \to [W, \Omega^2 Y]_* \to [W, \Omega F(f)]_* \to [W, \Omega X]_* \to [W, \Omega Y]_* \to [W, F(f)]_* \to [W, X]_* \to [W, Y]_*$$

which is a long exact sequence of pointed sets.

Proof. By 3.1 and 3.2, each consecutive three spots of the long fiber sequence form, up to homotopy equivalence, a fiber sequence. The result follows from 1.5. □

Note that $[W, \Omega X]_*$ is naturally a group and $[W, \Omega^2 X]_*$ is naturally an abelian group.

Now we recover the usual long exact sequence of homotopy groups of a pair.

Corollary 3.5. *Let $i\colon A \to X$ be a pointed map. Then there is a long exact sequence of homotopy groups*

$$\ldots \to \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \ldots$$

where $j\colon (X, a_0) \to (X, A)$ is given by the inclusion of the basepoint $\{a_0\} \hookrightarrow A$ and ∂ is the usual boundary map.

Proof. Consider the fiber sequence generated by $i\colon A \to X$ and apply the functor $[S^0, -]_*$ to obtain a long exact sequence

$$\ldots \to [S^0, \Omega^n F(i)]_* \to [S^0, \Omega^n A]_* \to [S^0, \Omega^n X]_* \to [S^0, \Omega^{n-1} F(i)]_* \to [S^0, \Omega^{n-1} A]_* \to \ldots.$$

Using the natural isomorphism

$$[S^0, \Omega^n X]_* \cong [\Sigma^n S^0, X]_* \cong [S^n, X]_*$$

along with the natural isomorphism

$$\pi_n(X, A) \cong \pi_{n-1}(F(i))$$

we see that the terms of the sequence are as claimed. One readily checks that the maps also coincide up to sign, which does not affect exactness of the sequence. □