MATH 442/842 – Algebraic Topology

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Abstract

These are the evolving lecture notes for the course MATH 442/842 – Algebraic Topology in the Winter 2025 semester. We are following Hatcher as main reference. The notes provide additional details and examples.

Contents

1 Spheres and disks

Definition 1.1. For $n \geq 0$, the standard ***n*-sphere** is

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

and the standard ***n*-disk** is

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

The boundary of the disk is a sphere of dimension one less:

$$\partial D^n = S^{n-1}.$$

Remark 1.2. Both S^n and D^n are closed and bounded subsets of Euclidean space, hence are compact, by the Heine–Borel theorem.

Problem 1.3. Show that a closed interval with its endpoints identified is homeomorphic to a circle: $[0,1]/0 \sim 1 \cong S^1$.

Solution. Consider the winding map

$$p \colon \mathbb{R} \twoheadrightarrow S^1$$
$$p(t) = (\cos(2\pi t), \sin(2\pi t))$$

which is continuous and surjective. Since $[0,1]$ is compact and S^1 is Hausdorff, p is a quotient map. The equivalence relation induced by p identifies the endpoints:

$$p(s) = p(t) \iff s = t \text{ or } s, t \in \{0,1\}.$$

Therefore p induces a homeomorphism $\overline{p} \colon [0,1]/0 \sim 1 \xrightarrow{\cong} S^1$, as illustrated in the diagram

$$\begin{tikzcd}
{[0,1]} \arrow[r, "p"] \arrow[d, "\text{quotient}"'] & S^1 \\
{[0,1]/0 \sim 1} \arrow[ur, "\cong", "\overline{p}"'] &
\end{tikzcd}$$

□

Exercise 1.4. Show that the n-disk with its boundary collapsed to a point is homeomorphic to the n-sphere S^n:

$$D^n/\partial D^n \cong S^n.$$

Problem 1.5. Show that there is a homeomorphism

$$(D^n \amalg D^n)/\!\sim\ \cong S^n$$

where the two disks are glued along their boundaries, i.e., the equivalence relation $\sim$ is generated by $x^{(1)} \sim x^{(2)}$ for all $x \in D^n$ with $\|x\| = 1$. Here the superscript denotes that $x^{(1)} \in D^n \amalg D^n$ lives in the first summand while $x^{(2)}$ lives in the second summand.

Solution. Writing $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$, consider the map $f_U \colon D^n \to S^n$ sending the disk to the upper hemisphere:

$$f_U(x) = (x, \sqrt{1 - \|x\|^2}).$$

Then f_U is continuous since its $n+1$ components are continuous, and in fact f_U is a homeomorphism onto the upper hemisphere, with inverse the projection $p_{\mathbb{R}^n} \colon S^n_{\text{upper}} \to D^n$ onto the first n coordinates. Indeed, a point $(x_1, \ldots, x_n, x_{n+1}) = (x, x_{n+1})$, where x denotes $(x_1, \ldots, x_n)$, is in the upper hemisphere S^n_{upper} if and only if it satisfies

$$\begin{cases} x_1^2 + \ldots + x_n^2 + x_{n+1}^2 = 1 = \|x\|^2 + x_{n+1}^2 \\ x_{n+1} \geq 0 \end{cases}$$

so that it is of the form $(x, \sqrt{1 - \|x\|^2})$ for some $x \in D^n$.

Likewise, consider the map $f_L \colon D^n \to S^n$ sending the disc homeomorphically onto the lower hemisphere:

$$f_L(x) = (x, -\sqrt{1 - \|x\|^2}).$$

Consider the map $f \colon D^n \amalg D^n \to S^n$ whose restrictions to the first and second summands are f_U and f_L respectively. Then f is continuous, since its restriction to each summand is continuous. Moreover, f is surjective, because of $S^n = S^n_{\text{upper}} \cup S^n_{\text{lower}}$.

However, f is not injective. Because the restrictions f_U and f_L are injective, non-injectivity can only happen when taking inputs from different summands:

$$\begin{aligned} f(x^{(1)}) = f(y^{(2)}) &\iff f_U(x) = f_L(y) \\ &\iff \left(x, \sqrt{1 - \|x\|^2}\right) = \left(y, -\sqrt{1 - \|y\|^2}\right) \\ &\iff x = y \text{ and } \|x\| = \|y\| = 1 \\ &\iff x^{(1)} \sim y^{(2)}. \end{aligned}$$

Therefore, f induces a continuous map from the quotient

$$\overline{f} \colon (D^n \amalg D^n)/\!\sim\ \cong S^n$$

which is surjective (because f is) and injective (because of the implication $f(x) = f(x') \implies x \sim x'$).

Since the disc $D^n \subset \mathbb{R}^n$ is closed and bounded, it is compact. Therefore the finite union $D^n \amalg D^n$ is compact, and so is its quotient $(D^n \amalg D^n)/\!\sim$. Since S^n is a metric space, it is Hausdorff. Now $\overline{f}$ is a continuous bijection from a compact space to a Hausdorff space, and is therefore a homeomorphism. □

Problem 1.6. Show that the punctured sphere $S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$.

Solution. Consider $\mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ as the hyperplane $x_{n+1} = 0$ inside $\mathbb{R}^{n+1}$. Take the "North pole" $p = (0, \ldots, 0, 1) \in S^n \subset \mathbb{R}^{n+1}$. We will produce a homeomorphism $f \colon \mathbb{R}^n \to S^n \setminus \{p\}$ using "stereographic projection". Define the maps

$$f \colon \mathbb{R}^n \to S^n \setminus \{p\}$$

where $f(x)$ is the unique point of $S^n \setminus \{p\}$ which lies on the straight line through x and p and

$$g \colon S^n \setminus \{p\} \to \mathbb{R}^n$$

where $g(y)$ is the unique point of $\mathbb{R}^n$ which lies on the straight line through y and p.

For all $x \in \mathbb{R}^n$, $g(f(x))$ is the unique point of $\mathbb{R}^n$ which lies on the straight line through $f(x)$ and p. By definition of f, x is a point of $\mathbb{R}^n$ which lies on the straight line through $f(x)$ and p. This proves $g(f(x)) = x$.

For all $y \in S^n \setminus \{p\}$, $f(g(y))$ is the unique point of $S^n \setminus \{p\}$ which lies on the straight line through $g(y)$ and p. By definition of g, y is a point of $S^n \setminus \{p\}$ which lies on the straight line through $g(y)$ and p. This proves $f(g(y)) = y$.

It remains to show that f and g are both continuous.

A straightforward calculation yields

$$f(x) = \frac{1}{\|x\|^2 + 1} \left[2x + (\|x\|^2 - 1)p \right]$$

which is a continuous function of x. Indeed, the norm $\|x\|$ is continuous in x, and the denominator satisfies $\|x\|^2 + 1 > 0$ for all $x \in \mathbb{R}^n$.

Another straightforward calculation yields

$$g(y) = \frac{1}{1 - y_{n+1}} \left[y - y_{n+1} p \right]$$

which is a continuous function of y. Indeed, the projection $\mathbb{R}^{n+1} \twoheadrightarrow \mathbb{R}$ onto each coordinate is continuous (so that y_{n+1} is a continuous function of y), and the denominator satisfies $1 - y_{n+1} > 0$ for all $y \in S^n \setminus \{p\}$. ☐

Remark 1.7. Problem 1.6 shows that the one-point compactification (also called Alexandroff compactification) of $\mathbb{R}^n$ is an n-sphere: $(\mathbb{R}^n)^+ \cong S^n$.

Definition. The (unreduced) **cone** on a space X is the space obtained by collapsing the "top" of the cylinder on X:

$$CX := (X \times I)/(X \times \{1\}).$$

Let $\iota \colon X \hookrightarrow CX$ denote the "inclusion of the base of the cone", that is, the composite

$$X \xrightarrow{(\mathrm{id}_X, 0)} X \times I \xrightarrow{\text{quotient}} CX.$$

Exercise 1.8. Show that the cone on a sphere is homeomorphic to a disk: $C(S^n) \cong D^{n+1}$, for $n \geq 0$. See Homework 1 Problem 1.

2 Homotopy

2.1 Definitions and properties

Definition 2.1. Let $f, g\colon X \to Y$ be (continuous) maps between spaces. A **homotopy** from f to g is a continuous map

$$H\colon X \times [0,1] \to Y$$

satisfying:

$$\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \end{cases}$$

for all $x \in X$.

We then say that f is **homotopic** to g, denoted $f \simeq g$.

In other words, a homotopy is a continuous deformation of f into g. We think of $t \in [0,1]$ as a "time parameter" or "deformation parameter". Denote by $h_t\colon X \to Y$ the map "at time t", i.e.

$$h_t(x) = H(x,t).$$

A homotopy from f to g can be displayed as a commutative diagram:

$$\begin{tikzcd}
X \arrow[d, "\mathrm{inc}_0"'] \arrow[dr, bend left, "f"] & \\
X \times I \arrow[r, "H"] & Y \\
X \arrow[u, "\mathrm{inc}_1"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

where $I = [0,1]$ denotes the unit interval.

Definition 2.2. The **cylinder** on a space X is the space $\mathrm{Cyl}(X) := X \times I$.

Exercise 2.3. Show that being homotopic $f \simeq g$ is an equivalence relation on the set

$$C(X,Y) = \{f\colon X \to Y \mid f \text{ is continuous}\}.$$

Denote the homotopy class of f by $[f]$.

To justify continuity of a concatenation of homotopies (at double speed), use the following "gluing lemma" from general topology.

Lemma 2.4. *Let $f\colon X \to Y$ be a function between spaces, and $A, B \subseteq X$ closed subsets with $A \cup B = X$. If the restrictions*

$$\begin{cases} f\,|_A\colon A \to Y \\ f\,|_B\colon B \to Y \end{cases}$$

are continuous, then $f\colon X \to Y$ is continuous.

Example 2.5. Any two maps $f, g\colon X \to \mathbb{R}^n$ are homotopic via the linear homotopy

$$H(x,t) = (1-t)f(x) + tg(x).$$

This works more generally for maps $f, g\colon X \to C$ where $C \subseteq \mathbb{R}^n$ is a convex subset.

Proposition 2.6. *Homotopy is compatible with composition, i.e., given homotopic maps $f, f'\colon X \to Y$ and homotopic maps $g, g'\colon Y \to Z$, the composites $X \to Z$ are also homotopic:*

$$\begin{cases} f \simeq f' \\ g \simeq g' \end{cases} \implies g \circ f \simeq g' \circ f'.$$

Proof sketch. Given homotopies $F\colon f \Rightarrow f'$ and $G\colon g \Rightarrow g'$, take the homotopy $H\colon X \times I \to Z$ given by

$$h_t = g_t \circ f_t.$$

□

The argument can be displayed in a 2-cell diagram:

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U1}] \arrow[r, bend right=50, "f'"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "F"] & Y \arrow[r, bend left=50, "g"{name=U2}] \arrow[r, bend right=50, "g'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "G"] & Z. \end{tikzcd}$$

Upshot: We can compose homotopy classes of maps, i.e., the formula

$$[g] \circ [f] := [g \circ f]$$

is well-defined.

Definition 2.7. The (naive) **homotopy category of topological spaces** $\mathrm{Ho}(\mathbf{Top})$ has as objects topological spaces and as morphisms homotopy classes of continuous maps

$$[f]\colon X \to Y.$$

Denote the hom sets by

$$\begin{aligned} [X,Y] &:= \{\text{homotopy classes of maps } [f]\colon X \to Y\} \\ &= \mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top})}(X,Y) \\ &= \mathrm{Hom}_{\mathbf{Top}}(X,Y)/\simeq . \end{aligned}$$

Example 2.8. For any space X, the set $[*, X] = \pi_0(X)$ consists of the path components of X.

2.2 Homotopy equivalence

Definition 2.9. A map $f\colon X \to Y$ is a **homotopy equivalence** if there is a map $g\colon Y \to X$ satisfying

$$\begin{cases} g \circ f \simeq \mathrm{id}_X \\ f \circ g \simeq \mathrm{id}_Y. \end{cases}$$

Such a map g is called a **homotopy inverse** of f. We then say that X and Y are **homotopy equivalent**, denoted $X \simeq Y$.

When a homotopy inverse $g\colon Y \to X$ exists, it need not be unique, but it is unique up to homotopy.

Exercise 2.10. Show that a composite of homotopy equivalences is a homotopy equivalence.

Deduce that "is homotopy equivalent to" $X \simeq Y$ is an equivalence relation among spaces.

Exercise 2.11. Show that a map $f\colon X \to Y$ is a homotopy equivalence if and only if it becomes an isomorphism $[f]\colon X \xrightarrow{\cong} Y$ in the homotopy category Ho(**Top**).

Example 2.12. Euclidean space is homotopy equivalent to a point: $\mathbb{R}^n \simeq *$. The unique map $f\colon \mathbb{R}^n \to *$ is a homotopy equivalence, with homotopy inverse (for instance) $g = c_0\colon * \to \mathbb{R}^n$. The linear homotopy

$$H(x,t) = (1-t)x$$

provides a homotopy from $\mathrm{id}_{\mathbb{R}^n}$ to $c_0\colon \mathbb{R}^n \to \mathbb{R}^n$, the constant map sending everything to 0.

The example illustrates the following notion.

Definition 2.13. A space X is **contractible** if it is homotopy equivalent to a point, i.e., $X \simeq *$.

A homotopy from id_X to a constant map $c_{x_0}\colon X \to X$ is called a **contraction** of X.

See Homework 1 Problem 2 for different characterizations of contractible spaces.

Definition 2.14. A map $f\colon X \to Y$ is **null-homotopic** if it is homotopic to a constant map $c_y\colon X \to Y$.

Exercise 2.15. Show that a map $f\colon X \to Y$ is null-homotopic if and only if it extends to the cone on X, as illustrated in the diagram

$$\begin{tikzcd} CX \arrow[dr, dashed] & \\ X \arrow[u, "\iota"] \arrow[r, "f"'] & Y. \end{tikzcd}$$

See Homework 1 Problem 1.

Example 2.16. The n-disk D^n is contractible, also using the linear homotopy $H(x,t) = (1-t)x$.

More generally, any convex subset $C \subseteq \mathbb{R}^n$ is contractible.

Yet more generally, any star-shaped subset $S \subseteq \mathbb{R}^n$ is contractible. **Star-shaped** means that there exists a point $x_0 \in S$ such that for all $x \in S$, the line segment joining x_0 and x is contained in S.

Example 2.17. The circle S^1 is not contractible. We will learn shortly that the circle has a non-trivial fundamental group $\pi_1(S^1) \cong \mathbb{Z} \neq 0$, hence cannot be contractible.

Remark 2.18. Given homotopy equivalent spaces $X \simeq Y$, if X is contractible, then so is Y:

$$Y \simeq X \simeq *.$$

2.3 Deformation retracts

Definition 2.19. A subspace $A \subseteq X$ is called a **(strong) deformation retract** of X if there is a homotopy $H \colon X \times I \to X$ satisfying:

$$\begin{cases} H(x,0) = x & \text{for all } x \in X \\ H(x,1) \in A & \text{for all } x \in X \\ H(a,t) = a & \text{for all } a \in A \text{ and } t \in I. \end{cases}$$

Such a homotopy H is called a **deformation retraction** of X onto A.

We say **weak deformation retract** if the last condition is only required for $t = 1$.

Warning 2.20. We follow the terminology of Hatcher [Hat02, §0], where a deformation retract is strong by default. Some authors use the term "deformation retract" to mean weak by default.

Example 2.21. For any $n \geq 0$, the origin $\{0\} \subseteq \mathbb{R}^n$ is a deformation retract of Euclidean space $\mathbb{R}^n$, via the deformation retraction

$$\begin{aligned} &H \colon \mathbb{R}^n \times I \to \mathbb{R}^n \\ &H(x,t) = (1-t)x. \end{aligned}$$

Similarly, the center $\{0\} \subset D^n$ is a deformation retract of the unit disk D^n.

Example 2.22. Any space X is a deformation retract of its cylinder $\mathrm{Cyl}(X)$, viewing X as the bottom of the cylinder $X \times \{0\}$, via the deformation retraction

$$\begin{aligned} &H \colon \mathrm{Cyl}(X) \times I \to \mathrm{Cyl}(X) \\ &H\left((x,s),t\right) = (x,(1-t)s). \end{aligned}$$

Example 2.23. For any $n \geq 1$, the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is a deformation retract of the punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$, via the deformation retraction

$$\begin{aligned} &H \colon (\mathbb{R}^n \setminus \{0\}) \times I \to \mathbb{R}^n \setminus \{0\} \\ &H(x,t) = (1-t)x + t\frac{x}{\|x\|}. \end{aligned}$$

Exercise: Check that the formula is well-defined, i.e., never takes the value $0 \in \mathbb{R}^n$.

Definition 2.24. Given maps $f,g \colon X \to Y$, a homotopy $H \colon X \times I \to Y$ is **relative** to a subspace $A \subseteq X$ if it is stationary on A, i.e., satisfies

$$H(a,t) = H(a,0) = f(a) \quad \text{for all } a \in A \text{ and } t \in I.$$

We write $f \simeq g \operatorname{rel} A$.

In particular, this forces the maps f and g to have the same restriction to A, i.e., $f|_A = g|_A \colon A \to Y$.

With this terminology, we can revisit Definition 2.19. A subspace $A \subseteq X$ with inclusion map $i \colon A \hookrightarrow X$ is:

- a **retract** of X if the inclusion map admits a retraction $r \colon X \to A$, i.e., a map satisfying $r \circ i = \mathrm{id}_A$.
- a **weak deformation retract** of X if moreover $i \circ r \simeq \mathrm{id}_X$ holds.
- a **(strong) deformation retract** of X if moreover $i \circ r \simeq \mathrm{id}_X \operatorname{rel} A$ holds.

Remark 2.25.

1. If A is a weak deformation retract of X, then in particular the inclusion $i \colon A \hookrightarrow X$ is a homotopy equivalence with homotopy inverse the retraction $r \colon X \to A$.
2. If A is merely a retract of X, their homotopy types can be very different. For instance, the one-point space $*$ is a retract of every (non-empty) space X.

3 Connected components as a homotopy invariant

Definition 3.1. Let X be a topological space. A **path** in X from x to y is a continuous map $\gamma\colon [0,1] \to X$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$.

The relation "there is a path from x to y" is an equivalence relation on X, whose equivalence classes are called the **path components** of X. Let $\pi_0(X)$ denote the set of path components of X.

Problem 3.2. Let $f\colon X \to Y$ be a map between spaces. Define an induced function on path components

$$f_*\colon \pi_0(X) \to \pi_0(Y),$$

sometimes denoted $\pi_0(f)$, by the formula $f_*[x] := [f(x)]$, where $[x]$ denotes the path component of a point $x \in X$.

(a) Show that the function f_* is well-defined.

(b) Show that the induced function makes $\pi_0\colon$ **Top** $\to$ **Set** into a functor.

(c) Show that π_0 is a *homotopy functor* in the following sense: homotopic maps $f \simeq g\colon X \to Y$ induce the same function

$$f_* = g_*\colon \pi_0(X) \to \pi_0(Y).$$

(d) Let $X \simeq Y$ be homotopy equivalent spaces. Show that the sets of path components $\pi_0(X)$ and $\pi_0(Y)$ are in bijection.

In particular, X is path-connected if and only if Y is.

Solution. See Homework 1 Problem 3. □

Problem 3.3. Let $\mathrm{Conn}(X)$ denote the set of connected components of a space X.

Let $f\colon X \to Y$ be a (continuous) map between spaces. Define an induced function on connected components

$$f_*\colon \mathrm{Conn}(X) \to \mathrm{Conn}(Y),$$

sometimes denoted $\mathrm{Conn}(f)$, by the formula $f_*[x] := [f(x)]$, where $[x]$ denotes the connected component of $x \in X$.

(a) Show that the function f_* is well-defined.

(b) Show that the induced function makes $\mathrm{Conn}\colon$ **Top** $\to$ **Set** into a functor.

(c) Show that Conn is *homotopy invariant* in the following sense: homotopic maps $f \simeq g\colon X \to Y$ induce the same function

$$f_* = g_*\colon \mathrm{Conn}(X) \to \mathrm{Conn}(Y).$$

(d) Let $X \simeq Y$ be homotopy equivalent spaces. Show that the sets of connected components $\mathrm{Conn}(X)$ and $\mathrm{Conn}(Y)$ are in bijection.

In particular, X is connected if and only if Y is.

Solution. Similar to Problem 3.2. For part (a), use the fact that the continuous image of a connected space is connected. For (c), use the fact that if there is a path between y and y', then y and y' lie in the same connected component. $\square$

Problem 3.4. (a) Denote by $\eta_X \colon \pi_0(X) \to \mathrm{Conn}(X)$ the function that assigns to a path component C the connected component that contains C. Show that η is a natural transformation.

Solution. Naturality means that for every map $f \colon X \to Y$, the following diagram of sets commutes:

$$\begin{tikzcd}
\pi_0(X) \arrow[r, "\eta_X"] \arrow[d, "\pi_0(f)"'] & \mathrm{Conn}(X) \arrow[d, "\mathrm{Conn}(f)"] \\
\pi_0(Y) \arrow[r, "\eta_Y"] & \mathrm{Conn}(Y).
\end{tikzcd}$$

Let $[x]_c$ denote the connected component of $x \in X$ and let $[x]_{pc}$ denote the path component of x. Then the function η_X is defined by the equation

$$\eta_X([x]_{pc}) = [x]_c.$$

For every $[x]_{pc} \in \pi_0(X)$, the following equations hold in $\mathrm{Conn}(Y)$:

$$\begin{aligned}
\mathrm{Conn}(f)\eta_X([x]_{pc}) &= \mathrm{Conn}(f)[x]_c \\
&= [f(x)]_c \\
\eta_Y \pi_0(f)([x]_{pc}) &= \eta_Y [f(x)]_{pc} \\
&= [f(x)]_c.
\end{aligned}$$

☐

(b) Let $X \simeq Y$ be homotopy equivalent spaces, where X satisfies the following condition: the path components of X coincide with its connected components. Show that Y also satisfies that condition.

Solution. The path components of X coincide with its connected components if and only if the surjection $\eta_X \colon \pi_0(X) \to \mathrm{Conn}(X)$ is a bijection. Now let $f \colon X \xrightarrow{\simeq} Y$ be a homotopy equivalence. By part (a), the following diagram commutes:

$$\begin{tikzcd}
\pi_0(X) \arrow[r, "\eta_X"] \arrow[d, "\pi_0(f)"', "\cong"] & \mathrm{Conn}(X) \arrow[d, "\mathrm{Conn}(f)", "\cong"'] \\
\pi_0(Y) \arrow[r, "\eta_Y"] & \mathrm{Conn}(Y),
\end{tikzcd}$$

where the downward maps are bijections, by Problems 3.2 and 3.3. Therefore, the top map η_X is bijective if and only if the bottom map η_Y is bijective. ☐

Remark 3.5. The topologist's sine curve

$$S = \{(x, \sin \frac{1}{x}) \mid x \in (0,1]\} \cup (\{0\} \times [-1,1]) \subset \mathbb{R}^2$$

does not satisfy said condition. To wit, S is connected, but it has two path components.

4 The fundamental group

Definition 4.1. Let X be a topological space. The **fundamental group** of X based at a point $x_0 \in X$ consists of the homotopy classes of loops based at x_0:

$$\pi_1(X, x_0) = \{\gamma \colon I \to X \mid \gamma(0) = \gamma(1) = x_0\} / \simeq .$$

The group structure is given by concatenation of loops.

Problem 4.2. Let (X, x_0) be a pointed space.

(a) Show that two loops $\alpha, \beta \colon S^1 \to X$ based at x_0 are unpointed homotopic (also called *freely* homotopic) if and only if their pointed homotopy classes $[\alpha]$ and $[\beta]$ are conjugate in the fundamental group $\pi_1(X, x_0)$.

Solution. ($\implies$) Let $H \colon S^1 \times I \to X$ be an (unpointed) homotopy from α to β. Keeping track of where the basepoint $v \in S^1$ is sent throughout the homotopy

$$\gamma(t) := H(v, t)$$

yields a loop $\gamma \colon I \to X$ based at x_0. Via the homeomorphism $S^1 \cong I/\partial I$, we can view H as a map $H \colon I \times I \to X$ that satisfies

$$H(0, t) = H(1, t) = \gamma(t)$$

for all $t \in I$. Hence, H restricts to the four edges of the square $I \times I$ as indicated in this picture:

$$\begin{tikzcd}
x_0 \arrow[r, "\beta"] & x_0 \\
x_0 \arrow[u, "\gamma"] \arrow[r, "\alpha"'] & x_0. \arrow[u, "\gamma"']
\end{tikzcd} \tag{1}$$

Since the square I^2 is contractible, it is in particular simply-connected, so that any two paths in I^2 with the same endpoints are homotopic (by Homework 2 Problem 2). Denote the path that goes along the bottom edge

$$\ell_{\text{bottom}} := (\mathrm{id}, 0) \colon I \to I^2$$

and likewise for the remaining three edges. There is a homotopy of paths in I^2

$$\ell_{\text{bottom}} \simeq \ell_{\text{left}} \cdot \ell_{\text{top}} \cdot \overline{\ell_{\text{right}}}.$$

Applying $H \colon I^2 \to X$ then yields a homotopy of paths in X

$$\alpha \simeq \gamma \cdot \beta \cdot \overline{\gamma}$$

and hence the equality in $\pi_1(X, x_0)$

$$[\alpha] = [\gamma \cdot \beta \cdot \overline{\gamma}] = [\gamma][\beta][\gamma]^{-1}.$$

($\Longleftarrow$) Let $[\alpha], [\beta] \in \pi_1(X, x_0)$ be conjugate elements, satisfying

$$[\alpha] = [\gamma][\beta][\gamma]^{-1}$$

for some $[\gamma] \in \pi_1(X, x_0)$. Pick representative loops α, β, and γ based at $x_0 \in X$. The equation

$$[\alpha][\gamma] = [\gamma][\beta]$$

in $\pi_1(X, x_0)$ means that there is a path homotopy $\alpha \cdot \gamma \simeq \gamma \cdot \beta$. Therefore, the map $h \colon \partial I^2 \to X$ with values indicated in the picture (1), viewed as two different paths from the corner $(0, 0)$ to $(1, 1)$, admits an extension to the square $H \colon I^2 \to X$. This map H provides an unpointed homotopy from α to β. □

Alternate solution for ($\Longrightarrow$). Let $v \in S^1$ denote the basepoint. The unpointed homotopy $H \colon S^1 \times I \to X$ from α to β yields an equation between their induced homomorphisms:

$$\begin{tikzcd}
\pi_1(S^1, v) \arrow[r, "\alpha_*"] \arrow[dr, "\beta_*"'] & \pi_1(X, \alpha(v)) = \pi_1(X, x_0) \\
& \pi_1(X, \beta(v)) = \pi_1(X, x_0) \arrow[u, "\beta_\gamma"', "\cong"]
\end{tikzcd}$$

where the unfortunate notation β_γ means the change-of-basepoint isomorphism induced by the path $\gamma \colon I \to X$. Applying this equation to the generator $[\mathrm{id}] \in \pi_1(S^1, v)$ yields:

$$\begin{aligned}
& \alpha_*([\mathrm{id}]) = \beta_\gamma \beta_*([\mathrm{id}]) \\
\iff & [\alpha] = \beta_\gamma([\beta]) \\
\iff & [\alpha] = [\gamma][\beta][\gamma]^{-1}. \qquad \square
\end{aligned}$$

(b) Show that a loop $\alpha\colon S^1 \to X$ based at x_0 is unpointed homotopic to the constant loop c_{x_0} if and only if α is pointed homotopic to c_{x_0}.

Solution. ($\Longleftarrow$) A pointed homotopy is a special kind of unpointed homotopy.

($\Longrightarrow$) Assume that the loop $\alpha\colon S^1 \to X$ is unpointed homotopic to the constant loop c_{x_0}. By part (a), their homotopy classes are conjugate in $\pi_1(X, x_0)$:

$$\begin{aligned} [\alpha] &= [\gamma][c_{x_0}][\gamma]^{-1} \\ &= [\gamma]1[\gamma]^{-1} \\ &= 1 \\ &= [c_{x_0}]. \end{aligned} \qquad \square$$

Alternate solution not using (a). By Homework 1 Problem 1, the null-homotopic map $\alpha\colon S^1 \to X$ admits an extension to the disk $\widetilde{\alpha}\colon D^2 \to X$. Since the disk D^2 is contractible, it is in particular simply-connected, so that the boundary inclusion $\iota\colon S^1 \hookrightarrow D^2$ is pointed homotopic to the constant loop $c_v\colon S^1 \to D^2$. Postcomposing with $\widetilde{\alpha}$ yields a pointed homotopy from

$$\widetilde{\alpha} \circ \iota = \widetilde{\alpha}|_{S^1} = \alpha$$

to

$$\widetilde{\alpha} \circ c_v = c_{\widetilde{\alpha}(v)} = c_{\alpha(v)} = c_{x_0}. \qquad \square$$

5 Topology of CW complexes

Problem 5.1. Let X be a space and $X = \sqcup_\alpha U_\alpha$ a *disjoint* union of open subsets $U_\alpha \subseteq X$. Show that X is the coproduct of the U_α, that is, the inclusions $U_\alpha \hookrightarrow X$ together induce a homeomorphism $X \cong \coprod_\alpha U_\alpha$.

Solution. Let $A \subseteq X$ be an open subset. For each index α, $A \cap U_\alpha$ is open in U_α, so that $A \subseteq \sqcup_\alpha U_\alpha$ is open in the coproduct topology. (This direction holds automatically.)

Conversely, let $A \subseteq \coprod_\alpha U_\alpha$ be an open subset in the coproduct topology. This means that A is a disjoint union of the form

$$A = \coprod_\alpha A_\alpha,$$

where $A_\alpha \subseteq U_\alpha$ is open in U_α for each index α. Since $U_\alpha \subseteq X$ is open in X, $A_\alpha \subseteq X$ is also open in X. Therefore $A \subseteq X$ is open in X, being the union of the open subsets A_α. ☐

Problem 5.2. Show that the following conditions on a space X are equivalent.

1. X is the coproduct of its path components.
2. Each path component of X is open in X.
3. Every point $x \in X$ has a path-connected neighborhood.

Solution. **(3) $\implies$ (2)** Let $x_0 \in X$ and let $C \subseteq X$ denote the path component of x_0. We want to show that C is open in X.

Let $x \in C$ and let $x \in U_x$ be a path-connected neighborhood of x. Then there is a path between x_0 and x, and between x and any point $y \in U_x$, hence also between x_0 and y. This shows the inclusion $U_x \subseteq C$, so that C is open in X.

(2) $\implies$ (1) Since X is the disjoint union of its path components, this implication follows from Problem 5.1.

(1) $\implies$ (2) In a coproduct $X = \coprod_\alpha X_\alpha$, each summand $X_\alpha \subseteq X$ is open.

(2) $\implies$ (3) Every point $x \in X$ lies in its path component $C_x \subseteq X$, which by assumption is open in X. Hence, C_x is a path-connected open neighborhood of x. ☐

Remark 5.3. A CW complex satisfies the equivalent conditions in Problem 5.2 and in fact much more: every point $x \in X$ admits an *arbitrarily small contractible* open neighborhood.

Problem 5.4. (a) Show that the conditions from Problem 5.2 imply the following condition: the path components of X coincide with its connected components.

Solution. Let $X = \coprod_{C \in \pi_0(X)} C$ be the decomposition of X as coproduct of its path components. Let $Z \subseteq X$ be a connected component of X. Since Z is connected, it must lie entirely within one summand $C \subseteq X$, i.e., the inclusion $Z \subseteq C$ holds. Since Z is itself a disjoint union of certain path components of X, the equality $Z = C$ holds. ☐

(b) Disprove the converse to part (a). In other words: Find a space X whose path components coincide with its connected components, but X is **not** the coproduct of its path components.

Solution. Consider the space

$$X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$$

as a subspace $X \subset \mathbb{R}$, illustrated in Figure 1. Then each point $\{x\} \subset X$ is a connected component of X, hence also a path component.

However, the point $\{0\} \subset X$ is not open in X, since every neighborhood of 0 contains infinitely many points of X. Therefore, X is not the coproduct $\coprod_{x \in X} \{x\}$. (This coproduct is a discrete space.) ☐

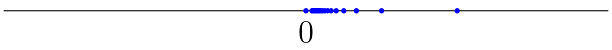


Figure 1: The space $X \subset \mathbb{R}$.

Problem 5.5. Show that the following spaces are **not** CW complexes.

(a) $X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ as a subspace $X \subset \mathbb{R}$.

Solution. As shown in Problem 5.4(b), X is not the coproduct of its path components, hence not a CW complex. ☐

(b) The topologist's sine curve:

$$X = \{(t, \sin\frac{1}{t}) \mid t \in (0,1]\} \cup (\{0\} \times [-1,1])$$

as a subspace $X \subset \mathbb{R}^2$, illustrated in Figure 2.

Solution. The space X consists of two path components but only one connected component. By Problem 5.4(a), X is not a CW complex. ☐

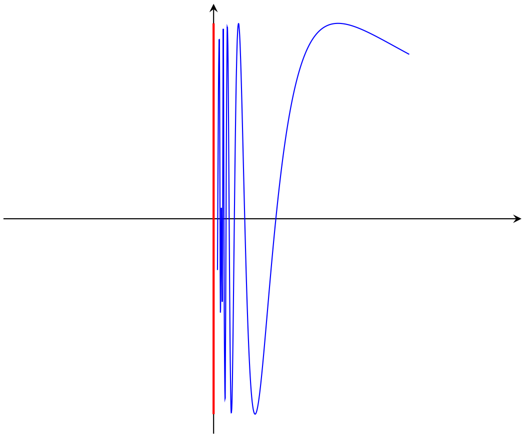


Figure 2: The topologist's sine curve.

(c) The Hawaiian earring

$$X = \bigcup_{n \in \mathbb{N}} C_n \subset \mathbb{R}^2 \tag{2}$$

where $C_n \subset \mathbb{R}^2$ denotes the circle centered at $(\frac{1}{n}, 0)$ with radius $\frac{1}{n}$. See Figure 3.

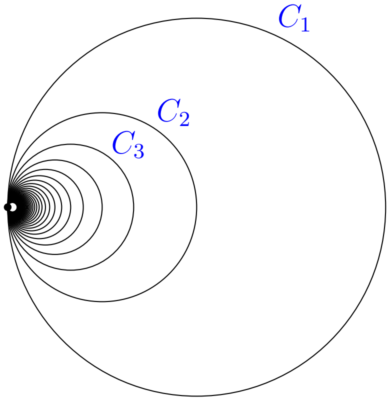


Figure 3: The Hawaiian earring.

Solution. The point $(0,0) \in X$ has **no** simply-connected neighborhood, by Lemma 5.7. In a CW complex, every point has arbitrarily small contractible neighborhoods, in particular at least one simply-connected neighborhood. □

(d) The "comb"

$$X = [0,1] \times \left(\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \right) \cup (\{0\} \times [0,1])$$

as a subspace $X \subset \mathbb{R}^2$, illustrated in Figure 4.

Solution. Consider the open ball $V = B\left((\frac{1}{2}, 0), 0.4\right)$ and a neighborhood U of $(\frac{1}{2}, 0)$ satisfying $U \subseteq V$. Then U is *not* path-connected. In a CW complex, every point has arbitrarily small contractible neighborhoods, in particular arbitrarily small path-connected neighborhoods. □

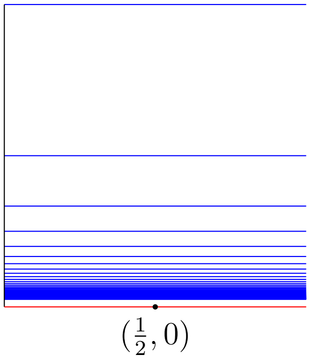


Figure 4: The comb.

Remark 5.6. The spaces in examples (a), (b), and (c) are not even homotopy equivalent to a CW complex. Proofs of those facts can be found in [MSE13]. However, the "comb" in example (d) is homotopy equivalent to a CW complex since it is contractible.

Lemma 5.7. *Let $X \subset \mathbb{R}^2$ denote the Hawaiian earring as in Equation (2). Let $U \subseteq X$ be a neighborhood of the point $(0,0) \in X$. Then the fundamental group $\pi_1(U,(0,0))$ is non-trivial.*

Proof. For each $n \in \mathbb{N}$, the inclusion of the circle $C_n \hookrightarrow X$ admits a retraction $r_n \colon X \twoheadrightarrow C_n$, namely the map given by $r_n(C_i) = \{(0,0)\}$ for all $i \neq n$. That is, r_n maps all the other circles C_i to the basepoint $(0,0)$.

Since U contains an open ball centered at $(0,0)$, the inclusion $C_m \subseteq U$ holds for m large enough. Then the inclusion $C_m \hookrightarrow U$ also admits a retraction, namely the restriction $r_m|_U \colon U \to C_m$, as illustrated in this diagram:

$$\begin{tikzcd}
C_m \arrow[r, hook] \arrow[rr, bend left, "\mathrm{id}_{C_m}"] \arrow[dr, hook] & X \arrow[r, "r_m"] & C_m \\
& U. \arrow[u, hook] \arrow[ur, "r_m|_U"'] &
\end{tikzcd}$$

Since $\pi_1(C_m,(0,0)) \cong \pi_1(S^1) \cong \mathbb{Z}$ is non-trivial and is a retract of $\pi_1(U,(0,0))$, the latter cannot be trivial. $\square$

Problem 5.8. Let X be a CW complex, with a given CW structure. Show that the following statements are equivalent.

1. X has (at least) a cell of dimension $d \geq n$.
2. There is an embedding $D^n \hookrightarrow X$.
3. There is an embedding $\mathbb{R}^n \hookrightarrow X$.

In particular, the dimension of X as a CW complex

$$\dim(X) = \sup\{n \mid e^n_\alpha \text{ is a cell of } X\}$$

does not depend on the CW structure, but only on the underlying space X.

Solution. The equivalence (2) $\iff$ (3) has nothing to do with CW complexes, but only with the fact that D^n and $\mathbb{R}^n$ embed into each other.

One embedding comes from the definition: the n-disk $D^n \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. Now let

$$B^n = B(0,1) = D^n \setminus \partial D^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

denote the open unit ball in $\mathbb{R}^n$. Let $h \colon \mathbb{R}^n \xrightarrow{\cong} B^n$ be a homeomorphism, for instance the scaling

$$h(x) = \frac{x}{1 + \|x\|}.$$

Then the composite

$$\mathbb{R}^n \xrightarrow[\cong]{h} B^n \xhookrightarrow{\text{inc}} D^n$$

is an embedding $\mathbb{R}^n \hookrightarrow D^n$.

(2) $\implies$ (3) Let $f \colon D^n \hookrightarrow X$ be an embedding. Then the composite $\mathbb{R}^n \hookrightarrow D^n \xrightarrow{f} X$ is an embedding.

(3) $\implies$ (2) Let $g \colon \mathbb{R}^n \hookrightarrow X$ be an embedding. Then the composite $D^n \hookrightarrow \mathbb{R}^n \xrightarrow{g} X$ is an embedding.

(1) $\implies$ (3) Let e^d_α be a d-cell of X with characteristic map $\Phi_\alpha \colon D^d \to X_d$. The restriction

$$B^d \xhookrightarrow{\Phi_\alpha|_{B^d}} X_d$$

is an embedding into the d-skeleton X_d, with image $\Phi_\alpha(B^d) = e^d_\alpha \subseteq X_d$. Composing with the inclusion $X_d \subseteq X$ yields an embedding $B^d \hookrightarrow X$. The inequality $d \geq n$ guarantees that there is an embedding $B^n \hookrightarrow B^d$, which yields an embedding $B^n \hookrightarrow B^d \hookrightarrow X$ by composition.

(2) $\implies$ (1) Let $f \colon D^n \hookrightarrow X$ be an embedding. By compactness of the disk D^n, the image $f(D^n) \subseteq X$ is also compact and thus lies in a finite subcomplex $K \subseteq X$. In other words, the map $f \colon D^n \hookrightarrow X$ factors through the inclusion $K \subseteq X$, as illustrated in this diagram:

$$\begin{tikzcd}
D^n \arrow[r, "f"] \arrow[dr, dashed, "f"'] & X \\
& K. \arrow[u, hook]
\end{tikzcd}$$

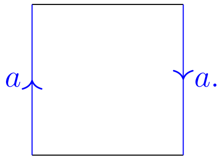
Let e_α^m be a cell of K of maximal dimension $m < \infty$. Then the "open" cell $e_\alpha^m \subseteq K_m$ is open in $K_m = K$. Therefore, its preimage $f^{-1}(e_\alpha^m) \subseteq D^n$ is open in D^n (and non-empty, without loss of generality), so that $f^{-1}(e_\alpha^m)$ contains a small open ball $\widetilde{B}^n \subseteq f^{-1}(e_\alpha^m)$. By restriction, we obtain an embedding

$$\widetilde{B}^n \xhookrightarrow{f|_{\widetilde{B}^n}} e_\alpha^m \cong B^m.$$

This is only possible if the dimensions satisfy $n \leq m$, by the Brouwer invariance of domain theorem [Hat02, Theorem 2B.3]. ☐

6 Surfaces

Definition 6.1. The **Möbius strip** M is the quotient space of the square $I^2/\sim$ with respect to the relation $(0,t) \sim (1, 1-t)$ for all $t \in I$. In other words, we identity two opposite sides as illustrated in this picture:



The *boundary* of M is the subspace $\partial M \subset M$ given by

$$\partial M = \{q(s,t) \mid t = 0 \text{ or } t = 1\} = q\left([0,1] \times \{0,1\}\right),$$

where $q \colon I^2 \twoheadrightarrow M$ denotes the quotient map.

Problem 6.2. Show that the boundary of the Möbius strip is homeomorphic to a circle: $\partial M \cong S^1$.

Solution. The quotient map $q \colon I^2 \twoheadrightarrow M$ is a closed map, by Lemma 6.3. Thus its restriction to the closed subspace $[0,1] \times \{0,1\} \subset I^2$ is also closed. Therefore, the continuous surjection

$$[0,1] \times \{0,1\} \xrightarrow{q|_{[0,1]\times\{0,1\}}} q([0,1] \times \{0,1\}) = \partial M$$

is also closed, hence a quotient map (of topological spaces). This yields the homeomorphisms

$$\begin{aligned} \partial M &= q([0,1] \times \{0,1\}) \\ &\cong [0,1] \times \{0,1\}/(0,0) \sim (1,1) \text{ and } (0,1) \sim (1,0) \\ &\cong [0,2]/0 \sim 2 \\ &\cong S^1. \end{aligned}$$

□

Lemma 6.3. *The quotient map $q \colon I^2 \twoheadrightarrow M$ is a closed map.*

Proof. Consider the homeomorphism

$$\begin{aligned} \varphi \colon \{0,1\} \times I &\xrightarrow{\cong} \{0,1\} \times I \\ \varphi(x,y) &= (1-x, 1-y). \end{aligned}$$

By definition, the Möbius strip is the quotient space

$$M = I^2/(x,y) \sim \varphi(x,y) \text{ for all } (x,y) \in \{0,1\} \times I.$$

For any subset $A \subseteq I^2$, the following equality holds:

$$q^{-1}q(A) = A \cup \varphi(A \cap (\{0,1\} \times I)).$$

If $A \subseteq I^2$ is closed, then $A \cap (\{0,1\} \times I)$ and $\varphi(A \cap (\{0,1\} \times I))$ are also closed, as is $q^{-1}q(A)$, as a finite union of closed subsets. Therefore, the subset $q(A) \subseteq I^2/\sim$ is closed, by definition of the quotient topology. □

Problem 6.4. Let $\iota\colon \partial M \hookrightarrow M$ denote the inclusion of the boundary of the Möbius strip.

(a) Compute the induced homomorphism

$$\iota_*\colon \pi_1(\partial M) \to \pi_1(M).$$

Solution. The projection onto the first coordinate

$$p\colon M \xrightarrow{\simeq} [0,1]/0 \sim 1 \cong S^1$$

is a homotopy equivalence. As generator of $\pi_1(\partial M) \cong \pi_1(S^1) \cong \mathbb{Z}$, we choose the homotopy class $[\alpha]$ of the path $\alpha\colon I \to \partial M$ given by

$$\alpha(s) = \begin{cases} (2s, 0) & \text{for } 0 \leq s \leq \frac{1}{2} \\ (2s-1, 1) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

For $\pi_1([0,1]/0 \sim 1) \cong \pi_1(S^1) \cong \mathbb{Z}$, we choose the usual generator, namely the homotopy class $[\beta]$ of the path $\beta\colon I \to [0,1]/0 \sim 1$ given by

$$\beta(s) = s.$$

Using those formulas, we compute the effect of the inclusion $\iota\colon \partial M \hookrightarrow M$ on fundamental groups:

$$p_*\iota_*([\alpha]) = [p\iota\alpha] = [\beta] \cdot [\beta] = 2[\beta] \in \pi_1(S^1).$$

In other words, the following diagram of groups commutes:

$$\begin{tikzcd}
\pi_1(\partial M) \arrow[r, "\iota_*"] \arrow[dr, "(p\iota)_*"'] \arrow[dd, dash, "\cong"'] & \pi_1(M) \arrow[d, "p_*"', "\cong"] \\
 & \pi_1(S^1) \arrow[d, dash, "\cong"] \\
\mathbb{Z} \arrow[r, "2"] & \mathbb{Z}.
\end{tikzcd}$$

□

Remark 6.5. The "equator" in the Möbius strip

$$Z := q\left([0,1] \times \{\frac{1}{2}\}\right) \subset M$$

is homeomorphic to a circle:

$$Z \cong [0,1]/0 \sim 1 \cong S^1.$$

The inclusion $Z \hookrightarrow M$ is homotopy inverse to the projection $p\colon M \to S^1 \cong Z$. This yields an explicit generator of $\pi_1(M) \cong \mathbb{Z}$, namely the homotopy class $[\beta']$ of the path $\beta'\colon I \to M$ given by

$$\beta'(s) = q(s, \frac{1}{2}),$$

as illustrated in Figure 5.

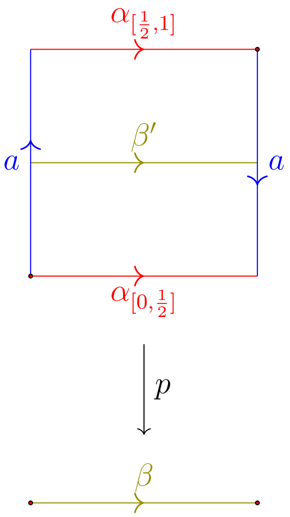


Figure 5: The projection $p\colon M \to S^1$.

(b) Show that ∂M is not a retract of M.

Solution. The induced homomorphism

$$\iota_* \colon \pi_1(\partial M) \to \pi_1(M)$$

admits no retraction. Indeed, there is no group homomorphism $r \colon \mathbb{Z} \to \mathbb{Z}$ that satisfies the equation

$$r \circ 2 = 1 \colon \mathbb{Z} \to \mathbb{Z}$$

since $(r \circ 2)(1) = r(2) = 2r(1) \in \mathbb{Z}$ is even, in particular $(r \circ 2)(1) \neq 1$.

Therefore, the map $\iota \colon \partial M \to M$ itself admits no retraction. □

Problem 6.6. Let $\varphi \colon S^1 \xrightarrow{\cong} \partial M \hookrightarrow M$ be the composite of a homeomorphism $S^1 \cong \partial M$ and the inclusion $\partial M \hookrightarrow M$. Show that the pushout $M \cup_\varphi D^2$ is homeomorphic to the projective plane $\mathbb{R}P^2$.

The space $M \cup_\varphi D^2$ is obtained from M by attaching a 2-cell along $\varphi \colon S^1 \to M$.

Solution. The projective plane is given (up to homeomorphism) by

$$\mathbb{R}P^2 = D^2/x \sim -x \text{ for } x \in \partial D^2.$$

Let $B \subset D^2$ be a small open ball inside the disk, for instance the metric ball $B = B(0, \frac{1}{2})$. We can view B as subspace of $\mathbb{R}P^2$ since the composite

$$\begin{tikzcd} B \arrow[r, hook, "\iota"] & D^2 \arrow[r, two heads, "q"] & D^2/\sim \; = \mathbb{R}P^2 \end{tikzcd}$$

is an embedding. The space $\mathbb{R}P^2 \setminus B$ has a boundary $\partial(\mathbb{R}P^2 \setminus B) \cong S^1$. By attaching a 2-cell along the boundary, we obtain the projective plane again:

$$(\mathbb{R}P^2 \setminus B) \cup_{\partial(\mathbb{R}P^2 \setminus B)} D^2 \cong \mathbb{R}P^2.$$

Hence, it suffices to show that $\mathbb{R}P^2 \setminus B$ is homeomorphic to the Möbius strip M. We will construct such a homeomorphism $\mathbb{R}P^2 \setminus B \cong M$ in steps as a composite of homeomorphisms.

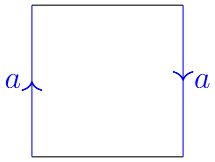


Figure 6: The Möbius strip M.

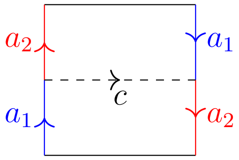


Figure 7: Relabeling.

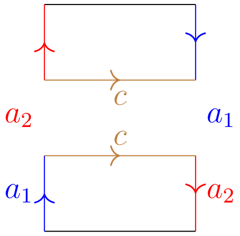


Figure 8: Cutting along c.

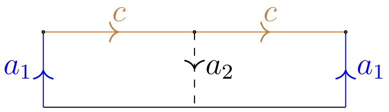


Figure 9: Pasting along a_2.

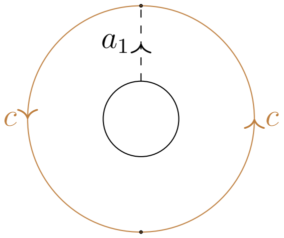


Figure 10: Pasting along a_1.

The last step yields the space $\mathbb{R}P^2 \setminus B$. $\square$

Problem 6.7. Show that the pushout $M \cup_{\partial M} M$ is homeomorphic to the Klein bottle K. This pushout is obtained by gluing two Möbius strips along their boundaries.

Solution. We will construct such a homeomorphism $M \cup_{\partial M} M \cong K$ in steps as a composite of homeomorphisms.

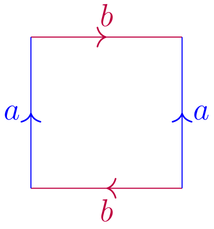


Figure 11: The Klein bottle K.

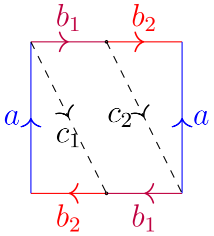


Figure 12: Relabeling.

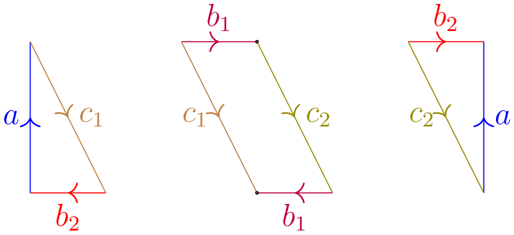


Figure 13: Cutting along c_1 and c_2.

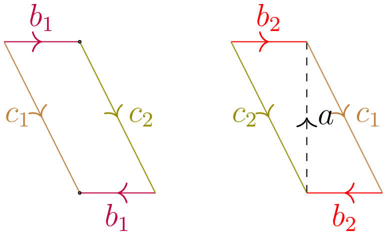


Figure 14: Pasting along a.

The two spaces in Figure 14 are homeomorphic to the Möbius strip M. Their boundaries are both parametrized by the loop $c_1 \cdot c_2$. Therefore, the displayed identification yields the space $M \cup_{\partial M} M$. □

Problem 6.8. Let M_g denote the orientable surface of genus $g \geq 0$. For instance, $M_1 \cong T^2$ is the torus. Note that its fundamental group $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ is abelian.

Show that for all $g \geq 2$, the fundamental group $\pi_1(M_g)$ is **not** abelian.

Solution. The fundamental group $\pi_1(M_g)$ has the following presentation with $2g$ generators and one relation:

$$\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

That group admits as a quotient group

$$\pi_1(M_g)/\langle b_1, \ldots, b_g \rangle_{\text{normal}} \cong \langle a_1, \ldots, a_g \mid 1 \rangle = \langle a_1, \ldots, a_g \rangle,$$

a free group F_g on g generators. For $g \geq 2$, the free group F_g is not abelian, which implies that $\pi_1(M_g)$ itself is not abelian. $\square$

7 Chain complexes and homology

This note is an introduction to chain complexes. More information can be found in [Hat02, §2.1], [May99, §12], [Wei94, §1.1], [Bre97, §IV.5], and of course Wikipedia.

7.1 Chain complexes

Definition 7.1. A **chain complex** C is a graded abelian group $\{C_n\}_{n \in \mathbb{Z}}$ together with maps $\partial \colon C_n \to C_{n-1}$ for all $n \in \mathbb{Z}$ called *boundary maps* satisfying $\partial^2 = 0$.

A chain complex C is **non-negatively graded** if it satisfies $C_n = 0$ for all $n < 0$.

A chain complex looks like this:

$$\begin{tikzcd}\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] & C_{n-1} \arrow[r] & \cdots\end{tikzcd}$$

A non-negatively graded chain complex looks like this:

$$\begin{tikzcd}\cdots \arrow[r] & C_2 \arrow[r, "\partial"] & C_1 \arrow[r, "\partial"] & C_0 \arrow[r] & 0.\end{tikzcd}$$

When working with non-negatively graded chain complexes, it is customary to stop in degree 0 and omit the group 0 in degree -1.

Definition 7.2. Let C be a chain complex.

- An element of C_n is called an ***n*-chain**.
- An element of $Z_n(C) := \ker(C_n \xrightarrow{\partial} C_{n-1})$ is called an ***n*-cycle**.
- An element of $B_n(C) := \operatorname{im}(C_{n+1} \xrightarrow{\partial} C_n)$ is called an ***n*-boundary**.
- The n^{th} **homology group** of C is the quotient

  $$H_n(C) := Z_n(C)/B_n(C),$$

  that is, cycles modulo boundaries.
- Two cycles $\alpha, \beta \in Z_n(C)$ are **homologous** if their difference $\alpha - \beta$ is a boundary, i.e., they represent the same homology class $[\alpha] = [\beta] \in H_n(C)$.

Remark 7.3. The condition $\partial^2 = 0$ is equivalent to $\operatorname{im}(\partial) \subseteq \ker(\partial)$. Thus an exact sequence is a special kind of chain complex: one with trivial homology, also called *acyclic*. The homology groups $H_*(C)$ measure the failure of exactness of the chain complex C.

Example 7.4. For an integer $n \geq 1$, consider the chain complex C displayed here:

$$\begin{tikzcd}
\text{degree} & 2 & 1 & 0 & -1 \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

The homology of C is

$$H_*(C) = \begin{cases} \mathbb{Z}/n, & * = 0 \\ 0, & * \neq 0. \end{cases}$$

Example 7.5. Consider the chain complex C displayed here:

$$\begin{tikzcd}
\text{degree} & 3 & 2 & 1 & 0 & -1 \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/4 \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r] & 0.
\end{tikzcd}$$

Note that the composite

$$2 \circ 2 = 4 \colon \mathbb{Z}/4 \to \mathbb{Z}/4$$

is 0, so that C is indeed a chain complex. The homology of C is

$$H_*(C) = \begin{cases} \mathbb{Z}/2, & * = 0 \\ 0, & * = 1 \\ \mathbb{Z}/2, & * = 2 \\ 0, & * \geq 3. \end{cases}$$

The generator of $H_0(C) \cong \mathbb{Z}/2$ is represented by the 0-cycle $1 \in \mathbb{Z}/4$. The generator of $H_2(C) \cong \mathbb{Z}/2$ is represented by the 2-cycle $2 \in \mathbb{Z}/4$.

7.2 Chain maps

Definition 7.6. Let C and D be chain complexes. A **chain map** $\varphi\colon C \to D$ is a family of group homomorphisms $\varphi_n\colon C_n \to D_n$ that commute with the boundary operators:

$$\begin{tikzcd}
C_n \arrow[r, "\varphi_n"] \arrow[d, "\partial"'] & D_n \arrow[d, "\partial"] \\
C_{n-1} \arrow[r, "\varphi_{n-1}"] & D_{n-1}
\end{tikzcd}$$

for all $n \in \mathbb{Z}$.

Notation 7.7. Let $\mathbf{Ch}(\mathbb{Z})$ denote the category of chain complexes of abelian groups and let $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ denote the full subcategory of non-negatively graded chain complexes.

For a commutative ring R, let $\mathbf{Ch}(R)$ denote the category of chain complexes of R-modules.

Exercise 7.8. Show that a chain map $\varphi\colon C \to D$ sends cycles to cycles

$$Z_n(C) \to Z_n(D)$$

and boundaries to boundaries

$$B_n(C) \to B_n(D)$$

for each degree $n \in \mathbb{Z}$. Deduce that a chain map induces a map on homology

$$H_n(\varphi)\colon H_n(C) \to H_n(D),$$

also denoted φ_*. Check that this assignment makes homology into a functor from chain complexes to abelian groups

$$H_n\colon \mathbf{Ch}(\mathbb{Z}) \to \mathbf{Ab}.$$

Exercise 7.9. Consider the chain map $\varphi\colon C \to D$ displayed in the diagram

$$\begin{tikzcd}
\text{degree} & \vdots \arrow[d] & \vdots \arrow[d] \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
1 & \mathbb{Z} \arrow[r, "1"] \arrow[d, "q"'] & \mathbb{Z} \arrow[d, "q"] \\
0 & \mathbb{Z}/4 \arrow[r, "q"] & \mathbb{Z}/2
\end{tikzcd}$$

where q denotes the various quotient maps. Here C is the left column and D is the right column.

(a) Compute the homology groups $H_*(C)$ and $H_*(D)$.

(b) Compute the map induced on homology

$$\varphi_* \colon H_n(C) \to H_n(D)$$

in all degrees n.

8 Chain homotopy

More information about chain homotopy can be found in [Hat02, §2.1], [May99, §12], [Wei94, §1.4], [Bre97, §IV.15], and of course Wikipedia.

8.1 Chain homotopy

Definition 8.1. Let C and D be chain complexes with boundary morphisms denoted d, and let $\varphi, \psi \colon C \to D$ be chain maps. A **chain homotopy** from φ to ψ is a sequence of group homomorphisms

$$h_n \colon C_n \to D_{n+1}$$

satisfying

$$\boxed{\psi - \varphi = dh + hd}.$$

That is, in degree n we have

$$\psi_n - \varphi_n = d^D_{n+1} h_n + h_{n-1} d^C_n. \tag{3}$$

The chain map φ is **null-homotopic** if it is chain homotopic to 0, i.e.:

$$\varphi = dh + hd.$$

A chain homotopy looks like this:

$$\begin{tikzcd}
\vdots \arrow[d, "d"'] & \vdots \arrow[d, "d"] \\
C_{n+1} \arrow[r, shift left, "\varphi_{n+1}"] \arrow[r, shift right, "\psi_{n+1}"'] \arrow[d, "d"'] \arrow[ur, "h_{n+1}"] & D_{n+1} \arrow[d, "d"] \\
C_n \arrow[r, shift left, "\varphi_n"] \arrow[r, shift right, "\psi_n"'] \arrow[d, "d"'] \arrow[ur, "h_n"] & D_n \arrow[d, "d"] \\
C_{n-1} \arrow[r, shift left, "\varphi_{n-1}"] \arrow[r, shift right, "\psi_{n-1}"'] \arrow[d, "d"'] \arrow[ur, "h_{n-1}"] & D_{n-1} \arrow[d, "d"] \\
\vdots \arrow[ur, "h_{n-2}"] & \vdots
\end{tikzcd}$$

Warning: The diagram does **not** commute! The chain homotopy equation (3) says that each difference $\psi_n - \varphi_n$ is built out of the two adjacent triangles forming a parallelogram.

Exercise 8.2. Show that chain homotopic maps induce the same map on homology:

$$\varphi \simeq \psi \implies H_n(\varphi) = H_n(\psi)\colon H_n(C) \to H_n(D)$$

for all $n \in \mathbb{Z}$.

Proposition 8.3. *Chain homotopy between chain maps $C \to D$:*

1. *is an equivalence relation.*
2. *is compatible with composition: Given chain maps $\varphi, \varphi'\colon C \to D$ and $\psi, \psi'\colon D \to E$, we have:*
$$\varphi \simeq \varphi' \quad \textit{and} \quad \psi \simeq \psi' \implies \psi \circ \varphi \simeq \psi' \circ \varphi'.$$
3. *is compatible with addition: Given chain maps $\varphi_1, \varphi_1', \varphi_2, \varphi_2'\colon C \to D$, we have:*
$$\varphi_1 \simeq \varphi_1' \quad \textit{and} \quad \varphi_2 \simeq \varphi_2' \implies \varphi_1 + \varphi_2 \simeq \varphi_1' + \varphi_2'.$$

Proof. Exercise. For part (3), it suffices to show that the null-homotopic chain maps $C \to D$ form a subgroup of $\mathrm{Hom}_{\mathbf{Ch}(\mathbb{Z})}(C, D)$. □

Exercise 8.4. Let $\mathbb{Z}[n]$ denote the chain complex with $\mathbb{Z}$ concentrated in degree n:

$$\begin{tikzcd}
 & \text{degree} & n+1 & n & n-1 & \\
\mathbb{Z}[n] = & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

(a) Show that a chain map from $\mathbb{Z}[n]$ picks out an n-cycle, i.e., there is a natural isomorphism of abelian groups
$$\mathrm{Hom}_{\mathbf{Ch}(\mathbb{Z})}(\mathbb{Z}[n], C) \cong Z_n(C).$$

(b) Show that two such chain maps $z, z'\colon \mathbb{Z}[n] \to C$ are chain homotopic if and only if they pick out homologous cycles, i.e., there is a natural isomorphism of abelian groups
$$\mathrm{Hom}_{\mathbf{Ch}(\mathbb{Z})}(\mathbb{Z}[n], C)/\simeq \;\cong H_n(C).$$

Here the left-hand side denotes the homotopy classes of chain maps $\mathbb{Z}[n] \to C$.

8.2 Chain homotopy equivalence

If you like homotopy equivalences between spaces, you'll love chain homotopy equivalences between chain complexes.

Definition 8.5. A chain map $\varphi\colon C \to D$ is a **chain homotopy equivalence** if there is a chain map $\psi\colon D \to C$ satisfying

$$\begin{cases} \psi \circ \varphi \simeq \mathrm{id}_C \\ \varphi \circ \psi \simeq \mathrm{id}_D. \end{cases}$$

Such a chain map $\psi\colon D \to C$ is called a **chain homotopy inverse** of φ. The chain complexes C and D are said to be **chain homotopy equivalent**, denoted $C \simeq D$.

Exercise 8.6. Show that a chain homotopy equivalence $\varphi\colon C \xrightarrow{\simeq} D$ induces isomorphisms on homology:

$$H_n(\varphi)\colon H_n(C) \xrightarrow{\cong} H_n(D)$$

for all $n \in \mathbb{Z}$.

Definition 8.7. A chain complex C is **contractible** if it is chain homotopy equivalent to the zero complex: $C \simeq 0$.

Equivalently, the identity of C is null-homotopic: $\mathrm{id}_C \simeq 0$.

Example 8.8. Recall that a chain complex is **acyclic** if it has trivial homology. Every contractible chain complex is acyclic, but the converse does not hold:

$$\begin{tikzcd}
C \simeq 0 \arrow[r, Rightarrow] & H_*(C) = 0. \arrow[l, Rightarrow, bend left, "\times"]
\end{tikzcd}$$

Example 8.9. Let $\alpha\colon A \xrightarrow{\cong} B$ be an isomorphism of abelian groups. Let C be the chain complex displayed here:

$$\begin{tikzcd}
 & \text{degree} & 2 & 1 & 0 & -1 \\
C = & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "\alpha", "\cong"'] & B \arrow[r] & 0.
\end{tikzcd}$$

Then C is contractible. A null-homotopy of id_C is given by

$$h_n = \begin{cases} \alpha^{-1}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

as displayed in this diagram:

$$\begin{tikzcd}
\text{degree} & \vdots \arrow[d] & \vdots \arrow[d] \\
2 & 0 \arrow[d] \arrow[r] \arrow[ur] & 0 \arrow[d] \\
1 & A \arrow[d, "\alpha"'] \arrow[r, "\mathrm{id}_A"] \arrow[ur] & A \arrow[d, "\alpha"] \\
0 & B \arrow[d] \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\alpha^{-1}" description] & B \arrow[d] \\
-1 & 0 \arrow[d] \arrow[r] \arrow[ur] & 0 \arrow[d] \\
 & \vdots \arrow[ur] & \vdots
\end{tikzcd}$$

Example 8.10. The chain complex:

$$\begin{tikzcd}
 & \text{degree} & 3 & 2 & 1 & 0 & -1 \\
C = & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "q", two heads] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

is acyclic but **not** contractible. Here $q\colon \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ denotes the quotient map.
Likewise, the chain complex:

$$\begin{tikzcd}
 & \text{degree} & 3 & 2 & 1 & 0 & -1 \\
C = & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r, "q", two heads] & \mathbb{Z}/n \arrow[r] & 0
\end{tikzcd}$$

is acyclic but **not** contractible.

9 Exact sequences and homology

Throughout these notes, we work with modules over a commutative ring R. In the case $R = \mathbb{Z}$, we are dealing with abelian groups, i.e., $\mathbb{Z}$-modules.

9.1 Exact sequences

Definition 9.1. A sequence of R-modules

$$\begin{tikzcd} \cdots \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] & A_n \arrow[r, "f_n"] & A_{n-1} \arrow[r] & \cdots \end{tikzcd} \tag{4}$$

is **exact at A_n** if the image of the previous map is the kernel of the next map:

$$\mathrm{im}(f_{n+1}) = \ker(f_n).$$

The sequence is called **exact** if it is exact at every position.

Remark 9.2. The inclusion $\mathrm{im}(f_{n+1}) \subseteq \ker(f_n)$ says that two consecutive maps compose to zero:

$$f_n \circ f_{n+1} = 0.$$

Thus an exact sequence is a special kind of chain complex, one whose homology is trivial: $H_*(A) = 0$.

Example 9.3. 1. The sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

is exact if and only if $A = 0$ holds.

2. The sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \end{tikzcd}$$

is exact if and only if f is a monomorphism, i.e., an injective map.

Note that exactness of the sequence means exactness at A, because that is the only position where exactness makes sense. Exactness at B is not defined since the sequence does not have a map out of B.

3. The sequence

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

is exact if and only if g is an epimorphism, i.e., a surjective map.

As before, exactness of the sequence means exactness at C, because that is the only position where exactness makes sense. Exactness at B is not defined since the sequence does not have a map into B.

4. The sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd}$$

is exact if and only if f is an isomorphism.

Warning 9.4. When saying that the sequence

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

is exact, we (and most authors) mean exact at B, because that is the only position where exactness makes sense, cf. Example 9.3.

Definition 9.5. A **short exact sequence** is an exact sequence of the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0. \end{tikzcd} \tag{5}$$

More explicitly:

- Exactness at A says that $f\colon A \hookrightarrow B$ is injective.
- Exactness at C says that $g\colon B \twoheadrightarrow C$ is surjective.
- Exactness at B says that the two maps are related by $\operatorname{im}(f) = \ker(g)$.

Remark 9.6. A sequence that extends infinitely in both directions as in Equation (4) is called a **long exact sequence**.

Example 9.7. 1. Any monomorphism of R-modules $f\colon A \hookrightarrow B$ extends to a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "f"] & B \arrow[r, two heads, "q"] & \operatorname{coker}(f) \arrow[r] & 0. \end{tikzcd}$$

Here $q\colon B \twoheadrightarrow B/\operatorname{im}(f) = \operatorname{coker}(f)$ denotes the quotient map.

2. Any epimorphism of R-modules $g\colon B \twoheadrightarrow C$ extends to a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker(g) \arrow[r, hook, "\text{inc}"] & B \arrow[r, two heads, "g"] & C \arrow[r] & 0. \end{tikzcd}$$

3. Since the maps appearing in a short exact sequence are of a special form, an arbitrary map of R-modules $f\colon A \to B$ need not appear in a short exact sequence. The next best thing is this: f extends to a 4-term exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r, hook, "\text{inc}"] & A \arrow[r, "f"] & B \arrow[r, two heads, "q"] & \operatorname{coker}(f) \arrow[r] & 0. \end{tikzcd}$$

Note that this construction generalizes the previous two parts.

9.2 The splitting lemma

Example 9.8. Given any R-modules A and C, the sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "\mathrm{inc}_A"] & A \oplus C \arrow[r, two heads, "\mathrm{proj}_C"] & C \arrow[r] & 0 \end{tikzcd} \tag{6}$$

is a short exact sequence. Here inc_A denotes the inclusion of the summand A and proj_C denotes the projection onto the factor C.

Definition 9.9. A short exact sequence (5) is called **split** if it is isomorphic to one of the form (6), i.e., there is an isomorphism $\varphi \colon B \xrightarrow{\cong} A \oplus C$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[dr, hook, "\mathrm{inc}_A"'] & B \arrow[r, "g"] \arrow[d, "\varphi", "\cong"'] & C \arrow[r] & 0. \\ & & A \oplus C \arrow[ur, two heads, "\mathrm{proj}_C"'] & & \end{tikzcd}$$

Proposition 9.10 (Splitting lemma)**.** *Given a short exact sequence* (5)*, the following conditions are equivalent.*

1. *The sequence is split.*
2. *The map $f \colon A \hookrightarrow B$ admits a retraction, i.e., a map $r \colon B \to A$ satisfying $r \circ f = \mathrm{id}_A$, as illustrated here:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[l, dashed, bend left, "r"] & C \arrow[r] & 0. \end{tikzcd}$$

3. *The map $g \colon B \twoheadrightarrow C$ admits a section, i.e., a map $s \colon C \to B$ satisfying $g \circ s = \mathrm{id}_C$, as illustrated here:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] \arrow[l, dashed, bend left, "s"] & 0. \end{tikzcd}$$

Proof. See [Hat02, §2.2.Split Exact Sequences] or [DF04, §10.5 Propositions 25, 26]. ☐

Example 9.11. Let $n \geq 2$ be an integer.

1. The sequence of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r, two heads, "q"] & \mathbb{Z}/n \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence, where $q \colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n$ denotes the quotient map.

2. The sequence of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/n \arrow[r, "n"] & \mathbb{Z}/n^2 \arrow[r, two heads, "q"] & \mathbb{Z}/n \arrow[r] & 0 \end{tikzcd}$$

is also a short exact sequence.

Neither of those two short exact sequences is split.

9.3 Exact sequences of homology groups

Proposition 9.12 (Snake lemma)**.** *Consider a morphism of short exact sequences of R-modules, i.e., a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & B_1 \arrow[r, "g_1"] \arrow[d, "\beta"] & C_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "f_0"] & B_0 \arrow[r, "g_0"] & C_0 \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. There is an induced 6*-term exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \ker(\alpha) \arrow[r, "H_1(f)"] & \ker(\beta) \arrow[r, "H_1(g)"] & \ker(\gamma) \arrow[dll, bend left, "\delta"'] & \\
& \operatorname{coker}(\alpha) \arrow[r, "H_0(f)"] & \operatorname{coker}(\beta) \arrow[r, "H_0(g)"] & \operatorname{coker}(\gamma) \arrow[r] & 0
\end{tikzcd}$$

where:

- *The maps $H_1(f)$ and $H_1(g)$ are the restrictions of f and g to the kernels;*
- *The maps $H_0(f)$ and $H_0(g)$ are the maps induced on cokernels by f_0 and g_0;*
- *The connecting homomorphism $\delta\colon \ker(\gamma) \to \operatorname{coker}(\alpha)$ is defined as follows.*

Given an element $x \in \ker(\gamma)$, pick a preimage of x under g_1, apply β, then pick a (unique) preimage under f_0, then take the equivalence class modulo $\operatorname{im}(\alpha)$. The formula is illustrated schematically here:

$$\begin{tikzcd}
& \widetilde{x} \arrow[d, mapsto, "\beta"] & x \arrow[l, dashed, "g_1^{-1}"'] \\
\overline{x} \arrow[d, mapsto, "q"] & \beta(\widetilde{x}) \arrow[l, dashed, "(f_0)^{-1}"'] & \\
q(\overline{x}) = \delta(x) \in \operatorname{coker}(\alpha). & &
\end{tikzcd}$$

Proof. Do it! It's a fun diagram chase. □

By mathematical law, I am obligated to refer you to the following explanation:

`https://www.youtube.com/watch?v=aXBNPjrvx-I`

Example 9.13. Consider the diagram of abelian groups with exact rows

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "1"] \arrow[d, "\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}^2 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

The middle map $\beta = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right] : \mathbb{Z} \to \mathbb{Z}^2$ has kernel 0 and cokernel

$$\mathbb{Z}^2 / \langle \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rangle \cong \mathbb{Z}$$

generated by the equivalence classes $[e_1] = [e_2]$. In this case, the 6-term exact sequence from the snake lemma

$$0 \longrightarrow \ker(\alpha) \longrightarrow \ker(\beta) \longrightarrow \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \longrightarrow \operatorname{coker}(\beta) \longrightarrow \operatorname{coker}(\gamma) \longrightarrow 0 \tag{7}$$

becomes

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\delta = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]} \mathbb{Z}^2 \xrightarrow{[1\ 1]} \mathbb{Z} \longrightarrow 0 \longrightarrow 0.$$

Example 9.14. Consider the diagram of abelian groups with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "4"] \arrow[d, "2"] & \mathbb{Z} \arrow[r, two heads, "q"] \arrow[d, "1"] & \mathbb{Z}/4 \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, two heads, "q"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

The left map $2 \colon \mathbb{Z} \to \mathbb{Z}$ is injective with cokernel $\mathbb{Z}/2$. The middle map $1 \colon \mathbb{Z} \to \mathbb{Z}$ is an isomorphism. The right map $q \colon \mathbb{Z}/4 \to \mathbb{Z}/2$ is surjective with kernel $\langle 2 \rangle \cong \mathbb{Z}/2$. The 6-term exact sequence (7) becomes

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\delta = 1} \mathbb{Z}/2 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0.$$

In this case, exactness of the sequence forces $\delta = 1$. Nevertheless, we can compute the connecting homomorphism δ explicitly using the formula. It is given on the generator $\overline{2} \in \ker(\mathbb{Z}/4 \xrightarrow{q} \mathbb{Z}/2)$ by $\delta(\overline{2}) = \overline{1}$, as illustrated schematically here:

$$\begin{tikzcd}
 & 2 \arrow[d, maps to, "1"] & \overline{2} \arrow[l, dashed, "q^{-1}"'] \\
1 \arrow[d, maps to, "q"] & 2 \arrow[l, dashed, "2^{-1}"'] & \\
\overline{1} = \delta(x) \in \operatorname{coker}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) = \mathbb{Z}/2. & &
\end{tikzcd}$$

Proposition 9.15. *For any short exact sequence of chain complexes*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "f"] & D \arrow[r, "g"] & E \arrow[r] & 0 \end{tikzcd}$$

there is a natural long exact sequence of homology groups

$$\begin{tikzcd} \cdots \arrow[r] & H_n(C) \arrow[r, "f_*"] & H_n(D) \arrow[r, "g_*"] & H_n(E) \arrow[r, "\delta"] & H_{n-1}(C) \arrow[r] & \cdots \end{tikzcd}$$

Proof. Similar diagram chase. See [Hat02, Theorem 2.16]. □

Remark 9.16. The snake lemma (Proposition 9.12) is a special case of Proposition 9.15, where the three chain complexes A, B, and C are concentrated in degrees 0 and 1. Indeed, the homology groups of such a chain complex A are:

$$\begin{aligned} H_1(A) &= Z_1(A)/B_1(A) = Z_1(A) = \ker(\alpha) \\ H_0(A) &= Z_0(A)/B_0(A) = A_0/\operatorname{im}(\alpha) = \operatorname{coker}(\alpha). \end{aligned}$$

The notation in Proposition 9.12 was chosen for that reason.

10 Cellular homology with coefficients

Given a CW-complex X, we know that its cellular chain complex $C_*^{\mathrm{CW}}(X)$ and singular chain complex $C_*(X)$ have isomorphic homology $H_*^{\mathrm{CW}}(X) \cong H_*(X)$. We want to generalize this statement to homology with coefficients. Along the way, we discuss some related material from homological algebra.

10.1 Direct approach

Proposition 10.1. *Let X be a CW-complex and G an abelian group. Then there is an isomorphism of homology with coefficients $H_*^{\mathrm{CW}}(X;G) \cong H_*(X;G)$. Moreover, this isomorphism is natural with respect to cellular maps $X \to Y$ and with respect to G (and all group homomorphisms).*

Proof. Recall that the isomorphism $H_n^{\mathrm{CW}}(X) \cong H_n(X)$ was obtained by showing that the two surjections illustrated in the diagram

$$\begin{tikzcd}
 & H_n(X_n) \arrow[dl, two heads] \arrow[dr, two heads] & \\
H_n^{\mathrm{CW}}(X) & & H_n(X)
\end{tikzcd}$$

have the same kernel. This was a consequence of the long exact sequences of the pairs (X_k, X_{k-1}), and the fact that the relative homology $H_*(X_k, X_{k-1})$ is concentrated in degree k. Homology with coefficients also has a (natural) long exact sequence associated to any pair, and the relative homology groups

$$\begin{aligned}
H_i(X_k, X_{k-1}; G) &\cong \widetilde{H}_i(X_k/X_{k-1}; G) \\
&\cong \widetilde{H}_i(\bigvee_{k\text{-cells}} S^k; G) \\
&\cong \bigoplus_{k\text{-cells}} \widetilde{H}_i(S^k; G) \\
&\cong \begin{cases} \bigoplus_{k\text{-cells}} G & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}
\end{aligned}$$

are also concentrated in degree k. Therefore, the proof for the case $G = \mathbb{Z}$ works here as well.

The naturality statements follow from naturality of the diagram

$$\begin{tikzcd}
 & H_n(X_n;G) \arrow[dl, two heads] \arrow[dr, two heads] & \\
H_n^{\mathrm{CW}}(X;G) & & H_n(X;G)
\end{tikzcd}$$

with respect to cellular maps $X \to Y$, and with respect to group homomorphisms $G \to G'$. $\square$

10.2 Approach using chain homotopy

Proposition 10.2. *Let C_* be a (possibly unbounded) chain complex of* free *abelian groups. Then C_* is quasi-isomorphic to its homology, in fact via a quasi-isomorphism $C_* \xrightarrow{\sim} H_*(C_*)$ (as opposed to a zig-zag).*

Proof. Consider[1] the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "d"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

which is split, since B_{n-1} is a free abelian group, being a subgroup of the free abelian group C_{n-1}. Choosing a splitting $C_n \cong Z_n \oplus B_{n-1}$ for each $n \in \mathbb{Z}$, the chain complex C_* is isomorphic (though not naturally) to the chain complex illustrated here:

$$\begin{tikzcd}
\vdots \arrow[d] & \text{degree} \\
Z_{n+1} \oplus B_n \arrow[d] & n+1 \\
Z_n \oplus B_{n-1} \arrow[d] & n \\
Z_{n-1} \oplus B_{n-2} \arrow[d] & n-1 \\
\vdots &
\end{tikzcd}$$

where the differential d_n is given by the inclusion $B_{n-1} \hookrightarrow Z_{n-1}$. Hence, there is an isomorphism of chain complexes $C_* \cong \bigoplus_{n\in\mathbb{Z}} C_*^{(n)}$ where $C_*^{(n)}$ denotes the tiny chain complex

$$\begin{tikzcd}
0 \arrow[d] & \\
B_n \arrow[d, hook] & n+1 \\
Z_n \arrow[d] & n \\
0 &
\end{tikzcd}$$

---

[1] Credit to Tyler Lawson for this explanation:
`http://mathoverflow.net/questions/10974/does-homology-detect-chain-homotopy-equivalence`

concentrated in degrees n and $n+1$. Consider $H_n(C_*)$ as a chain complex concentrated in degree n. The map $\varphi_n \colon C_*^{(n)} \to H_n(C_*)$ given by the quotient map $Z_n \twoheadrightarrow H_n(C_*) = Z_n/B_n$ in degree n is a chain map which is moreover a quasi-isomorphism. These maps assemble into a quasi-isomorphism

$$\bigoplus_{n\in\mathbb{Z}} \varphi_n \colon \bigoplus_{n\in\mathbb{Z}} C_*^{(n)} \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} H_n(C_*) = H_*(C_*).$$

as claimed. ☐

Recall the following fact from homological algebra.

Theorem 10.3 (Comparison theorem for projective resolutions)**.** *Let $\mathcal{A}$ be an abelian category, and let M be an object of $\mathcal{A}$, viewed as a chain complex concentrated in degree* 0*. Let P_* be a (non-negatively graded) chain complex of projective objects, with a chain map $f \colon P_* \to M$, and let D_* a (non-negatively graded) chain complex with a quasi-isomorphism $w \colon D_* \xrightarrow{\sim} M$. Then f admits a lift as in the diagram*

$$\begin{tikzcd}
 & D_* \arrow[d, "w", "\sim"'] \\
P_* \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & M
\end{tikzcd}$$

which is unique up to chain homotopy.

Proof. [Wei94, Theorem 2.2.6]. ☐

Example 10.4. In the category $\mathcal{A} = \mathbf{Ab}$ of abelian groups, an object is projective if and only if it is a free abelian group.

Proposition 10.5. *Let C_* and D_* be (possibly unbounded) chain complexes of* free *abelian groups.*

1. *If C_* and D_* have isomorphic homology $H_*(C_*) \cong H_*(D_*)$, then they are chain homotopy equivalent: $C_* \simeq D_*$.*
2. *If $f \colon C_* \xrightarrow{\sim} D_*$ is a quasi-isomorphism, then f is a chain homotopy equivalence.*

Proof. 1. Consider decompositions $C_* \cong \bigoplus_{n\in\mathbb{Z}} C_*^{(n)}$ and $D_* \cong \bigoplus_{n\in\mathbb{Z}} D_*^{(n)}$ as in the proof of Proposition 10.2. For each $n \in \mathbb{Z}$, consider the diagram of chain complexes

$$\begin{tikzcd}
 & D_*^{(n)} \arrow[d, "\psi_n", "\sim"'] \\
C_*^{(n)} \arrow[ur, dashed, "\widetilde{\varphi_n}"] \arrow[r, "\sim", "\varphi_n"'] & H_n(C_*) \cong H_n(D_*)
\end{tikzcd}$$

where a lift $\widetilde{\varphi_n} \colon C_*^{(n)} \to D_*^{(n)}$ exists, by Theorem 10.3. Reversing the roles of C_* and D_*, there also exists a lift $\widetilde{\psi_n} \colon D_*^{(n)} \to C_*^{(n)}$. Uniqueness of lifts up to chain homotopy shows that $\widetilde{\psi_n}$ is chain homotopy inverse to $\widetilde{\varphi_n}$. Therefore, the chain map

$$\bigoplus_{n \in \mathbb{Z}} \widetilde{\varphi_n} \colon \bigoplus_{n \in \mathbb{Z}} C_*^{(n)} \xrightarrow{\simeq} \bigoplus_{n \in \mathbb{Z}} D_*^{(n)}$$

is a chain homotopy equivalence, with chain homotopy inverse $\bigoplus_{n \in \mathbb{Z}} \widetilde{\psi_n}$.

2. For each $n \in \mathbb{Z}$, consider the diagram of chain complexes

$$\begin{tikzcd}
D_*^{(n)} \arrow[r, dashed, "\psi_n"] \arrow[d, "\sim"'] & C_*^{(n)} \arrow[d, "\sim"] \\
H_n(D_*) \arrow[r, "H_n(f)^{-1}", "\cong"'] & H_n(C_*)
\end{tikzcd}$$

where there exists a lift $\psi_n \colon D_*^{(n)} \to C_*^{(n)}$ (unique up to chain homotopy), by Theorem 10.3. These chain maps define a chain map $\psi \colon D_* \to C_*$ via the diagram

$$\begin{tikzcd}
\bigoplus_{n \in \mathbb{Z}} D_*^{(n)} \arrow[r, "\bigoplus_{n \in \mathbb{Z}} \psi_n"] \arrow[d, dash, "\cong"'] & \bigoplus_{n \in \mathbb{Z}} C_*^{(n)} \arrow[d, dash, "\cong"] \\
D_* \arrow[r, "\psi"] & C_*.
\end{tikzcd}$$

One readily checks that the restriction $f|_{C_*^{(n)}} \colon C_*^{(n)} \to D_*$ is chain homotopic to the composite

$$\begin{tikzcd}
C_*^{(n)} \arrow[r, "f|_{C_*^{(n)}}"] & D_* \arrow[r, two heads, "\text{proj}"] & D_*^{(n)} \arrow[r, hook, "\text{inc}"] & D_*
\end{tikzcd}$$

and that $\psi \colon D_* \to C_*$ is chain homotopy inverse to $f \colon C_* \to D_*$. □

Proposition 10.6. *The relation of chain homotopy is compatible with the tensor product of chain complexes. In other words, if the chain maps* $\varphi, \psi \colon C_* \to D_*$ *are chain homotopic and* $\varphi', \psi' \colon C'_* \to D'_*$ *are chain homotopic, then the chain maps*

$$\varphi \otimes \varphi', \psi \otimes \psi' \colon C_* \otimes C'_* \to D_* \otimes D'_*$$

are chain homotopic.

Proof. Using the factorizations illustrated in the diagram

$$\begin{tikzcd}
 & D_* \otimes C'_* \arrow[dr, "\mathrm{id}_{D_*} \otimes \varphi'"] & \\
C_* \otimes C'_* \arrow[ur, "\varphi \otimes \mathrm{id}_{C'_*}"] \arrow[rr, "\varphi \otimes \varphi'"] \arrow[dr, "\mathrm{id}_{C_*} \otimes \varphi'"'] & & D_* \otimes D'_* \\
 & C_* \otimes D'_* \arrow[ur, "\varphi \otimes \mathrm{id}_{D'_*}"'] &
\end{tikzcd}$$

it suffices to show that $\varphi \otimes \mathrm{id}_{C'_*}$ is chain homotopic to $\psi \otimes \mathrm{id}_{C'_*}$. Let $h \colon C_n \to D_{n+1}$ be a chain homotopy from φ to ψ, i.e., such that the equation $\psi - \varphi = dh + hd$ holds.

Let us check that $h \otimes \mathrm{id}_{C'_*} \colon (C_* \otimes C'_*)_n \to (D_* \otimes C'_*)_{n+1}$ is a chain homotopy from $\varphi \otimes \mathrm{id}_{C'_*}$ to $\psi \otimes \mathrm{id}_{C'_*}$. For any $x_i \in C_i$ and $x'_j \in C'_j$, with $i + j = n$, we have

$$\begin{aligned}
&d(h \otimes \mathrm{id}_{C'_*})(x_i \otimes x'_j) + (h \otimes \mathrm{id}_{C'_*})d(x_i \otimes x'_j) \\
&\qquad = d\left(hx_i \otimes x'_j\right) + (h \otimes \mathrm{id}_{C'_*})\left(dx_i \otimes x'_j + (-1)^{|x_i|} x_i \otimes dx'_j\right) \\
&\qquad = dhx_i \otimes x'_j + (-1)^{|hx_i|} hx_i \otimes dx'_j + hdx_i \otimes x'_j + (-1)^{|x_i|} hx_i \otimes dx'_j \\
&\qquad = dhx_i \otimes x'_j + (-1)^{i+1} hx_i \otimes dx'_j + hdx_i \otimes x'_j + (-1)^{i} hx_i \otimes dx'_j \\
&\qquad = dhx_i \otimes x'_j + hdx_i \otimes x'_j \\
&\qquad = (dh + hd)x_i \otimes x'_j \\
&\qquad = (\psi - \varphi)x_i \otimes x'_j \\
&\qquad = \psi x_i \otimes x'_j - \varphi x_i \otimes x'_j.
\end{aligned}$$

Therefore the equation

$$d(h \otimes \mathrm{id}_{C'_*}) + (h \otimes \mathrm{id}_{C'_*})d = \psi \otimes \mathrm{id}_{C'_*} - \varphi \otimes \mathrm{id}_{C'_*}$$

holds. □

Corollary 10.7. *If $\varphi \colon C_* \xrightarrow{\simeq} D_*$ and $\varphi' \colon C'_* \xrightarrow{\simeq} D'_*$ are chain homotopy equivalences, then their tensor product*

$$\varphi \otimes \varphi' \colon C_* \otimes C'_* \xrightarrow{\simeq} D_* \otimes D'_*$$

is a chain homotopy equivalence.

Proof. Let $\alpha \colon D_* \to C_*$ and $\alpha' \colon D'_* \to C'_*$ be chain homotopy inverses of φ and φ' respectively. Then

$$\alpha \otimes \alpha' \colon D_* \otimes D'_* \to C_* \otimes C'_*$$

is a chain homotopy inverse of $\varphi \otimes \varphi'$. □

The following proposition says that "any chain complex of free abelian groups will do", as long as it has the correct homology (with coefficients in $\mathbb{Z}$).

Proposition 10.8. *Let X be a space and C_* a chain complex of free abelian groups whose homology is isomorphic to the singular homology of X, i.e., $H_n(C_*) \cong H_n(X)$ holds for all n. Then for any abelian group G and any n, there are isomorphisms $H_n(C_* \otimes G) \cong H_n(X; G)$.*

Proof. The assumption is that the homology C_* is isomorphic to the homology of the singular chain complex $C_*(X)$. By Proposition 10.5, there is a chain homotopy equivalence $\varphi \colon C_* \xrightarrow{\simeq} C_*(X)$. By Corollary 10.7, the chain map

$$\varphi \otimes \mathrm{id}_G \colon C_* \otimes G \xrightarrow{\simeq} C_*(X) \otimes G$$

is a chain homotopy equivalence, in particular a quasi-isomorphism. □

Example 10.9. Let X be a Δ-complex, and $C_*^{\Delta}(X)$ the associated simplicial chain complex. Then there are isomorphisms $H_n^{\Delta}(X; G) \cong H_n(X; G)$. Naturality with respect to Δ-maps $X \to Y$ does not follow directly from the first part of Proposition 10.8.

However, recall that the isomorphism $H_n^{\Delta}(X) \cong H_n(X)$ is induced at the chain level by a quasi-isomorphism $\theta \colon C_*^{\Delta}(X) \xrightarrow{\sim} C_*(X)$, which is natural with respect to Δ-maps $X \to Y$. By the second part of Proposition 10.5, θ is in fact a chain homotopy equivalence. By Corollary 10.7, the chain map $\theta \otimes \mathrm{id}_G \colon C_*^{\Delta}(X) \otimes G \xrightarrow{\simeq} C_*(X) \otimes G$ is also a chain homotopy equivalence, and in particular induces isomorphisms $H_n^{\Delta}(X; G) \cong H_n(X; G)$. These isomorphisms are natural with respect to Δ-maps $X \to Y$, since the chain map θ is.

Example 10.10. Let X be a CW-complex, and $C_*^{\mathrm{CW}}(X)$ the associated cellular chain complex. Then there are isomorphisms $H_n^{\mathrm{CW}}(X; G) \cong H_n(X; G)$. Naturality with respect to cellular maps $X \to Y$ does not follow from Proposition 10.8.

10.3 Approach using the universal coefficient theorem

Recall the following fact.

Theorem 10.11 (Universal coefficient theorem)**.** *Let C_* be a chain complex of* free *abelian groups, and G an abelian group. Then for each $n \in \mathbb{Z}$, there is a short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & H_n(C_*) \otimes G \arrow[r, "\times"] & H_n(C_* \otimes G) \arrow[r] & \mathrm{Tor}\left(H_{n-1}(C_*), G\right) \arrow[r] & 0 \end{tikzcd}$$

which is natural in C_* and G. Moreover, the sequence is split, though the splitting is not natural.

Here, the map $\times \colon H_n(C_*) \otimes G \to H_n(C_* \otimes G)$ sends $[\alpha] \otimes g$ to the homology class $[\alpha \otimes g]$. The functor Tor denotes $\mathrm{Tor}_1^{\mathbb{Z}}$, just like our tensor product $\otimes$ denotes the tensor product $\otimes_{\mathbb{Z}}$ over the integers.

Corollary 10.12. *If $f \colon C_* \xrightarrow{\sim} D_*$ is a quasi-isomorphism between chain complexes of* free *abelian groups, then the map $f \otimes \mathrm{id}_G \colon C_* \otimes G \to D_* \otimes G$ is a quasi-isomorphism.*

Proof. By the universal coefficient theorem, for each $n \in \mathbb{Z}$, f induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_n(C_*) \otimes G \arrow[r, "\times"] \arrow[d, "H_n(f)\otimes G"', "\cong"] & H_n(C_* \otimes G) \arrow[r] \arrow[d, "H_n(f\otimes \mathrm{id}_G)"', "\therefore \cong"] & \mathrm{Tor}\left(H_{n-1}(C_*), G\right) \arrow[r] \arrow[d, "\mathrm{Tor}(H_{n-1}(f), \mathrm{id}_G)"', "\cong"] & 0 \\ 0 \arrow[r] & H_n(D_*) \otimes G \arrow[r, "\times"] & H_n(D_* \otimes G) \arrow[r] & \mathrm{Tor}\left(H_{n-1}(D_*), G\right) \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact. By assumption, $H_n(f)$ and $H_{n-1}(f)$ are isomorphisms, thus so are the downward maps $H_n(f) \otimes G$ and $\mathrm{Tor}\left(H_{n-1}(f), \mathrm{id}_G\right)$. By the 5-lemma, the downward map in the middle $H_n(f \otimes \mathrm{id}_G)$ is also an isomorphism. ☐

Example 10.13. Corollary 10.12 provides an alternate proof that the chain map

$$\theta \otimes \mathrm{id}_G \colon C_*^{\Delta}(X) \otimes G \xrightarrow{\sim} C_*(X) \otimes G$$

is a quasi-isomorphism, as discussed in Example 10.9.

What if an isomorphism in homology does not come from a chain map, as in the cellular homology theorem? Then we can still argue as follows.

Proposition 10.14. *If two (possibly unbounded) chain complexes of* free *abelian groups C_* and D_* have isomorphic homology $H_*(C_*) \cong H_*(D_*)$, then the chain complexes $C_* \otimes G$ and $D_* \otimes G$ have isomorphic homology.*

Proof. Using the splitting in the universal coefficient theorem, we have (non-natural) isomorphisms:

$$\begin{aligned} H_n(C_* \otimes G) &\cong H_n(C_*) \otimes G \oplus \mathrm{Tor}\left(H_{n-1}(C_*), G\right) \\ &\cong H_n(D_*) \otimes G \oplus \mathrm{Tor}\left(H_{n-1}(D_*), G\right) \\ &\cong H_n(D_* \otimes G). \end{aligned}$$

☐

Alternate proof. By Proposition 10.5, there exists a quasi-isomorphism $\varphi\colon C_* \xrightarrow{\sim} D_*$ (which is in fact a chain homotopy equivalence). By Corollary 10.12, the chain map $\varphi \otimes \mathrm{id}_G\colon C_* \otimes G \xrightarrow{\sim} D_* \otimes G$ is also a quasi-isomorphism. ☐

Remark 10.15. Section 10.3 is essentially doing the same thing as Section 10.2, from a more computational perspective. A key step for proving the universal coefficient theorem is to choose splittings of the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "d"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

like we did in the proof of Proposition 10.2.

A Categories and functors

These notes describe some basic category theory, focusing on examples of interest for algebraic topology. Some good general references on category theory include [ML98], [Rie17], and [Lei14].

A.1 Categories

Definition A.1. A **category** $\mathcal{C}$ consists of the following data:

1. A collection of **objects** $\mathrm{Ob}(\mathcal{C})$.
2. For all objects X and Y in $\mathcal{C}$, a set $\mathcal{C}(X, Y)$ of **morphisms** from X to Y.
3. For all objects X, Y, and Z in $\mathcal{C}$, a **composition** operation

$$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \xrightarrow{\circ} \mathcal{C}(X, Z).$$

   The composition of a pair of morphisms (g, f) is denoted $g \circ f$.

4. For all object X in $\mathcal{C}$, an **identity** morphism $1_X \in \mathcal{C}(X, X)$.

The following conditions are required to hold:

- Composition is associative, that is, for all morphisms $f \in \mathcal{C}(W, X)$, $g \in \mathcal{C}(X, Y)$, and $h \in \mathcal{C}(Y, Z)$, the equation
$$(h \circ g) \circ f = h \circ (g \circ f)$$
holds in $\mathcal{C}(W, Z)$.
- Identity morphisms are two-sided units for composition, that is, for all $f \in \mathcal{C}(X, Y)$, the equations
$$1_Y \circ f = f = f \circ 1_X$$
hold in $\mathcal{C}(X, Y)$.

Remark A.2. Here are some remarks about the terminology and notation.

1. The set of morphisms $\mathcal{C}(X, Y)$ is also denoted $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and called the **hom-set** from X to Y.
2. The identity morphism of X is also denoted id_X.
3. The composition symbol $\circ$ is sometimes omitted, writing gf instead of $g \circ f$.
4. A morphism $f \in \mathcal{C}(X, Y)$ is often denoted $f \colon X \to Y$ and sometimes called an *arrow* or a *map*. The object X is called the **source** (or *domain*) of f and Y is called the **target** (or *codomain*) of f.

5. The composition $gf\colon X \to Z$ of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is also displayed in a diagram:
$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z. \end{tikzcd}$$

6. A diagram **commutes** if all compositions with the same source and target are equal. For example, the following diagram commutes if and only if the equation $\varphi = gf$ holds:
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "\varphi"'] & & Z. \end{tikzcd}$$
The following diagram commutes if and only if the equation $gi = jf$ holds:
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "g"'] & Y. \end{tikzcd}$$

7. Diagrams are understood to be commutative unless otherwise noted. Drawing a non-commutative diagram without explanation is misleading.

Example A.3. 1. **Set** denotes the category of sets and functions $f\colon X \to Y$. Composition is the usual composition of functions, namely $(g \circ f)(x) = g(f(x))$. The identity $1_X\colon X \to X$ is the usual identity function, $1_X(x) = x$.

2. **Gp** denotes the category of groups and group homomorphisms $f\colon G \to H$. Here again, composition is the usual composition.

3. **Ab** denotes the category of abelian groups and group homomorphisms $f\colon A \to B$.

4. Let k be a field. $\mathbf{Vect}_k$ denotes the category of k-vector spaces and k-linear maps $f\colon V \to W$.

5. **Top** denotes the category of topological spaces and continuous maps $f\colon X \to Y$.

6. $\mathbf{Set}_*$ denotes the category of pointed sets and pointed functions $f\colon (X, x_0) \to (Y, y_0)$, that is, functions $f\colon X \to Y$ satisfying $f(x_0) = y_0$.

7. $\mathbf{Top}_*$ denotes the category of pointed topological spaces and pointed continuous maps $f\colon (X, x_0) \to (Y, y_0)$.

8. Ho(**Top**) denotes the homotopy category[2] of spaces. Objects are topological spaces, and morphisms are homotopy classes $[f]\colon X \to Y$ of continuous maps.

---

[2] This category Ho(**Top**) is sometimes called the *naive* homotopy category of spaces. In the literature, the term "homotopy category" of spaces sometimes refers to the category of CW complexes and homotopy classes of continuous maps between them.

9. Ho($\mathbf{Top}_*$) denotes the homotopy category of pointed spaces. Objects are pointed topological spaces, and morphisms are pointed homotopy classes $[f]\colon (X, x_0) \to (Y, y_0)$ of pointed continuous maps.

Definition A.4. A category is called **small** if the collection of objects is a set.

Remark A.5.

1. None of the categories listed in Example A.3 are small. For instance, the category **Set** is not small, since there is no "set of all sets", but rather a proper class Ob(**Set**) of all sets.
2. Some authors use a more general definition of category that allows for a proper class of morphisms $\mathcal{C}(X, Y)$, calling a category *locally small* if all the collections $\mathcal{C}(X, Y)$ are sets.

A.2 Isomorphisms

Definition A.6. A morphism $f\colon X \to Y$ is an **isomorphism** if it is invertible, i.e., there exists a morphism $g\colon Y \to X$ satisfying $gf = 1_X$ and $fg = 1_Y$.

Such a morphism g is called the **inverse** of f and is denoted $g = f^{-1}$. The fact that f is an isomorphism is sometimes denoted $f\colon X \xrightarrow{\cong} Y$.

Objects X and Y are **isomorphic** if there exists an isomorphism between them. This is often denoted $X \cong Y$.

Example A.7.

1. In **Set**, the isomorphisms are the bijective functions.
2. Likewise in **Gp**, **Ab**, $\mathbf{Vect}_k$, and $\mathbf{Set}_*$, a morphism $f\colon X \to Y$ is an isomorphism if and only if it is bijective. The inverse $f^{-1}\colon Y \to X$ is then the inverse function, which automatically preserves the given algebraic structure.
3. In **Top**, the isomorphisms are the homeomorphisms (by definition). Those are the continuous bijections $f\colon X \to Y$ whose inverse $f^{-1}\colon Y \to X$ is also bijective. *Not* every continuous bijection is a homeomorphism.
4. In Ho(**Top**), the isomorphisms are the homotopy equivalences (more precisely, the homotopy classes of homotopy equivalences). Two spaces X and Y are isomorphic in Ho(**Top**) if and only if they are homotopy equivalent.

Exercise A.8. Let $f\colon X \to Y$ be a morphism.

1. Let $g\colon Y \to X$ be a left inverse of f and $h\colon Y \to X$ a right inverse of f, i.e., the equations $gf = 1_X$ and $fh = 1_Y$ hold. Show that $g = h$ holds.

   This shows in particular that the inverse f^{-1} (if it exists) is uniquely determined by f.

2. Let $g, h\colon Y \to X$ be morphisms such that $gf\colon X \to X$ and $fh\colon Y \to Y$ are isomorphisms. Show that f is an isomorphism.

Definition A.9. A **groupoid** is a category in which every morphism is an isomorphism.

Example A.10. A groupoid with one object is the same as a group.

Example A.11. The **fundamental groupoid** of a topological space X is the category $\Pi_1(X)$ defined as follows. The objects of $\Pi_1(X)$ are the elements of X, i.e., the points in the space. For $x, y \in X$, the morphisms in $\Pi_1(X)$ from x to y are the homotopy classes $[\gamma]$ of paths from x to y, that is, $\gamma\colon I \to X$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$.

Composition in $\Pi_1(X)$ is the concatenation of paths $[\alpha] \bullet [\beta] = [\alpha \bullet \beta]$. The identity of $x \in X$ is the homotopy class $[c_x]$ of the constant path $c_x\colon I \to X$.

The fundamental group $\pi_1(X, x_0)$ based at a point $x_0 \in X$ is the automorphism group

$$\pi_1(X, x_0) = \Pi_1(X)(x_0, x_0).$$

Invertibility of a morphism was defined by two conditions, which we now treat separately.

Definition A.12. Let $i\colon A \to X$ and $r\colon X \to A$ be morphisms satisfying $ri = 1_A$, as illustrated in this diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "1_A"'] & X \arrow[r, "r"] & A. \end{tikzcd}$$

Then r is called a **retraction** of i, and i is called a **section** of r. The object A is called a **retract** of X.

In other words, a section is a morphism that has a left inverse. A section is also called a **split monomorphism**. A retraction is a morphism that has a right inverse. A retraction is also called a **split epimorphism**.

Example A.13. In **Set**, a function $f\colon X \to Y$ admits a section if and only if f is surjective. This statement is equivalent to the axiom of choice.

A function $f\colon X \to Y$ admits a retraction if and only if f is injective. (This statement does *not* rely on the axiom of choice.)

Example A.14. In **Ab** (and in **Gp**), every split epimorphism is surjective, but not every surjective morphism $f\colon A \twoheadrightarrow B$ admits a section. For instance:

- The quotient map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n$ admits no section, since the only map from $\mathbb{Z}/n$ to $\mathbb{Z}$ is the trivial map $0\colon \mathbb{Z}/n \to \mathbb{Z}$.
- The quotient map $\mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ does not admit a section.

Likewise, every split monomorphism is injective, but not every injective morphism $f\colon A \hookrightarrow B$ admits a retraction. For instance:

- The map $n\colon \mathbb{Z} \hookrightarrow \mathbb{Z}$ (multiplication by n) admits no retraction.
- The non-trivial map $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ (multiplication by 2) admits no retraction.

Example A.15. In $\mathbf{Vect}_k$, every surjective morphism $f\colon V \twoheadrightarrow W$ admits a section. Every injective morphism $f\colon V \hookrightarrow W$ admits a retraction.

Example A.16. In **Top**, every split epimorphism is surjective, but not every surjective map $f\colon X \twoheadrightarrow Y$ admits a section. For instance:

1. The quotient map $q\colon D^n \twoheadrightarrow D^n/\partial D^n \cong S^n$ admits no section.
2. The winding map $p\colon \mathbb{R} \twoheadrightarrow S^1$ admits no section. Recall that p is defined by the formula:
$$p(t) = (\cos(2\pi t), \sin(2\pi t)) \in S^1.$$
3. The continuous bijection $p|_{[0,1)}\colon [0,1) \to S^1$ admits no section, since a section could only be the inverse function $S^1 \to [0,1)$, which is not continuous.

Example A.17. In **Top**, every split monomorphism $i\colon A \hookrightarrow X$ is an embedding, but not every embedding admits a retraction. For instance:

1. The boundary of the disk $S^{n-1} = \partial D^n \hookrightarrow D^n$ is not a retract of D^n.
2. The boundary of the Möbius strip $\partial M \hookrightarrow M$ is not a retract of M.
3. The 1-skeleton of the torus $S^1 \vee S^1 \hookrightarrow T^2$ is not a retract of T^2.

Example A.18. Recall that a retract of a contractible space is contractible (§0 Exercise 9). Also, every finite CW complex can be embedded into Euclidean space [Hat02, Corollary A.10]. Assuming that the CW complex X is not contractible, such an embedding $i\colon X \hookrightarrow \mathbb{R}^N$ does not admit a retraction. How sad would topology be if all finite CW complexes were contractible.

Example A.19. Here are examples of retracts in **Top**.

1. Let X be a space and $x_0 \in X$. Then the point $\{x_0\} \hookrightarrow X$ is a retract of X, with retraction (necessarily) the constant map $X \to \{x_0\}$.
2. Let X and Y be (non-empty) spaces. We can view X and Y as retracts of the product $X \times Y$. If we pick a point $y_0 \in Y$, the map

   $$X \xrightarrow{(1_X, y_0)} X \times Y$$

   admits a retraction, for instance the projection $p_X\colon X \times Y \to X$.
3. If X and Y are pointed, then we can also view X and Y as retracts of the wedge sum $X \vee Y$. The inclusion $X \hookrightarrow X \vee Y$ admits a retraction, for instance the pointed map $r_X\colon X \vee Y \twoheadrightarrow X$ with restrictions

   $$\begin{cases} r_X|_X = 1_X \\ r_X|_Y = x_0, \end{cases}$$

   where $x_0 \in X$ denotes the basepoint.

A.3 Functors

Definition A.20. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

1. A function $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
2. For all objects X and Y in $\mathcal{C}$, a function
$$F\colon \mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y)).$$
In other words, the functor F assigns to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$.

The following conditions are required to hold:

- F preserves composition. That is, for all morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$, the equation
$$F(g \circ f) = F(g) \circ F(f)$$
holds in $\mathcal{D}(F(X),F(Z))$.
- F preserves identity morphisms. That is, for all object X in $\mathcal{C}$, the equation
$$F(1_X) = 1_{F(X)}$$
holds in $\mathcal{D}(F(X),F(X))$.

Example A.21.

1. To a topological space $(X,\mathcal{T})$, we can assign the underlying set X. This yields the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$, which forgets the topology.
2. We can endow a set X with the discrete topology, where every subset $A \subseteq X$ is open. This yields a functor $\mathrm{Dis}\colon \mathbf{Set} \to \mathbf{Top}$.
3. Likewise, we can endow a set X with the *indiscrete* topology, where $\emptyset$ and X are the only open subsets. This yields another functor $\mathrm{Ind}\colon \mathbf{Set} \to \mathbf{Top}$.
4. To a topological space X, we can assign the set $\mathrm{Conn}(X)$ of its connected components. This yields a functor $\mathrm{Conn}\colon \mathbf{Top} \to \mathbf{Set}$; see the notes "Connected components as a homotopy invariant" under January 18.
5. Likewise, we can assign to X the set $\pi_0(X)$ of its path components. This yields a functor $\pi_0\colon \mathbf{Top} \to \mathbf{Set}$; see Homework 1 Problem 3.
6. To a pointed set (X,x_0), we can assign its underlying set X. This yields the forgetful functor $U\colon \mathbf{Set}_* \to \mathbf{Set}$, which forgets the basepoint. The forgetful functor $U\colon \mathbf{Top}_* \to \mathbf{Top}$ is defined similarly.

7. For a space X, consider the pointed space $X_+ := X \amalg \{*\}$, with a disjoint basepoint $* \in X_+$. This yields a functor $(-)_+ \colon \mathbf{Top} \to \mathbf{Top}_*$. The functor $(-)_+ \colon \mathbf{Set} \to \mathbf{Set}_*$ is defined similarly.

8. The fundamental group yields a functor $\pi_1 \colon \mathbf{Top}_* \to \mathbf{Gp}$, which assigns to a pointed space (X, x_0) its fundamental group $\pi_1(X, x_0)$ based at x_0.

9. The functor $\mathrm{Cyl} \colon \mathbf{Top} \to \mathbf{Top}$ assigns to a space X the *cylinder* on X, defined by $\mathrm{Cyl}(X) := X \times I$.

10. By construction of the homotopy category, there is a quotient functor

$$q \colon \mathbf{Top} \to \mathrm{Ho}(\mathbf{Top}),$$

which identifies homotopic maps. More precisely, for all space X, we have $q(X) = X$, and for all spaces X and Y, the function

$$\mathbf{Top}(X, Y) \xrightarrow{q} \mathbf{Top}(X, Y)/\simeq_{\text{homotopy}} = \mathrm{Ho}(\mathbf{Top})(X, Y)$$

is the canonical quotient function.

Remark A.22. In most of the examples, we only described the effect of the functor on objects. The effect on morphisms is straightforward.

Remark A.23. The homotopy invariance of the functors $\mathrm{Conn} \colon \mathbf{Top} \to \mathbf{Set}$, $\pi_0 \colon \mathbf{Top} \to \mathbf{Set}$, and $\pi_1 \colon \mathbf{Top}_* \to \mathbf{Gp}$ can be interpreted as follows. Each of these functors induces a functor from the respective homotopy category:

$$\begin{aligned} \mathrm{Conn} \colon & \mathrm{Ho}(\mathbf{Top}) \to \mathbf{Set} \\ \pi_0 \colon & \mathrm{Ho}(\mathbf{Top}) \to \mathbf{Set} \\ \pi_1 \colon & \mathrm{Ho}(\mathbf{Top}_*) \to \mathbf{Gp}. \end{aligned}$$

In contrast, the forgetful functor $U \colon \mathbf{Top} \to \mathbf{Set}$ is *not* homotopy invariant and hence does not induce a functor $\mathrm{Ho}(\mathbf{Top}) \to \mathbf{Set}$. The underlying function of a homotopy class $[f] \colon X \to Y$ is not well-defined!

Here are a few examples from algebra.

Example A.24. 1. To a vector space $(V, +, \cdot)$ over a field k, one can assign the underlying abelian group $(V, +)$. This yields a forgetful functor $U \colon \mathbf{Vect}_k \to \mathbf{Ab}$, which forgets the scalar multiplication.

2. An abelian group A can be viewed in particular as a group. This defines a functor $\iota \colon \mathbf{Ab} \to \mathbf{Gp}$, the inclusion of a *subcategory.*

3. Let G be a group and $[G, G] \leq G$ the subgroup generated by commutators. The quotient group $G_{\mathrm{ab}} := G/[G, G]$ is called the **abelianization** of G. This yields a functor $(-)_{\mathrm{ab}} \colon \mathbf{Gp} \to \mathbf{Ab}$.

4. The free group $F^{\text{gp}}(S)$ on a set S yields a functor $F^{\text{gp}} \colon \mathbf{Set} \to \mathbf{Gp}$.

5. Similarly, the free abelian group on a set S

$$F^{\text{ab}}(S) \cong \bigoplus_{s \in S} \mathbb{Z}$$

yields a functor $F^{\text{ab}} \colon \mathbf{Set} \to \mathbf{Ab}$.

Exercise A.25. Let S be a set. Show that there is a natural isomorphism

$$\left(F^{\text{gp}}(S)\right)_{\text{ab}} \cong F^{\text{ab}}(S).$$

In other words, the abelianization of the free group on S is the free abelian group on S.

Exercise A.26. Show that any functor preserves isomorphisms, sections, and retractions. That is, if $f \colon X \to Y$ is an isomorphism (resp. a section, a retraction), then so is $F(f) \colon F(X) \to F(Y)$.

In particular, if A is a retract of X, then $F(A)$ is a retract of $F(X)$.

We have already used that trick. Using the functor $\pi_1 \colon \mathbf{Top}_* \to \mathbf{Gp}$, we showed that the boundary $\partial D^2 = S^1$ of the disk is not a retract of D^2, and that the boundary ∂M of the Möbius strip is not a retract of M.

B Products and coproducts

B.1 Products

Definition B.1. Let X and Y be objects of a category $\mathcal{C}$. A **product** of X and Y is an object P together with morphisms $p_X \colon P \to X$ and $p_Y \colon P \to Y$ that satisfies the following (universal) property: For all object W with morphisms $f_X \colon W \to X$ and $f_Y \colon W \to Y$, there exists a unique morphism

$$f \colon W \to P$$

satisfying $p_X f = f_X$ and $p_Y f = f_Y$, as illustrated in the diagram:

$$\begin{tikzcd}
 & W \arrow[d, dashed, "\exists! f"] \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_Y"] & \\
 & X \times Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"'] & \\
X & & Y.
\end{tikzcd}$$

The objects X and Y are called the **factors** of the product, and the morphism $p_X \colon P \to X$ the **projection** onto the factor X. A product is usually denoted $X \times Y$, in view of Lemma B.2. We denote $f = (f_X, f_Y) \colon W \to X \times Y$ the morphism determined by the morphisms $f_X \colon W \to X$ and $f_Y \colon W \to Y$. The morphisms f_X and f_Y are sometimes called the *components* or *coordinates* of the morphism $f \colon W \to X \times Y$.

Lemma B.2. *The product is unique up to unique isomorphism. That is, if P' is another product of X and Y with projections $p'_X \colon P' \to X$ and $p'_Y \colon P' \to Y$, then there exists a unique isomorphism $\varphi \colon P \xrightarrow{\cong} P'$ satisfying $p'_X \varphi = p_X$ and $p'_Y \varphi = p_Y$, as illustrated in this diagram:*

$$\begin{tikzcd}
P \arrow[rr, "\varphi", "\cong"'] \arrow[dr, bend right, "p_Y"] \arrow[ddr, bend right=40, "p_X"'] & & P' \arrow[dl, bend left, "p'_Y"'] \arrow[ddl, bend left=40, "p'_X"] \\
 & Y & \\
 & X. &
\end{tikzcd}$$

Proof. Once in their lifetime, every person should solve this exercise, which is very satisfying. □

Remark B.3. There are many isomorphisms $P \xrightarrow{\cong} P'$. Uniqueness is achieved if we require compatibility with the projections p_X, p_Y, p'_X, p'_Y. The projections are part of the structure of a product.

Remark B.4. The argument in the proof of Lemma B.2 holds more generally for any construction that satisfies a *universal property*, for instance coproducts, pullbacks, pushouts, etc. Such constructions are unique up to unique isomorphism.

The notion of product can be generalized to an arbitrary family of objects.

Definition B.5. Let $\{X_i\}_{i \in I}$ be a family of objects in a category $\mathcal{C}$, where I denotes an indexing set. A **product** of the family $\{X_i\}_{i \in I}$ is an object $\prod_{i \in I} X_i$ together with a morphism $p_i \colon \prod_{j \in I} X_j \to X_i$ for each $i \in I$ that satisfies the following (universal) property: For all object W with a morphism $f_i \colon W \to X_i$ for each $i \in I$, there exists a unique morphism

$$f \colon W \to \prod_{i \in I} X_i$$

satisfying $p_i f = f_i$ for each $i \in I$, as illustrated in the diagram:

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[dr, "f_i"'] & \prod_{j \in I} X_j \arrow[d, "p_i"] \\
& X_i.
\end{tikzcd}$$

In slogan form: "A morphism into a product is the same as a morphism into each factor."

Definition B.6. An object Z in a category $\mathcal{C}$ is **terminal** if for all object X, there is a unique morphism $X \to Z$.

A terminal object is sometimes denoted $*$ or 1.

Example B.7. A terminal object is a product of the empty family of objects, that is, a product of no objects. In particular, a terminal object (if it exists) is unique up to unique isomorphism.

Example B.8. In **Set**, the product is the usual Cartesian product of sets $\prod_{i \in I} X_i$. Any singleton $\{x\}$ is terminal.

Likewise in **Set**$_*$, the product is the Cartesian product, with componentwise basepoint. That is, the product $\prod_{i \in I} (X_i, x_i)$ in **Set**$_*$ is the pointed set

$$\prod_{i \in I} (X_i, x_i) = \left(\prod_{i \in I} X_i, (x_i)_{i \in I} \right).$$

Example B.9. In **Top**, the product of spaces is the Cartesian product of the underlying sets endowed with the product topology. Any one-point space $\{x\}$ is terminal.

In **Top**$_*$, the product is as in **Top**, with componentwise basepoint.

Exercise B.10. Recall that the Cartesian product $\prod_{i \in I} X_i$ of spaces also admits the **box topology**, which has as a base the collection of all "open boxes":

$$\mathcal{B}_{\text{box}} = \{ \prod_{i \in I} U_i \mid U_i \subseteq X_i \text{ is open} \}.$$

Now let $\{X_i\}_{i \in I}$ be a family of non-trivial topological spaces (*trivial* topology is a synonym for *indiscrete* topology).

(a) Show that the box topology on $\prod_{i \in I} X_i$ is strictly finer than the product topology, that is, the inclusion of topologies $\mathcal{T}_{\text{prod}} \subset \mathcal{T}_{\text{box}}$ is strict.

(b) Show explicitly that $\left(\prod_{i \in I} X_i, \mathcal{T}_{\text{box}}\right)$ together with the projections $p_i \colon \prod_{j \in I} X_j \to X_i$ is *not* a product of the spaces X_i in **Top**. In other words, this data does *not* satisfy the universal property of a product.

Exercise B.11. Show that the product $\prod_{i \in I} X_i$ in the homotopy category Ho(**Top**) is as in **Top**.

Example B.12. In **Gp**, the product of a family of groups $\{G_i\}_{i \in I}$ is the Cartesian product $\prod_{i \in I} G_i$ with componentwise multiplication. The trivial group 0 is terminal.

The same description of the product holds in **Ab**. Likewise in $\mathbf{Vect}_k$, the product of a family of vector spaces $\{V_i\}_{i \in I}$ is the Cartesian product $\prod_{i \in I} V_i$ with componentwise addition and scalar multiplication.

B.2 Coproducts

Coproduct is the notion dual to the notion of product. *Dual* means that we reverse all the arrows.

Definition B.13. Let X and Y be objects in a category $\mathcal{C}$. A **coproduct** of X and Y is an object C together with morphisms $\iota_X \colon X \to C$ and $\iota_Y \colon Y \to C$ that satisfies the following (universal) property: For all object Z with morphisms $f_X \colon X \to Z$ and $f_Y \colon Y \to Z$, there exists a unique morphism

$$f \colon C \to Z$$

satisfying $f\iota_X = f_X$ and $f\iota_Y = f_Y$, as illustrated in the diagram:

$$\begin{tikzcd}
X \arrow[dr, "\iota_X"] \arrow[ddr, bend right, "f_X"'] & & Y \arrow[dl, "\iota_Y"'] \arrow[ddl, bend left, "f_Y"] \\
& C \arrow[d, dashed, "\exists! f"] & \\
& Z. &
\end{tikzcd}$$

The coproduct is sometimes called the *sum*. A coproduct is often denoted $X \amalg Y$. The objects X and Y are called the **summands** of the coproduct, and the morphism $\iota_X \colon X \to X \amalg Y$ is called the **inclusion** of the summand X.

The notion of coproduct can be generalized to an arbitrary family of objects.

Definition B.14. Let $\{X_i\}_{i \in I}$ be a family of objects in a category $\mathcal{C}$, where I denotes an indexing set. A **coproduct** of the family $\{X_i\}_{i \in I}$ is an object $\coprod_{i \in I} X_i$ together with a morphism $\iota_i \colon X_i \to \coprod_{j \in I} X_j$ for each $i \in I$ that satisfies the following (universal) property: For all obbject Z with a morphism $f_i \colon X_i \to Z$ for each $i \in I$, there exists a unique morphism

$$f \colon \coprod_{i \in I} X_i \to Z$$

satisfying $f\iota_i = f_i$ for each $i \in I$, as illustrated in the diagram:

$$\begin{tikzcd}
\coprod_{j \in I} X_j \arrow[r, "f"] & Z \\
X_i. \arrow[u, "\iota_i"] \arrow[ur, "f_i"'] &
\end{tikzcd}$$

In slogan form: “A morphism from a coproduct is the same as a morphism from each summand.”

Definition B.15. An object A in a category $\mathcal{C}$ is **initial** if for all object X, there is a unique morphism $A \to X$.

An initial object is sometimes denoted $\emptyset$ or 0.

Example B.16. An initial object is a coproduct of the empty family, that is, a coproduct of no objects.

Definition B.17. An object in a category $\mathcal{C}$ is a **zero object** if it is both initial and terminal.

A zero object is sometimes denoted 0.

Exercise B.18. Show that a category $\mathcal{C}$ has a zero object if and only if $\mathcal{C}$ has an initial object $\emptyset$, a terminal object $*$, and the unique morphism $\emptyset \to *$ is an isomorphism.

Example B.19. In **Set**, the coproduct is the usual disjoint union of sets $\coprod_{i \in I} X_i = \sqcup_{i \in I} X_i$. The empty set $\emptyset$ is initial.

The category **Set** does not have a zero object, since the morphism $\emptyset \to *$ is not bijective, hence not an isomorphism.

Example B.20. In $\mathbf{Set}_*$, the coproduct of pointed sets is the wedge sum

$$\coprod_{i \in I} (X_i, x_i) = \bigvee_{i \in I} X_i$$

with the (well-defined) basepoint $[x_i] \in \bigvee_{i \in I} X_i$.

Any one-element pointed set $\{x\}$ is initial and terminal, i.e., a zero object in $\mathbf{Set}_*$.

Example B.21. In **Top**, the coproduct of spaces is the disjoint union of underlying sets endowed with the coproduct topology. The empty space $\emptyset$ is initial. As in **Set**, there is no zero object in **Top**.

Exercise B.22.

(a) Find an example of space X decomposed as a disjoint union $X = A \sqcup B$ where however X is *not* the coproduct $A \amalg B$. Refer explicitly to the coproduct topology.

(b) Now show that X together with the inclusions $A \hookrightarrow X$ and $B \hookrightarrow X$ is *not* a coproduct of A and B in **Top**. In other words, this data does *not* satisfy the universal property of a coproduct.

Example B.23. In $\mathbf{Top}_*$, the coproduct of pointed spaces is the wedge sum

$$\coprod_{i \in I} (X_i, x_i) = \bigvee_{i \in I} X_i$$

with the (well-defined) basepoint $[x_i] \in \bigvee_{i \in I} X_i$. Any one-point pointed space $\{x\}$ is initial and terminal, i.e., a zero object in $\mathbf{Top}_*$.

Example B.24. In Ho(**Top**), the coproduct $\coprod_{i \in I} X_i$ is as in **Top**. This follows from the natural homeomorphism in **Top**

$$(\coprod_{i \in I} X_i) \times [0,1] \cong \coprod_{i \in I} (X_i \times [0,1]).$$

Example B.25. In **Gp**, the coproduct is the free product of groups

$$G \amalg H = G * H.$$

This also holds for the coproduct of an infinite family

$$\coprod_{i \in I} G_i = \mathop{\Huge *}_{i \in I} G_i.$$

The trivial group 0 is initial and terminal, i.e., a zero object in **Gp**.

Example B.26. In **Ab**, the coproduct is the direct sum of abelian groups

$$A \amalg B = A \oplus B.$$

This also holds for the coproduct of an infinite family

$$\coprod_{i \in I} A_i = \bigoplus_{i \in I} A_i.$$

The trivial group 0 is initial and terminal, i.e., a zero object in **Ab**.

Likewise in $\mathbf{Vect}_k$, the coproduct is the direct sum of vector spaces

$$\coprod_{i \in I} V_i = \bigoplus_{i \in I} V_i.$$

Remark B.27. In **Ab** (and in $\mathbf{Vect}_k$), finite products and coproducts coincide, that is, the canonical map

$$A \oplus B \xrightarrow{\cong} A \times B$$

is an isomorphism. We call the direct sum in **Ab** a *biproduct*, since $A \oplus B$ is both the coproduct and the product of A and B in **Ab**. However, this only holds for finite families. For an infinite family of (non-trivial) abelian groups $\{A_i\}_{i \in I}$, the canonical map

$$\bigoplus_{i \in I} A_i \to \prod_{i \in I} A_i$$

is *not* an isomorphism, but rather an isomorphism onto the subgroup

$$\{(x_i)_{i \in I} \in \prod_{i \in I} A_i \mid x_i \neq 0 \text{ for finitely many indices } i \in I\}.$$

Remark B.28. In **Gp**, coproducts and products are very different. Consider for instance the cyclic group $\mathbb{Z}/2$. The product

$$\mathbb{Z}/2 \times \mathbb{Z}/2$$

contains four elements, the so-called *Klein four-group*. On the other hand, the coproduct (free product) $\mathbb{Z}/2 * \mathbb{Z}/2$ contains infinitely many elements

$$1, a, b, ab, ba, aba, bab, abab, baba, \ldots.$$

More precisely, $\mathbb{Z}/2 * \mathbb{Z}/2$ admits the presentation

$$\begin{aligned}\mathbb{Z}/2 * \mathbb{Z}/2 &= \langle a, b \mid a^2 = 1, b^2 = 1 \rangle \\ &\cong \langle r, s \mid s^2 = 1, srs = r^{-1} \rangle\end{aligned}$$

with $r = ab$ and $s = a$. This group is called the *infinite dihedral group* [Hat02, §1.2.Free products of groups]. The canonical map

$$\mathbb{Z}/2 * \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2$$

has a huge kernel, namely all words with an even number of a's and b's, for instance $baba$.

Remark B.29. Remarks B.27 and B.28 also show that the notions of product and coproduct depend crucially on the category. If we view $\mathbb{Z}/2$ as an object in **Gp**, then the coproduct in **Gp**

$$\mathbb{Z}/2 \amalg \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$$

is the free product. If we view $\mathbb{Z}/2$ instead as an object in **Ab**, then the coproduct in **Ab**

$$\mathbb{Z}/2 \amalg \mathbb{Z}/2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is the direct sum.

Coproducts in **Gp** and in **Ab** are related as follows.

Exercise B.30. Let G and H be groups. Show that there is a natural isomorphism of abelian groups

$$(G * H)_{\mathrm{ab}} \cong G_{\mathrm{ab}} \oplus H_{\mathrm{ab}}.$$

More generally for any family of groups:

$$(\mathop{\ast}_{i \in I} G_i)_{\mathrm{ab}} \cong \bigoplus_{i \in I} (G_i)_{\mathrm{ab}}.$$

In other words, the abelianization functor $(-)_{\mathrm{ab}} \colon \mathbf{Gp} \to \mathbf{Ab}$ preserves coproducts.

C Pullbacks and pushouts

C.1 Pullbacks

Definition C.1. Let $g \colon X \to Z$ and $h \colon Y \to Z$ be morphisms in a category $\mathcal{C}$. A **pullback** of g and h is an object P together with morphisms $p_X \colon P \to X$ and $p_Y \colon P \to Y$ satisfying the equation $gp_X = hp_Y$ and the following (universal) property: For all object W with morphisms $f_X \colon W \to X$ and $f_Y \colon W \to Y$ satisfying $gf_X = hf_Y$, there exists a unique morphism

$$f \colon W \to P$$

satisfying $p_X f = f_X$ and $p_Y f = f_Y$, as illustrated in the diagram

$$\begin{tikzcd}
W \arrow[rrd, bend left, "f_Y"] \arrow[rdd, bend right, "f_X"'] \arrow[rd, dashed, "\exists!\, f"] & & \\
& P \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "h"] \\
& X \arrow[r, "g"'] & Z.
\end{tikzcd}$$

The pullback is denoted $X \times_Z Y$. The morphism determined by $f_X \colon W \to X$ and $f_Y \colon W \to Y$ is denoted by $f = (f_X, f_Y) \colon W \to X \times_Z Y$. A pullback diagram is often denoted by a corner symbol:

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d] \arrow[rd, phantom, "\lrcorner"] & Y \arrow[d] \\
X \arrow[r] & Z.
\end{tikzcd}$$

Example C.2. If $Z = *$ is a terminal object, then the pullback $X \times_* Y = X \times Y$ is the product of the objects X and Y.

Example C.3. In **Set**, the pullback

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] \arrow[rd, phantom, "\lrcorner"] & Y \arrow[d, "h"] \\
X \arrow[r, "g"'] & Z
\end{tikzcd}$$

can be described as follows:

$$X \times_Z Y = \{(x, y) \in X \times Y \mid g(x) = h(y)\},$$

where $p_X \colon X \times_Z Y \to X$ and $p_Y \colon X \times_Z Y \to Y$ are the usual projections:

$$\begin{cases} p_X(x,y) = x \\ p_Y(x,y) = y. \end{cases}$$

Example C.4. Let X be a set and $A, B \subseteq X$ subsets. Their pullback is the intersection $A \cap B$. More precisely, the diagram of inclusions

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\
A \arrow[r, "\iota_A"'] & X
\end{tikzcd}$$

is a pullback diagram in **Set**.

More generally, let $p \colon E \to X$ be an arbitrary function between sets. Then the diagram

$$\begin{tikzcd}
p^{-1}(A) \arrow[r] \arrow[d, "p|_{p^{-1}(A)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
A \arrow[r, hook, "\iota_A"'] & X
\end{tikzcd}$$

is a pullback diagram. We can view this pullback as the "restriction of p over A", or a "restriction in the target".

Example C.5. In $\mathbf{Set}_*$, the pullback

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "h"] \\
X \arrow[r, "g"'] & Z
\end{tikzcd}$$

is as in **Set**, namely the set

$$X \times_Z Y = \{(x,y) \in X \times Y \mid g(x) = h(y)\},$$

with componentwise basepoint

$$(x_0, y_0) \in X \times_Z Y.$$

Note that the maps $g \colon (X, x_0) \to (Z, z_0)$ and $h \colon (Y, y_0) \to (Z, z_0)$ are pointed, so that the following equations hold:

$$g(x_0) = z_0 = h(y_0).$$

This ensures that the point $(x_0, y_0) \in X \times Y$ lies in the subset

$$(x_0, y_0) \in \{(x,y) \in X \times Y \mid g(x) = h(y)\}.$$

Example C.6. In **Gp** and in **Ab**, the pullback

$$\begin{tikzcd}
G \times_K H \arrow[r, "p_Y"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "\psi"] \\
G \arrow[r, "\varphi"'] & K
\end{tikzcd}$$

is as in **Set**, namely the set

$$G \times_K H = \{(g, h) \in G \times H \mid \varphi(g) = \psi(h)\}$$

with componentwise multiplication. In other words, we view $G \times_K H \subseteq G \times H$ as a subgroup.

For instance, the kernel of a homomorphism $\varphi \colon G \to K$ is obtained as the pullback in the diagram

$$\begin{tikzcd}
\ker(\varphi) \arrow[r] \arrow[d, hook, "\text{inclusion}"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
G \arrow[r, "\varphi"'] & K.
\end{tikzcd}$$

Example C.7. In **Top**, the pullback

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "h"] \\
X \arrow[r, "g"'] & Z
\end{tikzcd}$$

is as in **Set**, namely the set

$$X \times_Z Y = \{(x, y) \in X \times Y \mid g(x) = h(y)\}$$

viewed as a subspace $X \times_Z Y \subseteq X \times Y$.

Example C.8. Let $p \colon E \to X$ be a map in **Set** or in **Top**, and $x \in X$. The preimage

$$p^{-1}(x) \subseteq E$$

is called the **fiber** of p over the point $x \in X$, sometimes denoted $E_x := p^{-1}(x)$. The fiber can be obtained as a pullback:

$$\begin{tikzcd}
p^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
* \arrow[r, "x"'] & X.
\end{tikzcd}$$

Example C.9. Pullbacks also appear in the definition of a covering space $p\colon \widetilde{X} \to X$. For a subset $U \subseteq X$, consider the pullback diagram

$$\begin{tikzcd}
p^{-1}(U) \arrow[r] \arrow[d, "p|_{p^{-1}(U)}"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{X} \arrow[d, "p"] \\
U \arrow[r, hook, "\iota_U"'] & X,
\end{tikzcd}$$

where $\iota_U\colon U \hookrightarrow X$ denotes the inclusion. An open neighborhood $U \subseteq X$ is called a **trivializing neighborhood** for p if there is a homeomorphism $\varphi\colon p^{-1}(U) \xrightarrow{\cong} \coprod_{\alpha\in J} U$ making the following diagram commute:

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "\varphi", "\cong"'] \arrow[d, "p|_{p^{-1}(U)}"'] & \coprod_{\alpha\in J} U \arrow[dl, "\nabla"] \\
U. &
\end{tikzcd}$$

Here, $\nabla\colon \coprod_\alpha U \to U$ denotes the fold map (a.k.a. the codiagonal), the morphism whose restriction to each summand is the identity $1_U\colon U \to U$. Such a homeomorphism φ is called a **local trivialization** of p over U. Each summand U of $\coprod_{\alpha\in J} U$ is called a **sheet** of p over U. The set J is an indexing set for the sheets.

Remark C.10. The pullback is also called *fiber product*, for the following reason. Let $p\colon E \to X$ and $p'\colon E' \to X$ be arbitrary morphisms in **Top** or in **Set**. By construction, the pullback $E \times_X E'$ comes with a map $E \times_X E' \to X$, namely the composite in the diagram

$$\begin{tikzcd}
E \times_X E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E' \arrow[d, "p'"] \\
E \arrow[r, "p"'] & X.
\end{tikzcd}$$

The fiber of this map $E \times_X E' \to X$ over a point $x \in X$ is

$$\begin{aligned}
(E \times_X E')_x &= \{(e,e') \in E \times_X E' \mid p(e) = p'(e') = x\} \\
&= \{(e,e') \in E \times_X E' \mid p(e) = x \text{ and } p'(e') = x\} \\
&\cong \{e \in E \mid p(e) = x\} \times \{e' \in E' \mid p'(e') = x\} \\
&= E_x \times E'_x,
\end{aligned}$$

the product of the respective fibers E_x and E'_x over x.

Remark C.11. The homotopy category Ho(**Top**) is missing several pullbacks. That is, there are diagrams

$$\begin{tikzcd} & Y \arrow[d, "h"] \\ X \arrow[r, "g"'] & Z \end{tikzcd}$$

in Ho(**Top**) that don't admit a pullback. Proving this requires a bit more homotopy theory, which goes beyond the scope of this course [Mat16].

C.2 Pushouts

Pushouts provide a general way to build an object by "gluing different pieces together".

Definition C.12. Let $g\colon W \to X$ and $h\colon W \to Y$ be morphisms in a category $\mathcal{C}$. A **pushout** of g and h is an object P together with morphisms $\iota_X\colon X \to P$ and $\iota_Y\colon Y \to P$ satisfying the equation $\iota_X g = \iota_Y h$ and the following (universal) property: For all object Z with morphisms $f_X\colon X \to Z$ and $f_Y\colon Y \to Z$ satisfying $f_X g = f_Y h$, there exists a unique morphism

$$f\colon P \to Z$$

satisfying $f\iota_X = f_X$ and $f\iota_Y = f_Y$, as illustrated in the diagram

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "g"'] & Y \arrow[d, "\iota_Y"] \arrow[ddr, bend left, "f_Y"] & \\
X \arrow[r, "\iota_X"'] \arrow[drr, bend right, "f_X"'] & P \arrow[dr, dashed, "\exists!\, f"] & \\
& & Z.
\end{tikzcd}$$

The pushout is denoted $X \cup_W Y$. A pushout diagram is often denoted with a corner symbol:

$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\
X \arrow[r] & X \cup_W Y.
\end{tikzcd}$$

Example C.13. If $W = \emptyset$ is an initial object, then the pushout $X \cup_\emptyset Y = X \amalg Y$ is the coproduct of the objects X and Y.

Example C.14. In **Set**, the pushout

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "\iota_Y"] \\
X \arrow[r, "\iota_X"'] & X \cup_W Y
\end{tikzcd}$$

can be described as a quotient of the disjoint union:

$$X \cup_W Y = (X \sqcup Y)\,/g(w) \sim h(w) \text{ for each } w \in W.$$

Here, the morphisms $\iota_X\colon X \to X \cup_W Y$ and $\iota_Y\colon Y \to X \cup_W Y$ are induced by the usual inclusions:

$$\begin{cases} \iota_X(x) = [x] \\ \iota_Y(y) = [y]. \end{cases}$$

Example C.15. Let X be a set and $A, B \subseteq X$ subsets. Their union $A \cup B$ can be obtained as a pushout:

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\
A \arrow[r] & A \cup B.
\end{tikzcd}$$

Note that the pullback diagram

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\iota_B"] \\
A \arrow[r, "\iota_A"'] & X
\end{tikzcd}$$

is rarely a pushout, namely, if and only if $A \cup B = X$ holds.

Example C.16. Let $A \subseteq X$ be a subset. Then the diagram

$$\begin{tikzcd}
A \arrow[r, hook, "\iota_A"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, two heads, "q"] \\
* \arrow[r] & X/A
\end{tikzcd}$$

is a pushout, where the quotient map $q\colon X \twoheadrightarrow X/A$ collapses the subset A to a point.

Example C.17. In $\mathbf{Set}_*$, the pushout

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\
X \arrow[r] & X \cup_W Y
\end{tikzcd}$$

is as in **Set**. The basepoints $x_0 \in X$ and $y_0 \in Y$ automatically get identified, because of the relation

$$x_0 = g(w_0) \sim h(w_0) = y_0.$$

Example C.18. In **Top**, the pushout

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "\iota_Y"] \\
X \arrow[r, "\iota_X"'] & X \cup_W Y
\end{tikzcd}$$

is as in **Set**, namely the set

$$X \cup_W Y = (X \amalg Y) \, / \, g(w) \sim h(w) \text{ for each } w \in W$$

endowed with the quotient topology. Here $X \amalg Y$ denotes the disjoint union endowed with the coproduct topology, i.e., the coproduct in **Top**.

Example C.19. Let X be a space and $A, B \subseteq X$ subspaces. The pullback

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & X \end{tikzcd} \tag{8}$$

is rarely a pushout. If A and B do not cover the space X, then the diagram (8) cannot be a pushout, by the remark in Example C.15. And when $A \cup B = X$ holds, the diagram (8) is a pushout in **Set**, but *not* automatically a pushout in **Top**.

Exercise C.20. Let X be a space with a cover $X = A \cup B$.

(a) If $A, B \subseteq X$ are open in X, show that the diagram (8) is a pushout in **Top**.

(b) If $A, B \subseteq X$ are closed in X, show that the diagram (8) is a pushout in **Top**.

(c) Find an example of cover $X = A \cup B$ such that the diagram (8) is *not* a pushout in **Top**.

Example C.21. Attaching an n-cell along an attaching map $\varphi \colon S^{n-1} \to X$ is the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ D^n \arrow[r, "\Phi"'] & X \cup_\varphi D^n. \end{tikzcd}$$

More explicitly, attaching an n-cell yields the space

$$X \cup_\varphi D^n = (X \amalg D^n) \, / \, w \sim \varphi(w) \text{ for each } w \in \partial D^n = S^{n-1}$$

endowed with the quotient topology. The pushout diagram also yields the characteristic map of the cell $\Phi \colon D^n \to X \cup_\varphi D^n$.

Example C.22. Let X be a CW complex with skeleta $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$. By definition, the n-skeleton X_n is obtained from X_{n-1} by attaching n-cells, i.e., X_n is the pushout

$$\begin{tikzcd} \coprod_{\alpha \in J_n} S^{n-1} \arrow[r, "(\varphi_\alpha)"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X_{n-1} \arrow[d] \\ \coprod_{\alpha \in J_n} D^n \arrow[r, "(\Phi_\alpha)"'] & X_n. \end{tikzcd}$$

Here J_n denotes the indexing set of the n-cells of X. For each $\alpha \in J_n$, $\varphi_\alpha \colon S^{n-1} \to X_{n-1}$ denotes the attaching map of the corresponding n-cell e^n_α. Those attaching maps together define the map

$$(\varphi_\alpha)_{\alpha \in J_n} \colon \coprod_{\alpha \in J_n} S^{n-1} \to X_{n-1}$$

at the top of the diagram.

Remark C.23. The homotopy category Ho(**Top**) is missing several pushouts. That is, there are diagrams

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "g"'] & Y \\ X & \end{tikzcd}$$

in Ho(**Top**) that do not admit a pushout; cf. Remark C.11.

Example C.24. In **Gp**, the pushout is the amalgamated product of groups:

$$\begin{tikzcd} K \arrow[r, "\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\ulcorner"] & H \arrow[d] \\ G \arrow[r] & G *_K H. \end{tikzcd}$$

More precisely, $G *_K H$ is the quotient group

$$G *_K H = (G * H) / \langle \{ \varphi(k)\psi(k)^{-1} \mid k \in K \} \rangle_{\text{normal}}$$

where $\langle S \rangle_{\text{normal}}$ denotes the normal subgroup generated by the subset S.

Example C.25. In **Ab**, the pushout is given as follows:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d] \\ B \arrow[r] & \operatorname{coker}(\varphi, -\psi). \end{tikzcd}$$

More precisely, the pushout is the quotient group

$$\begin{aligned} B \cup_A C &= (B \oplus C) / \langle \{ (\varphi(a), 0) - (0, \psi(a)) \mid a \in A \} \rangle \\ &= B \oplus C / \langle \{ (\varphi(a), -\psi(a)) \mid a \in A \} \rangle \\ &= B \oplus C / \operatorname{im}(\varphi, -\psi) \\ &= \operatorname{coker}(\varphi, -\psi) \end{aligned}$$

where $(\varphi, -\psi)\colon A \to B \oplus C$ denotes the map with components $\varphi\colon A \to B$ and $-\psi\colon A \to C$. In other words, a pushout in **Ab** can be viewed as an exact sequence:

$$\begin{tikzcd} A \arrow[r, "{(\varphi,-\psi)}"] & B \oplus C \arrow[r, two heads] & \operatorname{coker}(\varphi, -\psi) \arrow[r] & 0. \end{tikzcd}$$

Example C.26. The pushout of the diagram in **Ab**

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \\ 0 & \end{tikzcd}$$

is the following:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ 0 \arrow[r] & \operatorname{coker}(\varphi) \end{tikzcd}$$

where $B \twoheadrightarrow \operatorname{coker}(\varphi) = B / \operatorname{im}(\varphi)$ denotes the canonical quotient map.

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