MATH 441/841 - General Topology Worksheets

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Abstract

These are collected worksheets from the course MATH 441/841 - General Topology in the Fall 2021 semester. They contain practice problems to supplement the lectures and the projects.

Contents

1 Metrizability of countable products

Problem 1.1. Let (X, d) be a metric space. Consider the function $\rho \colon X \times X \to \mathbb{R}$ defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(a) Show that ρ is a metric on X.

(b) Show that the metric ρ from part (a) induces the same topology on X as the original metric d.

Remark 1.2. We could also have used the formula $\rho(x, y) = \min\{d(x, y), 1\}$. The goal was just to find a metric ρ which is topologically equivalent to d and is bounded. With that other metric ρ, Problem 1.1 is found in [Mun00, Theorem 20.1]; for our choice of metric ρ, it is found in [Mun00, Exercise 20.11].

Problem 1.3. Let $\{(X_i, d_i)\}_{i \in \mathbb{N}}$ be a countable family of metric spaces, where each metric d_i is bounded by 1, that is:

$$d_i(x_i, y_i) \leq 1 \quad \text{for all } x_i, y_i \in X_i.$$

Write $X := \prod_{i \in \mathbb{N}} X_i$ and consider the function $d \colon X \times X \to \mathbb{R}$ defined by

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i).$$

(a) Show that d is a metric on X. (First check that d is a well-defined function.)

(b) Show that the metric d from part (a) induces the product topology on $X = \prod_{i \in \mathbb{N}} X_i$.

Problem 1.3 can be found in [Mun00, Exercise 21.3(b)].

Together, Problems 1.1 and 1.3 show the following.

Theorem 1.4. *Let $\{X_i\}_{i \in \mathbb{N}}$ be a countable family of metrizable spaces. Then their product $\prod_{i \in \mathbb{N}} X_i$ is metrizable.*

2 Locally compact spaces

Problem 2.1. Show that the space $\mathbb{Q}$ of rational numbers, with its standard topology, is **not** locally compact.

Problem 2.2. Let X be a set. The **particular point topology** on X with "particular point" $p \in X$ is defined as

$$\mathcal{T} = \{S \subseteq X \mid p \in S \text{ or } S = \emptyset\}.$$

One readily checks that $\mathcal{T}$ is indeed a topology.

(a) Show that X (endowed with the particular point topology) is locally compact.

(b) Show that X is compact if and only if X is finite.

(c) Show that X is Lindelöf if and only if X is countable.

(d) Assuming X is uncountable, find a *compact* subspace $K \subseteq X$ whose closure $\overline{K}$ is not compact, in fact not even Lindelöf.

3 Compact metric spaces

Problem 3.1. Let (X, d) be a metric space and $A \subseteq X$ a subset. Define the **ϵ-neighborhood** of A as the set

$$B(A, \epsilon) := \{x \in X \mid d(x, A) < \epsilon\}.$$

(a) Show that the ϵ-neighborhood of A is

$$B(A, \epsilon) = \bigcup_{a \in A} B(a, \epsilon)$$

i.e., the union of all open balls of radius ϵ around points $a \in A$.

(b) Assume that A is *compact* and let $U \subseteq X$ be an open subset containing A. Show that some ϵ-neighborhood of A is contained in U, that is, $B(A, \epsilon) \subseteq U$ for some $\epsilon > 0$.

Problem 3.2. (a) Let (X, d) be a metric space which is totally bounded. Show that any subspace $A \subseteq X$ is also totally bounded. Explicitly: For every $\epsilon > 0$, A can be covered by finitely many balls of radius ϵ **centered at points of $\boldsymbol{A}$**.

(b) Let (X, d) be a metric space and let $A \subseteq X$ be a subspace which is totally bounded. Show that its closure $\overline{A} \subseteq X$ is also totally bounded.

(c) Let (X, d) be a complete metric space. Show that a subspace $A \subseteq X$ is compact if and only if it is closed (in X) and totally bounded.

(d) Let (X, d) be a complete metric space. Show that a subspace $A \subseteq X$ is totally bounded if and only if its closure $\overline{A} \subseteq X$ is compact.

(This condition is called being *relatively compact* in X.)

4 Completeness and uniform continuity

Throughout this worksheet, let $f\colon X \to Y$ be a *uniformly* continuous map between *metric* spaces.

Problem 4.1. Show that f sends Cauchy sequences to Cauchy sequences. In other words, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, show that $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

Problem 4.2. Assuming moreover that f is a homeomorphism and Y is complete, show that X is complete.

Problem 4.3. Find an example where f is a homeomorphism and X is complete, but Y is *not* complete. (Don't forget to show that your example f is uniformly continuous.)

Remark 4.4. Problem 4.2 implies that if two metric spaces are uniformly isomorphic, then one is complete if and only if the other is complete. In other words, completeness depends on more than just the topology, but at most on the uniform type.

5 The order topology

Definition 5.1. A **totally ordered set** is a partially ordered set $(X, \leq)$ where any two elements are comparable: for all $x, y \in X$, we have either $x \leq y$ or $y \leq x$.

Definition 5.2. Let $(X, \leq)$ be a totally ordered set. The **order topology** on X is the topology generated by "open rays"

$$(a, \infty) := \{x \in X \mid x > a\}$$
$$(-\infty, a) := \{x \in X \mid x < a\}$$

for any $a \in X$.

Problem 5.3. Show that the order topology on a totally ordered set is always T_1.

Problem 5.4. Show that the order topology on $\mathbb{R}$ with its usual order $\leq$ is the standard (metric) topology on $\mathbb{R}$.

Definition 5.5. An **interval** in a partially ordered set $(X, \leq)$ is a subset $I \subseteq X$ such that for all $x, y \in I$, the condition $x \leq z \leq y$ implies $z \in I$.

Problem 5.6. Let $(X, \leq)$ be a totally ordered set endowed with the order topology. Show that every connected subspace $A \subseteq X$ is an interval in X.

Problem 5.7. Find an example of totally ordered set $(X, \leq)$, endowed with the order topology, and an interval $A \subseteq X$ which is not a connected subspace.

6 The compact-open topology

On this worksheet, function spaces are endowed with the compact-open topology unless otherwise noted.

Problem 6.1. Let X be a compact topological space, and (Y, d) a metric space. Consider the uniform metric

$$d(f, g) := \sup_{x \in X} d\left(f(x), g(x)\right)$$

on the set of continuous maps $C(X, Y)$.

Show that the topology on $C(X, Y)$ induced by the uniform metric is the compact-open topology.

Problem 6.2. Let X and Y be topological spaces. Let $f, g\colon X \to Y$ be two continuous maps. Show that a homotopy from f to g induces a (continuous) path from f to g in the space of continuous maps $C(X, Y)$.

More precisely, let $F(X, Y)$ denote the *set of all functions* from X to Y. There is a natural bijection of sets:

$$\varphi\colon F(X \times [0,1], Y) \xrightarrow{\cong} F\left([0,1], F(X, Y)\right)$$

sending a function $H\colon X \times [0,1] \to Y$ to the function $\varphi(H)\colon [0,1] \to F(X, Y)$ defined by

$$\varphi(H)(t) = H(-, t) =: h_t.$$

Your task is to show that if a function $H\colon X \times [0,1] \to Y$ is continuous, then the following two conditions hold:

1. $h_t\colon X \to Y$ is continuous for all $t \in [0,1]$;
2. The corresponding function $\varphi(H)\colon [0,1] \to C(X, Y)$ is continuous.

Remark 6.3. If X is *locally compact Hausdorff*, then the converse holds as well: the two conditions guarantee that $H\colon X \times [0,1] \to Y$ is continuous. In that case, a homotopy from f to g is really the same as a path from f to g in the function space $C(X, Y)$.

Problem 6.4. (a) Let X and Y be topological spaces, where Y is Hausdorff. Show that $C(X, Y)$ is Hausdorff.

(b) Assume that there exists a topological space X such that $C(X, Y)$ is Hausdorff. Show that Y is Hausdorff.

Problem 6.5. (a) Let X, Y, and Z be topological spaces. Let $g\colon Y \to Z$ be a continuous map. Show that the induced map "postcomposition by g"

$$\begin{aligned} g_*\colon C(X, Y) &\to C(X, Z) \\ f &\mapsto g_*(f) = g \circ f \end{aligned}$$

is continuous.

(b) Let W, X, and Y be topological spaces. Let $d\colon W \to X$ be a continuous map. Show that the induced map "precomposition by d"

$$\begin{aligned} d^*\colon C(X,Y) &\to C(W,Y) \\ f &\mapsto d^*(f) = f \circ d \end{aligned}$$

is continuous.

Problem 6.6. Let X and Y be topological spaces, where X is *Hausdorff*. Let $\mathcal{S}$ be a subbasis for the topology of Y. Show that the collection

$$\{V(K,S) \mid K \subseteq X \text{ compact}, \, S \in \mathcal{S}\}$$

is a subbasis for the compact-open topology on $C(X,Y)$.

The notation above is $V(K,S) = \{f \in C(X,Y) \mid f(K) \subseteq S\}$.

Problem 6.7. Consider the real line $\mathbb{R}$ and the rationals $\mathbb{Q}$ with their standard (metric) topology. Consider the evaluation map

$$e\colon \mathbb{Q} \times C(\mathbb{Q},\mathbb{R}) \to \mathbb{R}.$$

Let $f\colon \mathbb{Q} \to \mathbb{R}$ be a constant function (say, $f \equiv 0$), and let $q \in \mathbb{Q}$. Show that the evaluation map e is *not* continuous at $(q,f) \in \mathbb{Q} \times C(\mathbb{Q},\mathbb{R})$.

Hint: You may want to use the fact that all compact subsets of $\mathbb{Q}$ have empty interior, and the fact that $\mathbb{Q}$ is completely regular (since it is normal).

7 Compactness in function spaces

On this worksheet, function spaces are endowed with the compact-open topology unless otherwise noted.

Problem 7.1. Let X be a topological space and (Y, d) a metric space. For each compact subset $K \subseteq X$, consider the pseudometric on $C(X, Y)$ defined by

$$d_K(f, g) = \sup_{x \in K} d\left(f(x), g(x)\right)$$

and its associated open balls $B_K(f, \epsilon) = \{g \in C(X, Y) \mid d_K(f, g) < \epsilon\}$.

Show that the collection of all open balls

$$\mathcal{B} = \{B_K(f, \epsilon) \mid K \subseteq X \text{ compact}, \ f \in C(X, Y), \epsilon > 0\}$$

forms a basis for a topology on $C(X, Y)$. More explicitly:

1. $\mathcal{B}$ covers $C(X, Y)$;
2. Finite intersections of members of $\mathcal{B}$ are unions of members of $\mathcal{B}$.

The following proposition will be relevant to Problem 7.4. No need to prove the proposition in your write-up.

Proposition 7.2. *1. Given a pseudometric d on a set X, there is a topologically equivalent pseudometric ρ on X which is bounded above by 1.*

For example, the formulas $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ or $\rho(x, y) = \min\{d(x, y), 1\}$ work.

2. Given a countable family of pseudometrics $\{d_n\}_{n \in \mathbb{N}}$ on X which are bounded above by 1, the formula

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x, y) \tag{1}$$

defines a pseudometric d on X.

3. The topology $\mathcal{T}_d$ on X induced by d is the topology generated by $\bigcup_{n \in \mathbb{N}} \mathcal{T}_{d_n}$. More explicitly, this is the topology generated by the collection of all open balls

$$\{B_n(x, \epsilon) \mid n \in \mathbb{N}, x \in X, \epsilon > 0\}$$

where we used the notation $B_n(x, \epsilon) := \{y \in X \mid d_n(x, y) < \epsilon\}$.

Definition 7.3. A family of pseudometrics $\{d_\alpha\}_{\alpha \in A}$ on a set X is **separating** if the following implication holds:

$$d_\alpha(x, y) = 0 \text{ for all } \alpha \in A \implies x = y.$$

In other words, for any distinct points $x \neq y$, there is an index $\alpha \in A$ satisfying $d_\alpha(x, y) > 0$.

Problem 7.4. Let X be a set and $\{d_n\}_{n\in\mathbb{N}}$ a countable family of pseudometrics on X which are bounded above by 1. Let d be the pseudometric on X defined by the formula (1) as in the proposition above.

Show that d is a metric if and only if the family of pseudometrics $\{d_n\}_{n\in\mathbb{N}}$ is separating.

Problem 7.5. Let (Y, d) be a metric space and consider the mapping space $C(\mathbb{R}, Y)$. For all $n \in \mathbb{N}$, consider the compact interval $[-n, n] \subset \mathbb{R}$ and the associated pseudometric

$$d_n(f, g) = \sup_{x\in[-n,n]} d\left(f(x), g(x)\right).$$

(a) Show that the family of pseudometrics $\{d_n\}_{n\in\mathbb{N}}$ on $C(\mathbb{R}, Y)$ is separating.

(b) Show that the topology $\mathcal{T}$ on $C(\mathbb{R}, Y)$ generated by $\bigcup_{n\in\mathbb{N}} \mathcal{T}_{d_n}$ is the topology of compact convergence.

Problem 7.6. Let X and Y be topological spaces, where Y is Hausdorff.

(a) Consider the set Y^X of *all* functions from X to Y, endowed with the topology of pointwise convergence. Recall that via the correspondence $Y^X \cong \prod_{x\in X} Y$, this corresponds to the product topology.

Show that a collection of functions $\mathcal{F} \subseteq Y^X$ is compact if and only if the following two conditions hold:

1. $\mathcal{F}$ is closed in Y^X.
2. For all $x \in X$, the projection $p_x(\mathcal{F}) = \{f(x) \mid f \in \mathcal{F}\} \subseteq Y$ has compact closure in Y.

(b) Let $C(X, Y)$ be endowed with the compact-open topology, and let $\mathcal{F} \subseteq C(X, Y)$ be a compact subspace. Show that $\mathcal{F}$ satisfies the conditions 1. and 2. listed in part (a), that is:

1. $\mathcal{F}$ viewed as a subset of Y^X is closed **with respect to the topology of pointwise convergence**.
2. For all $x \in X$, the projection $p_x(\mathcal{F}) = \{f(x) \mid f \in \mathcal{F}\} \subseteq Y$ has compact closure in Y.

Problem 7.7 (Munkres Exercise 47.1)**.** For each of the following subsets $\mathcal{F} \subset C(\mathbb{R}, \mathbb{R})$, say if $\mathcal{F}$ is equicontinuous of not, and prove your answer.

(a) $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ where $f_n(x) = x + \sin nx$.

(b) $\mathcal{F} = \{g_n \mid n \in \mathbb{N}\}$ where $g_n(x) = n + \sin x$.

(c) $\mathcal{F} = \{h_n \mid n \in \mathbb{N}\}$ where $h_n(x) = |x|^{\frac{1}{n}}$.

(d) $\mathcal{F} = \{k_n \mid n \in \mathbb{N}\}$ where $k_n(x) = n \sin\left(\frac{x}{n}\right)$.

8 The Baire category theorem

Problem 8.1. Let X be a topological space.

(a) Show that the following properties of a subset $A \subseteq X$ are equivalent.

1. The closure of A in X has empty interior: $\text{int}(\overline{A}) = \emptyset$.
2. For all non-empty open subset $U \subseteq X$, there is a non-empty open subset $V \subseteq U$ satisfying $V \cap A = \emptyset$.

A subset $A \subseteq X$ satisfying these equivalent properties is called **nowhere dense** in X.

(b) Show that the following properties of the space X are equivalent.

1. Any countable intersection of open dense subsets is dense. In other words, if each $U_n \subseteq X$ is open and dense in X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.
2. Any countable union of closed subsets with empty interior has empty interior. In other words, if each $C_n \subseteq X$ is closed in X and satisfies $\text{int}(C_n) = \emptyset$, then their union satisfies $\text{int}\left(\bigcup_{n=1}^{\infty} C_n\right) = \emptyset$.

A space X satisfying these equivalent properties is called a **Baire space**.

Problem 8.2. Show that a topological space X is of second category in itself if and only if any countable intersection of open dense subsets of X is non-empty.

Problem 8.3 (Uniform boundedness principle)**.** Let X be a Baire space and $S \subseteq C(X, \mathbb{R})$ a collection of real-valued continuous functions on X which is pointwise bounded: for each $x \in X$, there is a bound $M_x \in \mathbb{R}$ satisfying

$$|f(x)| \leq M_x \text{ for all } f \in S.$$

Show that there is a non-empty open subset $U \subseteq X$ on which the collection S is uniformly bounded: there is a bound $M \in \mathbb{R}$ satisfying

$$|f(x)| \leq M \text{ for all } x \in U \text{ and all } f \in S.$$

9 Connected components as a homotopy invariant

Definition 9.1. Let X be a topological space. A **path** in X from x to y is a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

The relation "there is a path from x to y" is an equivalence relation on X, whose equivalence classes are called the **path components** of X. Let $\pi_0(X)$ denote the set of path components of X.

Problem 9.2. Let $f\colon X \to Y$ be a (continuous) map between spaces. Define an induced function on path components

$$f_*\colon \pi_0(X) \to \pi_0(Y),$$

sometimes denoted $\pi_0(f)$, by the formula $f_*[x] := [f(x)]$, where $[x]$ denotes the path component of a point $x \in X$.

(a) Show that the function f_* is well-defined.

(b) Show that the function f_* is compatible with composition and identities. In other words, the equation

$$(gf)_* = g_* f_*$$

holds for all maps $f\colon X \to Y$ and $g\colon Y \to Z$, and the equation

$$(\mathrm{id}_X)_* = \mathrm{id}_{\pi_0(X)}$$

holds for every space X.

(c) Show that π_0 is *homotopy invariant* in the following sense: homotopic maps $f \simeq g\colon X \to Y$ induce the same function

$$f_* = g_*\colon \pi_0(X) \to \pi_0(Y).$$

(d) Let $X \simeq Y$ be homotopy equivalent spaces. Show that the sets of path components $\pi_0(X)$ and $\pi_0(Y)$ are in bijection.

In particular, X is path-connected if and only if Y is.

Problem 9.3. Let $\mathrm{Conn}(X)$ denote the set of connected components of a space X.

Let $f\colon X \to Y$ be a (continuous) map between spaces. Define an induced function on connected components

$$f_*\colon \mathrm{Conn}(X) \to \mathrm{Conn}(Y),$$

sometimes denoted $\mathrm{Conn}(f)$, by the formula $f_*[x] := [f(x)]$, where $[x]$ denotes the connected component of $x \in X$.

(a) Show that the function f_* is well-defined.

(b) Show that the function f_* is compatible with composition and identities. In other words, the equation

$$(gf)_* = g_* f_*$$

holds for all maps $f\colon X \to Y$ and $g\colon Y \to Z$, and the equation

$$(\mathrm{id}_X)_* = \mathrm{id}_{\pi_0(X)}$$

holds for every space X.

(c) Show that Conn is *homotopy invariant* in the following sense: homotopic maps $f \simeq g\colon X \to Y$ induce the same function

$$f_* = g_* \colon \operatorname{Conn}(X) \to \operatorname{Conn}(Y).$$

(d) Let $X \simeq Y$ be homotopy equivalent spaces. Show that the sets of connected components $\operatorname{Conn}(X)$ and $\operatorname{Conn}(Y)$ are in bijection.

In particular, X is connected if and only if Y is.

Problem 9.4. (a) Denote by $\eta_X \colon \pi_0(X) \to \operatorname{Conn}(X)$ the function that assigns to a path component C the connected component that contains C. Show that η is *natural* in the following sense: for every map $f\colon X \to Y$, the following diagram of sets commutes:

$$\begin{array}{ccc} \pi_0(X) & \xrightarrow{\eta_X} & \operatorname{Conn}(X) \\ {\scriptstyle \pi_0(f)}\downarrow & & \downarrow{\scriptstyle \operatorname{Conn}(f)} \\ \pi_0(Y) & \xrightarrow{\eta_Y} & \operatorname{Conn}(Y). \end{array}$$

(b) Let $X \simeq Y$ be homotopy equivalent spaces, where X satisfies the following condition: the path components of X coincide with its connected components. Show that Y also satisfies that condition.

Remark 9.5. The topologist's sine curve

$$S = \{(x, \sin \frac{1}{x}) \mid x \in (0,1]\} \cup (\{0\} \times [-1,1]) \subset \mathbb{R}^2$$

does not satisfy said condition. To wit, S is connected, but it has two path components.

References

[Mun00] J. R. Munkres, *Topology*, Second Edition, Prentice-Hall, Inc., Upper Saddle River, NJ, 2000.

