MATH 441/841 - General Topology

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Abstract

These are the lecture notes for the course MATH 441/841 - General Topology in the Fall 2021 semester. We were following Munkres as main reference. The notes provide more details and examples.

Contents

1 Generating a topology 3

2 Infinite products and coproducts 6

3 Quotient spaces 11

4 Closure and limit points 15

5 Sequences 20

6 Hausdorff spaces 24

7 Countability axioms 27

8 Compactness 32

9 The tube lemma and applications 37

10 Sequential compactness 43

11 Complete metric spaces 49

12 Banach fixed point theorems 52

13 The evaluation map **58**

14 Nets **60**

15 Compactness via nets **65**

16 Tychonoff's theorem **69**

17 Separation axioms **75**

18 Urysohn's lemma **82**

1 Generating a topology

1.1 Bases and subbases

Definition 1.1. Let $(X, \mathcal{T})$ be a topological space. A **base** (or *basis*) for the topology $\mathcal{T}$ of X is a collection $\mathcal{B}$ of subsets of X satisfying

$$\mathcal{T} = \left\{ \bigcup_{\alpha} B_\alpha \mid B_\alpha \in \mathcal{B} \right\}.$$

That is, open sets are precisely unions of members of $\mathcal{B}$.

Example 1.2. Let (X, d) be a metric space. Then the collection of open balls

$$\mathcal{B} = \{B(x, r) \mid x \in X, r > 0\}$$

is a base for the metric topology. See Homework 1 Problem 1(b).

Exercise 1.3. Let X be a topological space and $\mathcal{B}$ a collection of open subsets of X. Show that $\mathcal{B}$ is a base of the topology if and only if for every open subset $U \subseteq X$ and $x \in U$, there exists a basic open $B \in \mathcal{B}$ satisfying $x \in B \subseteq U$.

Example 1.4. Let (X, d) be a metric space. The collection of open balls with rational radius

$$\{B(x, r) \mid x \in X, r > 0, r \in \mathbb{Q}\}$$

is a base for the metric topology. The collection of open balls

$$\{B(x, \frac{1}{n}) \mid x \in X, n \in \mathbb{N}\}$$

is also a base for the metric topology.

In particular, a given topology can have many different bases.

Exercise 1.5. Let X be a set. Show that a collection $\mathcal{B}$ of subsets of X is a base for some topology on X if and only if $\mathcal{B}$ satisfies the following conditions:

1. $\mathcal{B}$ covers X, that is, $\bigcup_{B \in \mathcal{B}} B = X$.
2. Finite intersections are unions: For any $B, B' \in \mathcal{B}$, we have $B \cap B' = \bigcup_\alpha B_\alpha$ for some family $\{B_\alpha\}$ of members of $\mathcal{B}$.

Definition 1.6. Let $(X, \mathcal{T})$ be a topological space. A **subbase** for the topology $\mathcal{T}$ of X is a collection $\mathcal{S}$ of subsets of X satisfying

$$\mathcal{T} = \left\{ \bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha} S_{\alpha,i} \mid S_{\alpha,i} \in \mathcal{S} \right\}.$$

That is, finite intersections of members of $\mathcal{S}$ form a base for the topology.

Remark 1.7. The number n_α of members in the finite intersection is allowed to be zero. The intersection of an empty family of subsets of X is X.

Likewise, the arbitrary union is allowed to be the union of an empty family, with the index α running over an empty indexing set. The union of an empty family of subsets of X is the empty subset $\emptyset \subseteq X$.

1.2 Comparing topologies

For a given set X, topologies on X can be partially ordered by inclusion.

Definition 1.8. Let X be a set, and $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies on X. We say $\mathcal{T}_1$ is **smaller** than $\mathcal{T}_2$, denoted $\mathcal{T}_1 \leq \mathcal{T}_2$, if the inclusion $\mathcal{T}_1 \subseteq \mathcal{T}_2$ holds, viewed as subsets of the power set $\mathcal{P}(X)$. In other words, every $\mathcal{T}_1$-open is also $\mathcal{T}_2$-open.

One can also say that $\mathcal{T}_2$ is **larger** than $\mathcal{T}_1$.

Some references say that $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, while $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$.

Remark 1.9. The indiscrete topology $\mathcal{T}_{\text{ind}} = \{\emptyset, X\}$ is the least element in that partial order, whereas the discrete topology $\mathcal{T}_{\text{dis}} = \mathcal{P}(X)$ is the greatest element. In other words, the inequalities

$$\mathcal{T}_{\text{ind}} \leq \mathcal{T} \leq \mathcal{T}_{\text{dis}}$$

hold for any topology $\mathcal{T}$ on X.

Remark 1.10. By definition, the inequality $\mathcal{T}_1 \leq \mathcal{T}_2$ holds if and only if the identity function

$$\mathrm{id} \colon (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$$

is continuous. Note the reversal, mapping "from fine to coarse".

The poset of topologies on X has arbitrary meets (infima), described explicitly in the following proposition.

Proposition 1.11. *Let $\{\mathcal{T}_\beta\}$ be a family of topologies on X. Then the intersection $\bigcap_\beta \mathcal{T}_\beta$ is a topology on X, and therefore the infimum of the family $\{\mathcal{T}_\beta\}$.*

Proof. Exercise. □

Remark 1.12. If we consider an empty family of topologies, then their intersection is

$$\bigcap_\beta \mathcal{T}_\beta = \mathcal{P}(X) = \mathcal{T}_{\text{dis}},$$

which is a topology on X. Thus, the proposition also holds in that case.

Definition 1.13. Let X be a set and $\mathcal{S}$ be a collection of subsets of X. The **topology generated by** $\mathcal{S}$ (if it exists) is the smallest topology $\mathcal{T}_\mathcal{S}$ containing $\mathcal{S}$. In other words, it satisfies $\mathcal{S} \subseteq \mathcal{T}_\mathcal{S}$ and for any other topology $\mathcal{T}'$ containing $\mathcal{S}$, we have $\mathcal{T}_\mathcal{S} \leq \mathcal{T}'$.

Note that this universal property makes $\mathcal{T}_\mathcal{S}$ unique, if it exists.

Proposition 1.14. *For any collection of subsets $\mathcal{S}$, the topology $\mathcal{T}_\mathcal{S}$ exists.*

Proof. The topology

$$\mathcal{T}_\mathcal{S} = \bigcap_{\substack{\text{topologies } \mathcal{T} \\ \text{with } \mathcal{S} \subseteq \mathcal{T}}} \mathcal{T}$$

has the required properties. □

The following proposition provides an explicit description of $\mathcal{T}_{\mathcal{S}}$.

Proposition 1.15. *The topology generated by* $\mathcal{S}$ *is*

$$\mathcal{T}_{\mathcal{S}} = \left\{ \bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha} S_{\alpha,i} \mid S_{\alpha,i} \in \mathcal{S} \right\},$$

i.e., the topology for which $\mathcal{S}$ *is a subbase.*

Proof. Homework 2 Graduate Problem 3. □

2 Infinite products and coproducts

2.1 Infinite products

Definition 2.1. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of topological spaces. The **product topology** $\mathcal{T}_{\text{prod}}$ on the Cartesian product $\prod_\alpha X_\alpha$ is the smallest topology making all projection maps $p_\beta \colon \prod_\alpha X_\alpha \to X_\beta$ continuous.

In other words, the product topology is generated by subsets of the form $p_\beta^{-1}(U_\beta)$ for $U_\beta \subseteq X_\beta$ open.

Those generating open subsets are "large strips" $p_\beta^{-1}(U_\beta) = \prod_\alpha U_\alpha$ with

$$U_\alpha = \begin{cases} U_\beta & \text{if } \alpha = \beta \\ X_\alpha & \text{if } \alpha \neq \beta. \end{cases}$$

A base for the product topology $\mathcal{T}_{\text{prod}}$ is the collection of "large boxes"

$$\left\{ \prod_\alpha U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open, and } U_\alpha = X_\alpha \text{ except for at most finitely many } \alpha \right\}.$$

Proposition 2.2. *The topological space* $(\prod_\alpha X_\alpha, \mathcal{T}_{\text{prod}})$ *along with the projections* $p_\beta \colon \prod_\alpha X_\alpha \to X_\beta$ *satisfies the universal property of a product.*

Explicitly: Given a space Z *and continuous maps* $f_\alpha \colon Z \to X_\alpha$ *for all* $\alpha \in I$*, there exists a unique continuous map* $f \colon Z \to \prod_\alpha X_\alpha$ *satisfying* $p_\alpha \circ f = f_\alpha$ *for all* α*.*

Proof. The continuous maps $f_\alpha \colon Z \to X_\alpha$ are in particular functions, so that there is a unique function $f \colon Z \to \prod_\alpha X_\alpha$ whose components are $p_\alpha \circ f = f_\alpha$. In other words, f is given by

$$f(z) = (f_\alpha(z))_{\alpha \in A}.$$

It remains to check that f is continuous. The product topology is generated by subsets of the form $p_\beta^{-1}(U_\beta)$ for $U_\beta \subseteq X_\beta$ open. Its preimage under f is

$$\begin{aligned} f^{-1}\left(p_\beta^{-1}(U_\beta)\right) &= (p_\beta \circ f)^{-1}(U_\beta) \\ &= f_\beta^{-1}(U_\beta) \end{aligned}$$

which is open in Z since $f_\beta \colon Z \to X_\beta$ is continuous. □

Definition 2.3. The **box topology** $\mathcal{T}_{\text{box}}$ on the Cartesian product $\prod_\alpha X_\alpha$ is the topology for which the collection of all "boxes"

$$\{\prod_\alpha U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open}\}$$

is a base.

Note that we always have $\mathcal{T}_{\text{prod}} \leq \mathcal{T}_{\text{box}}$, and equality holds for finite products. For an infinite product, the inequality is usually strict.

Exercise 2.4. Let $\{X_\alpha\}$ be a family of discrete spaces, where each X_α contains at least two points.

(a) Show that the box topology on $\prod_\alpha X_\alpha$ is discrete.

(b) Show that the product topology on $\prod_\alpha X_\alpha$ is *not* discrete.

Definition 2.5. A map between topological spaces $f\colon X \to Y$ is called an **open** map if for every open subset $U \subseteq X$, its image $f(U) \subseteq Y$ is open in Y.

Exercise 2.6. Show that the projection maps $p_\beta\colon \prod_\alpha X_\alpha \to X_\beta$ are open maps in the box topology (and therefore also in the product topology).

2.2 Disjoint unions

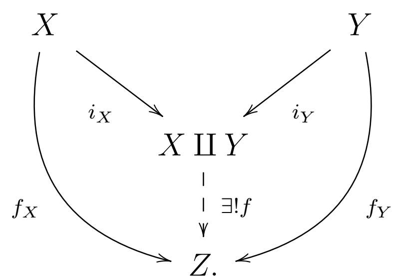
In this section, we describe a construction which is dual to the product. The discussion will be eerily similar to that of products, because the ideas are the same, and because of copy-paste. We first review disjoint unions of sets.

Let X and Y be sets. The disjoint union of X and Y is the set

$$X \amalg Y = \{w \mid w \in X \text{ or } w \in Y\}.$$

It comes equipped with the inclusion maps $i_X \colon X \to X \amalg Y$ and $i_Y \colon Y \to X \amalg Y$ from each summand. This explicit description of $X \amalg Y$ is made more meaningful by the following proposition.

Proposition 2.7. *The disjoint union of sets $X \amalg Y$ along with inclusion maps i_X and i_Y is the* ***coproduct*** *of sets, i.e., it satisfies the following universal property. For any set Z along with maps $f_X \colon X \to Z$ and $f_Y \colon Y \to Z$, there is a unique map $f \colon X \amalg Y \to Z$ whose restrictions are $f \circ i_X = f_X$ and $f \circ i_Y = f_Y$, in other words making the following diagram commute:*



Proof. Given f_X and f_Y, define $f \colon X \amalg Y \to Z$ by

$$f(w) := \begin{cases} f_X(w) & \text{if } w \in X \\ f_Y(w) & \text{if } w \in Y, \end{cases}$$

which satisfies $f \circ i_X = f_X$ and $f \circ i_Y = f_Y$.

To prove uniqueness, note that any element $w \in X \amalg Y$ is in one of the summands:

$$w = \begin{cases} i_X(w) & \text{if } w \in X \\ i_Y(w) & \text{if } w \in Y. \end{cases}$$

Therefore, any function $g \colon X \amalg Y \to Z$ can be written as

$$g(w) = \begin{cases} g(i_X(w)) = (g \circ i_X)(w) & \text{if } w \in X \\ g(i_Y(w)) = (g \circ i_Y)(w) & \text{if } w \in Y \end{cases}$$

so that g is *determined* by its restrictions $g \circ i_X$ and $g \circ i_Y$. □

In slogans: "A map out of $X \amalg Y$ is the same data as a map out of X and a map out of Y".

Yet another slogan: "$X \amalg Y$ is the closest set equipped with a map from X and a map from Y."

As usual with universal properties, this characterizes $X \amalg Y$ up to unique isomorphism.

2.3 Coproduct topology

The next goal is to define the coproduct $X \amalg Y$ of topological spaces X and Y such that it satisfies the analogous universal property in the category of topological spaces.

In other words, we want to find a topology on the disjoint union $X \sqcup Y$ such that the inclusion maps

$$\begin{cases} i_X \colon X \to X \amalg Y \\ i_Y \colon Y \to X \amalg Y \end{cases}$$

are *continuous*, and such that for any topological space Z along with *continuous* maps $f_X \colon X \to Z$ and $f_Y \colon Y \to Z$, there is a unique *continuous* map $f \colon X \amalg Y \to Z$ whose restrictions are $f \circ i_X = f_X$ and $f \circ i_Y = f_Y$.

Definition 2.8. Let X and Y be topological spaces. The **coproduct topology** is the largest topology on $X \amalg Y$ making the inclusions $i_X \colon X \to X \amalg Y$ and $i_Y \colon Y \to X \amalg Y$ continuous.

This means that a subset $U \subseteq X \amalg Y$ is open if and only if $i_X^{-1}(U)$ is open in X and $i_Y^{-1}(U)$ is open in Y. More concretely, noting $i_X^{-1}(U) = U \cap X$ and $i_Y^{-1}(U) = U \cap Y$, open subsets can be described as

$$\mathcal{T}_{\text{coprod}} = \{U = U_X \sqcup U_Y \mid U_X \subseteq X \text{ is open and } U_Y \subseteq Y \text{ is open}\}.$$

This definition works for an infinite disjoint union as well.

Definition 2.9. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of topological spaces. The **coproduct topology** $\mathcal{T}_{\text{coprod}}$ on the disjoint union $\bigsqcup_\alpha X_\alpha$ is the largest topology making all inclusion maps $i_\beta \colon X_\beta \to \coprod_\alpha X_\alpha$ continuous.

This means that a subset $U \subseteq \coprod_\alpha X_\alpha$ is open if and only if $i_\alpha^{-1}(U)$ is open in X_α for all $\alpha \in A$ More concretely, noting $i_\alpha^{-1}(U) = U \cap X_\alpha$, open subsets can be described as

$$\mathcal{T}_{\text{coprod}} = \left\{ U = \bigsqcup_\alpha U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open for all } \alpha \right\}.$$

That is, open subsets are disjoint unions of open subsets from each of the summands.

Proposition 2.10. *The topological space* $(\coprod_\alpha X_\alpha, \mathcal{T}_{\text{coprod}})$ *along with the inclusions* $i_\beta \colon X_\beta \to \coprod_\alpha X_\alpha$ *is a coproduct of topological spaces.*

Explicitly: Given a space Z *and continuous maps* $f_\alpha \colon X_\alpha \to Z$ *for all* $\alpha \in I$*, there exists a unique continuous map* $f \colon \coprod_\alpha X_\alpha \to Z$ *satisfying* $f \circ i_\alpha = f_\alpha$ *for all* α.

Proof. The continuous maps $f_\alpha \colon X_\alpha \to Z$ are in particular functions, so that there is a unique function from the disjoint union $f \colon \bigsqcup_\alpha X_\alpha \to Z$ whose restrictions are $f \circ i_\alpha = f_\alpha$. In other words, f is given by

$$f(w) = f(i_\alpha(w)) = f_\alpha(w)$$

where α is the unique index for which $w \in X_\alpha$.

It remains to check that f is continuous. Let $U \subseteq Z$ be open and consider its preimage $f^{-1}(U) \subseteq \coprod_\alpha X_\alpha$. To show that this subset is open, it suffices to check that its restriction to each summand is open:

$$\begin{aligned} i_\alpha^{-1}\left(f^{-1}(U)\right) &= (f \circ i_\alpha)^{-1}(U) \\ &= f_\alpha^{-1}(U) \end{aligned}$$

is indeed open in X_α since $f_\alpha \colon X_\alpha \to Z$ is continuous. □

Upshot: A map $f\colon \coprod_\alpha X_\alpha \to Z$ is continuous if and only if its restriction $f \circ i_\alpha \colon X_\alpha \to Z$ to each summand is continuous.

Proposition 2.11. *Each summand $X_\beta \subseteq \coprod_\alpha X_\alpha$ is open in the coproduct topology.*

Proof. Write $X_\beta = \coprod_\alpha U_\alpha$ where

$$U_\alpha = \begin{cases} X_\beta & \text{if } \alpha = \beta \\ \emptyset & \text{if } \alpha \neq \beta \end{cases}$$

is open in X_α for all α. □

More generally, the same proof shows that each inclusion map $i_\beta \colon X_\beta \to \coprod_\alpha X_\alpha$ is an open map. In fact, this characterizes the coproduct topology.

Exercise 2.12. Show that a topology on the disjoint union $\bigsqcup_\alpha X_\alpha$ is the coproduct topology *if* and only if it makes all the inclusion maps $i_\beta \colon X_\beta \to \coprod_\alpha X_\alpha$ continuous and open.

We can reinterpret the backward implication ($\Longleftarrow$) as follows. If a space X is a disjoint union $X = \bigsqcup_\alpha U_\alpha$ of *open* subsets $U_\alpha \subseteq X$, then X is the coproduct $X = \coprod_\alpha U_\alpha$, where each $U_\alpha \subseteq X$ has the subspace topology.

Exercise 2.13. (a) Consider the subset $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ viewed as a subspace of the real line $\mathbb{R}$. As a *set*, X is the disjoint union of the singletons $\{0\}$ and $\{\frac{1}{n}\}$ for all $n \in \mathbb{N}$. However, show that X does *not* have the coproduct topology on $\{0\} \amalg \coprod_{n\in\mathbb{N}} \{\frac{1}{n}\}$.

(b) Consider the subset $X = \mathbb{R} \setminus \mathbb{Z} \subset \mathbb{R}$ viewed as a subspace of the real line $\mathbb{R}$. As a *set*, X is the disjoint union of intervals $(n, n+1)$ for all $n \in \mathbb{Z}$. Show that X has the coproduct topology

$$X = \coprod_{n\in\mathbb{Z}} (n, n+1).$$

Exercise 2.14. Let X be a space and let J be a set, viewed as a discrete space. Show that the canonical bijection

$$X \times J \cong \coprod_{j\in J} X$$

is a homeomorphism, where $X \times J$ has the product topology and $\coprod_{j\in J} X$ has the coproduct topology.

3 Quotient spaces

3.1 Quotient topology

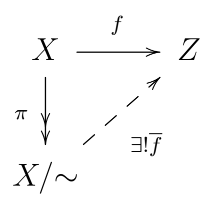
Definition 3.1. Let X be a topological space and $\sim$ an equivalence relation on X, along with the canonical projection $\pi\colon X \twoheadrightarrow X/{\sim}$. The **quotient topology** on $X/{\sim}$ is the largest topology making π continuous.

Explicitly, a subset $U \subseteq X/{\sim}$ is open if and only if its preimage $\pi^{-1}(U) \subseteq X$ is open in X.

Proposition 3.2. *With the quotient topology on $X/{\sim}$, a map $g\colon X/{\sim} \to Z$ is continuous if and only if the composite $g \circ \pi\colon X \to Z$ is continuous.*

Proof. Exercise. □

Proposition 3.3. *The space $X/{\sim}$ endowed with the quotient topology satisfies the universal property of a quotient. More precisely, the projection $\pi\colon X \twoheadrightarrow X/{\sim}$ is continuous, and for any continuous map $f\colon X \to Z$ which is constant on equivalence classes, there is a unique continuous map $\overline{f}\colon X/{\sim} \to Z$ satisfying $f = \overline{f} \circ \pi$, i.e., making the diagram*



commute.

Proof. By the universal property of the projection map in sets, there is a unique function $\overline{f}\colon X/{\sim} \to Z$ such that $f = \overline{f} \circ \pi$. It remains to check that $\overline{f}$ is continuous. By Proposition 3.2, the fact that $\overline{f} \circ \pi$ is continuous guarantees that $\overline{f}$ is continuous. □

Exercise 3.4. Show that the interval with its endpoints identified $[0,1]/0 \sim 1$ is homeomorphic to the circle S^1.

Exercise 3.5. Consider the quotient group $\mathbb{R}/\mathbb{Z}$, where the equivalence relation on $\mathbb{R}$ is $x \sim x' \iff x - x' \in \mathbb{Z}$. Show that the quotient space is homeomorphic to the circle:

$$\mathbb{R}/\mathbb{Z} \cong S^1.$$

Definition 3.6. For $n \geq 0$, the standard ***n*-sphere** is

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

and the standard ***n*-disk** is

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

The boundary of the disk is a sphere of dimension one less:

$$\partial D^n = S^{n-1}.$$

Exercise 3.7. Show that the n-disk with its boundary collapsed to a point is homeomorphic to the n-sphere S^n:

$$D^n/\partial D^n \cong S^n.$$

3.2 Quotient maps

Definition 3.8. Let X and Y be topological spaces. A map $q\colon X \to Y$ is called a **quotient map** or **identification map** if it is, up to homeomorphism, of the form $\pi\colon X \twoheadrightarrow X/\sim$ where $X/\sim$ is endowed with the quotient topology. More precisely, q is a quotient map if there exists an equivalence relation $\sim$ on X and a homeomorphism $\varphi\colon X/\sim \xrightarrow{\cong} Y$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ & {\scriptstyle\pi}\searrow & \uparrow{\scriptstyle\varphi}\ {\scriptstyle\cong} \\ & & X/\sim \end{array}$$

commute.

The definition implies that q must be continuous and surjective, and that the equivalence relation $\sim$ on X must be the one induced by q, namely $x \sim x'$ if and only if $q(x) = q(x')$.

How to recognize quotient maps? In sets, a quotient map is the same as a surjection. However, in topological spaces, being continuous and surjective is *not* enough to be a quotient map. The crucial property of a quotient map is that open sets $U \subseteq X/\sim$ can be "detected" by looking at their preimage $\pi^{-1}(U) \subseteq X$.

Proposition 3.9. *Let $q\colon X \twoheadrightarrow Y$ be a surjective continuous map satisfying that $U \subseteq Y$ is open if and only if its preimage $q^{-1}(U) \subseteq X$ is open. Then q is a quotient map.*

Proof. Let $\sim$ be the equivalence relation on X induced by q, that is, $x \sim x'$ if and only if $q(x) = q(x')$. By definition, $q\colon X \to Y$ is constant on equivalence classes. By the universal property of the quotient space $X/\sim$, there is a unique continuous map $\overline{q}\colon X/\sim \to Y$ satisfying $\overline{q} \circ \pi = q$, i.e., making the diagram

$$\begin{array}{ccc} X & \overset{q}{\twoheadrightarrow} & Y \\ {\scriptstyle\pi}\downarrow & \nearrow{\scriptstyle\exists!\overline{q}} & \\ X/\sim & & \end{array}$$

commute. By construction, $\overline{q}$ is now bijective. To prove that it is a homeomorphism, it remains to show that it is an open map.

Let $U \subseteq X/\sim$ be open. We want to show that $\overline{q}(U) \subseteq Y$ is open. By assumption, q has the property of "detecting" open subsets of Y, i.e., it suffices to check that the preimage $q^{-1}(\overline{q}(U)) \subseteq X$ is open. This preimage is

$$\begin{aligned} q^{-1}(\overline{q}(U)) &= (\overline{q} \circ \pi)^{-1}(\overline{q}(U)) \\ &= \pi^{-1}\overline{q}^{-1}(\overline{q}(U)) \\ &= \pi^{-1}(U) && \text{since } \overline{q} \text{ is injective} \end{aligned}$$

which is open in X since $\pi\colon X \twoheadrightarrow X/\sim$ is continuous. $\square$

Example 3.10. Let $f\colon X \to Y$ be a continuous bijection. If f is a quotient map, then it must be a homeomorphism.

As discussed on September 7, a continuous bijection need *not* be a homeomorphism. Hence those examples were also examples of continuous surjections that are *not* quotient maps.

Example 3.11. The winding map

$$p\colon \mathbb{R} \twoheadrightarrow S^1$$
$$p(t) = (\cos(2\pi t), \sin(2\pi t))$$

is a quotient map. Its restriction

$$p|_{[0,1]}\colon [0,1] \twoheadrightarrow S^1$$

is also a quotient map. However, recall that the restriction

$$p|_{[0,1)}\colon [0,1) \twoheadrightarrow S^1$$

is *not* a quotient map.

Exercise 3.12. Let $f\colon X \twoheadrightarrow Y$ be a surjective continuous map.

(a) If f is an open map, show that f is a quotient map.

(b) If f is a closed map, show that f is a quotient map.

(c) Find an example of a *quotient* map $q\colon \mathbb{R} \twoheadrightarrow Y$ which is neither an open map nor a closed map.

Exercise 3.13. Consider the quotient group $\mathbb{R}/\mathbb{Q}$, where the equivalence relation on $\mathbb{R}$ is $x \sim x' \iff x - x' \in \mathbb{Q}$. Show that the quotient topology on $\mathbb{R}/\mathbb{Q}$ is indiscrete, i.e., only the empty set $\emptyset$ and the entire set $\mathbb{R}/\mathbb{Q}$ are open.

4 Closure and limit points

4.1 Closure

Definition 4.1. Let X be a topological space and $A \subseteq X$ a subset. The **closure** of A, denoted $\overline{A}$, is the smallest closed subset of X that contains A.

A point $x \in \overline{A}$ is called a **closure point** of A.

Lemma 4.2. *The closure of any subset $A \subseteq X$ exists.*

Proof. The closed subset

$$\overline{A} = \bigcap_{\substack{C \subseteq X \text{ closed} \\ A \subseteq C}} C$$

satisfies the required property. □

Proposition 4.3. *A point $x \in X$ satisfies $x \in \overline{A}$ if and only if every neighborhood of x intersects A.*

Remark 4.4. The following conditions are also equivalent to $x \in \overline{A}$:

- Every open neighborhood of x intersects A.
- Every basic open neighborhood of x intersects A (with respect to some base of the topology on X).

The equivalence relies on the usual argument (cf. Homework 2 Problem 1):

- Every neighborhood of x contains an open neighborhood of x (by definition of neighborhood).
- Every open neighborhood of x contains a basic open neighborhood of x.

Definition 4.5. Let (X, d) be a metric space and $A \subseteq X$ a subset. The **distance** from a point $x \in X$ to the subset A is

$$d(x, A) := \inf_{a \in A} d(x, a).$$

Proposition 4.6. *Let (X, d) be a metric space and $A \subseteq X$ a subset. Then a point $x \in X$ satisfies $x \in \overline{A}$ if and only if $d(x, A) = 0$ holds.*

Proof. Consider the equivalent conditions:

$x \in \overline{A}$.

$\iff$ For all $\epsilon > 0$, we have $B(x, \epsilon) \cap a \neq \emptyset$.

$\iff$ For all $\epsilon > 0$, there is a point $a \in A$ satisfying $d(x,a) < \epsilon$.

$\iff$ $\inf_{a \in A} d(x,a) = 0 = d(x,A)$. □

Definition 4.7. Let X be a topological space. A subset $A \subseteq X$ is called **dense** in X if its closure is all of X, that is, $\overline{A} = X$.

Proposition 4.8. *A subset $A \subseteq X$ is dense in X if and only if every non-empty open subset of X contains a point of A.*

Proof. Consider the equivalent conditions on A:

A is dense in X, that is, $\overline{A} = X$.

$\iff$ For all $x \in X$, we have $x \in \overline{A}$.

$\iff$ For all $x \in X$ and for all open neighborhood U of x, we have $U \cap A \neq \emptyset$.

$\iff$ For all non-empty open subset $U \subseteq X$, we have $U \cap A \neq \emptyset$. □

Remark 4.9. By the same argument as in Remark 4.4, A is dense in X if and only if every non-empty *basic* open subset of X contains a point of A (for some base of the topology on X).

Example 4.10. The rational numbers $\mathbb{Q} \subset \mathbb{R}$ are dense in $\mathbb{R}$.

The dyadic rational numbers

$$\mathbb{Z}[\frac{1}{2}] = \{\frac{p}{2^k} \mid p \in \mathbb{Z}, k \geq 0\}$$

are also dense in $\mathbb{R}$.

Example 4.11. Let X be a set equipped with the cofinite topology. Then any infinite subset $A \subseteq X$ is dense in X.

4.2 Limit points

Definition 4.12. In a topological space X, a **punctured neighborhood** of a point $x \in X$ is a subset of the form $N \setminus \{x\}$ where $N \subseteq X$ is a neighborhood of x.

Example 4.13. In $\mathbb{R}$, the subset $(3,4) \cup (4,6]$ is a punctured neighborhood of 4, namely $(3,6] \setminus \{4\}$.

Definition 4.14. Let X be a topological space and $A \subseteq X$ a subset. A point $x \in X$ is a **limit point** of A if every punctured neighborhood of x intersects A. That is, for every neighborhood N of x, we have $(N \setminus \{x\}) \cap A \neq \emptyset$.

The set of limit points of A is sometimes denoted A'.

Proposition 4.15. *Let X be a topological space and $A \subseteq X$ a subset. Then a point $x \in X$ is a closure point of A if and only if x belongs to A or is a limit point of A:*

$$\overline{A} = A \cup A'.$$

Example 4.16. In the real line $\mathbb{R}$, consider the subset $A = \{2\} \cup (5,7]$. Its set of limit points is $A' = [5,7]$ and its closure is $\overline{A} = \{2\} \cup [5,7]$.

Now consider the subset $B = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Its set of limit points is $B' = \{0\}$ and its closure is $\overline{B} = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$.

Remark 4.17. The following conditions on a subset $A \subseteq X$ are equivalent:

- $A \subseteq X$ is closed.
- A equals its closure: $A = \overline{A}$.
- A contains its closure points: $\overline{A} \subseteq A$.
- A contains its limit points: $A' \subseteq A$.

Definition 4.18. Let X be a topological space and $A \subseteq X$ a subset. A point $a \in A$ is an **isolated point** of A if a admits a neighborhood N that contains no other point of a:

$$N \cap A = \{a\}.$$

The negation of Definition 4.14 yields the following.

Proposition 4.19. *A point $a \in A$ is an isolated point of A if and only if it is not a limit point of A:*

$$\{\textit{isolated points of } A\} = A \setminus A'.$$

Example 4.20. In Example 4.16, the only isolated point of $A = \{2\} \cup (5,7]$ is the point 2.

In contrast, all points of $B = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ are isolated points of B.

4.3 Interior

Definition 4.21. Let X be a topological space and $A \subseteq X$ a subset. The **interior** of A, denoted $\operatorname{int}(A)$, is the largest open subset of X contained in A.

Lemma 4.22. *The interior of any subset $A \subseteq X$ exists.*

Proof. The open subset

$$\operatorname{int}(A) = \bigcup_{\substack{U \subseteq X \text{ open} \\ U \subseteq A}} U$$

satisfies the required property. □

Definition 4.23. A point $x \in X$ is an **interior point** of A if there exists an open subset $U \subseteq X$ satisfying $x \in U \subseteq A$.

With that terminology, the interior $\operatorname{int}(A)$ is the set of all interior points of A.

Example 4.24. In $\mathbb{R}$, the interior of the subset $A = \{2\} \cup (5, 7]$ is $\operatorname{int}(A) = (5, 7)$.

The subset $\mathbb{Q} \subset \mathbb{R}$ has empty interior: $\operatorname{int}(\mathbb{Q}) = \emptyset$.

Example 4.25. In $\mathbb{R}^n$, the interior of the unit disk is the open unit disk:

$$\operatorname{int}(D^n) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}.$$

Remark 4.26. The following conditions on a subset $A \subseteq X$ are equivalent:

- $A \subseteq X$ is open.
- A equals its interior: $A = \operatorname{int}(A)$.
- Every point of A is an interior point of A.

Proposition 4.27. *Let X be a topological space and $A \subseteq X$ a subset. Then we have*

$$\operatorname{int}(A)^c = \overline{A^c}$$

as well as

$$(\overline{A})^c = \operatorname{int}(A^c).$$

Proof. Let us describe the complement of the interior:

$$
\begin{aligned}
\operatorname{int}(A)^c &= \left(\bigcup_{\substack{U \subseteq X \text{ open} \\ U \subseteq A}} U \right)^c && \text{by Lemma 4.22} \\
&= \bigcap_{\substack{U \subseteq X \text{ open} \\ U \subseteq A}} U^c && \text{by De Morgan's law} \\
&= \bigcap_{\substack{U \subseteq X \text{ open} \\ U^c \supseteq A^c}} U^c \\
&= \bigcap_{\substack{C \subseteq X \text{ closed} \\ A^c \subseteq C}} C && \text{relabeling } C := U^c \\
&= \overline{A^c} && \text{by Lemma 4.2.}
\end{aligned}
$$

The prove the second part, apply the first part to the subset A^c:

$$\operatorname{int}(A^c)^c = \overline{A}$$

and take complements. □

5 Sequences

5.1 Continuity via the closure

Proposition 5.1. *A function $f\colon X \to Y$ between topological spaces is continuous if and only if the inclusion $f(\overline{A}) \subseteq \overline{f(A)}$ holds for every subset $A \subseteq X$.*

Proof. Recall that f is continuous if and only if for every closed subset $C \subseteq Y$, the preimage $f^{-1}(C) \subseteq X$ is closed.

($\Longrightarrow$) Let $A \subseteq X$ be a subset. We want to show the inclusion

$$\begin{aligned} & f(\overline{A}) \subseteq \overline{f(A)} \\ \iff & \overline{A} \subseteq f^{-1}(\overline{f(A)}). \end{aligned}$$

Since f is continuous and $\overline{f(A)} \subseteq Y$ is closed, the preimage $f^{-1}(\overline{f(A)}) \subseteq X$ is closed. Moreover, the following inclusion holds:

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}).$$

Since $\overline{A}$ is the smallest closed subset containing A, we conclude $\overline{A} \subseteq f^{-1}(\overline{f(A)})$, as desired.

($\Longleftarrow$) Let $C \subseteq Y$ be a closed subset. We want to show that the preimage $f^{-1}(C) \subseteq X$ is closed, or equivalently:

$$\begin{aligned} & \overline{f^{-1}(C)} \subseteq f^{-1}(C) \\ \iff & f(\overline{f^{-1}(C)}) \subseteq C. \end{aligned}$$

Let us show that that inclusion does hold:

$$\begin{aligned} f(\overline{f^{-1}(C)}) &\subseteq \overline{f(f^{-1}(C))} && \text{by assumption on } f \\ &\subseteq \overline{C} && \text{since } f(f^{-1}(C)) \subseteq C \\ &= C && \text{since } C \text{ is closed.} \quad \square \end{aligned}$$

5.2 Sequences

Definition 5.2. A **sequence** $(x_n)_{n\in\mathbb{N}}$ in a topological space X is a family of points $x_n \in X$ indexed by the natural numbers $n \in \mathbb{N}$.

Definition 5.3. A sequence $(x_n)_{n\in\mathbb{N}}$ **converges** to a point $x \in X$ if for every neighborhood U of x, there exists an index $N \in \mathbb{N}$ such that $x_n \in U$ holds for all $n \geq N$.

We write $x_n \to x$ or $x_n \xrightarrow{n\to\infty} x$.

Definition 5.4. Let $U \subseteq X$ be a subset. A sequence $(x_n)_{n\in\mathbb{N}}$ is **eventually** in U if there exists an index $N \in \mathbb{N}$ such that $x_n \in U$ holds for all $n \geq N$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is **frequently** (or *often*) in U if for every index $m \in \mathbb{N}$, there is an index $k \geq m$ such that $x_k \in U$ holds.

In this terminology, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ if for every neighborhood U of x, the sequence $(x_n)_{n\in\mathbb{N}}$ is eventually in U.

Example 5.5. If a sequence $(x_n)_{n\in\mathbb{N}}$ is eventually constant at a point $x \in X$, then $(x_n)_{n\in\mathbb{N}}$ converges to x.

Remark 5.6. Changing finitely many terms of a sequence does not change its convergence behavior. If (x_n) and (y_n) are sequences in X that eventually coincide, i.e., there is an index $N \in \mathbb{N}$ such that $x_n = y_n$ holds for all $n \geq N$, then we have $x_n \to x$ if and only if $y_n \to x$.

Example 5.7. In a discrete space X, the only convergent sequences $(x_n)_{n\in\mathbb{N}}$ are those that are eventually constant. Indeed, given $x_n \to x$, the sequence $(x_n)_{n\in\mathbb{N}}$ must be eventually in the neighborhood $\{x\}$ of x.

Exercise 5.8. Let X be a set endowed with the cocountable topology. Show that the only convergent sequences $(x_n)_{n\in\mathbb{N}}$ in X are those that are eventually constant.

Proposition 5.9. *In a metric space (X, d), a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if the distance $d(x_n, x)$ converges to 0:*

$$x_n \to x \iff d(x_n, x) \to 0.$$

Proof. Consider the following chain of equivalent conditions.

$$\begin{aligned}
& x_n \to x. \\
\overset{\text{def}}{\iff}\ & \text{For all neighborhood } U \text{ of } x, \text{ the sequence } (x_n) \text{ is eventually in } U. \\
\iff\ & \text{For all radius } \epsilon > 0, \text{ the sequence } (x_n) \text{ is eventually in the open ball } B(x,\epsilon). \\
\overset{\text{def}}{\iff}\ & \text{For all } \epsilon > 0, \text{ the distance } d(x_n, x) \text{ is eventually less than } \epsilon. \\
\iff\ & d(x_n, x) \to 0.
\end{aligned}$$

□

Warning! In a topological space, the limit of a sequence need *not* be unique.

Example 5.10. In an indiscrete space X, every sequence $(x_n)_{n\in\mathbb{N}}$ converges to every point $x \in X$.

Example 5.11. Consider $\mathbb{R}$ endowed with the cofinite topology and consider the sequence defined by $x_n = n$ for all $n \in \mathbb{N}$. Then that sequence $(x_n)_{n\in\mathbb{N}}$ converges to every point $x \in \mathbb{R}$.

The same argument works for any sequence $(x_n)_{n\in\mathbb{N}}$ with distinct values x_n.

We will see that in a Hausdorff space, the limit of a sequence is unique (if it exists).

5.3 Limit point versus limit of a sequence

Proposition 5.12. *Let X be a space and $A \subseteq X$ a subset. If $(a_n)_{n\in\mathbb{N}}$ is a sequence in A that converges to a point $x \in X$, then x is a closure point of X:*

$$a_n \to x \implies x \in \overline{A}.$$

Proof. Let U be a neighborhood of x. By the condition $a_n \to x$, there is an index $N \in \mathbb{N}$ such that $a_n \in U$ holds for all $n \geq N$. In particular, we have $a_N \in U \cap A \neq \emptyset$. □

In a metric space, the converse holds.

Proposition 5.13. *Let (X, d) be a metric space, $A \subseteq X$ a subset, and $x \in \overline{A}$. Then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A converging to x.*

Proof. For each $n \in \mathbb{N}$, the open ball $B(x, \frac{1}{n})$ must intersect A, since x is a closure point of A. Pick a point

$$a_n \in B(x, \frac{1}{n}) \cap A \neq \emptyset.$$

Then the sequence $(a_n)_{n\in N}$ in A converges to x:

$$\begin{aligned} & d(a_n, x) < \frac{1}{n} \xrightarrow{n\to\infty} 0 \\ \implies & d(a_n, x) \to 0 \\ \implies & a_n \to x \end{aligned}$$

using Proposition 5.9. □

The analogous statement is *not* true in a general topological space: not all closure points of A are limits of sequences in A.

Definition 5.14. Let X be a topological space and $A \subseteq X$ a subset. The **sequential closure** of A is the set of all limits of sequences in A, denoted

$$\overline{A}^{\text{seq}} = \{x \in X \mid \text{there is a sequence } (a_n)_{n\in\mathbb{N}} \text{ in } A \text{ with } a_n \to x\}.$$

The inclusions

$$A \subseteq \overline{A}^{\text{seq}} \subseteq \overline{A}$$

always hold, the first one because of constant sequences, the second one by Proposition 5.12.

Example 5.15. Let X be a set endowed with the cocountable topology. For any subset $A \subseteq X$, we have $\overline{A}^{\text{seq}} = A$, by Exercise 5.8.

On the other hand, any uncountably infinite subset A is dense in X, i.e., $\overline{A} = X$. For such a subset, the inclusion

$$\overline{A}^{\text{seq}} \subset \overline{A}$$

is proper.

6 Hausdorff spaces

6.1 Definitions and examples

Definition 6.1. A topological space X is **Hausdorff** (or T_2) if for any distinct points $x, y \in X$, there exist open subsets $U, V \subset X$ satisfying $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

In other words, distinct points can be separated by neighborhoods.

Example 6.2. Every metric space is Hausdorff. Given distinct points $x, y \in X$, we have $d(x, y) > 0$. Take the open balls $U = B(x, \frac{d(x,y)}{2})$ and $V = B(y, \frac{d(x,y)}{2})$. Then U and V are disjoint, by the triangle inequality.

Example 6.3. Every discrete space X is Hausdorff. Given distinct points $x, y \in X$, take the open neighborhoods $U = \{x\}$ and $V = \{y\}$.

Example 6.4. An indiscrete space X with at least two points is not Hausdorff. Indeed, the only neighborhood of a point x is the entire set X.

Example 6.5. Let X be a set endowed with the cofinite topology. Then the following are equivalent.

1. X is Hausdorff.
2. X is discrete.
3. X is finite.

Proof. The equivalence (2) $\iff$ (3) was proved on Homework 1 Problem 2. The implication (2) $\implies$ (1) is Example 6.3. Let us prove (1) $\implies$ (3) by contraposition.

Assume that X is infinite and pick any two points $x, y \in X$. Let U and V be neighborhoods of x and y respectively. Then U and V are cofinite:

$$U = X \setminus \{x_1, \ldots, x_m\}$$
$$V = X \setminus \{y_1, \ldots, y_n\}.$$

Since X is infinite, the intersection

$$U \cap V = X \setminus \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$

must contain a point (in fact, infinitely many). □

Proposition 6.6. *1. A subspace of a Hausdorff space is Hausdorff.*

2. An arbitrary product of Hausdorff spaces is Hausdorff.

3. An arbitrary coproduct of Hausdorff spaces is Hausdorff.

Proof. (1) Let X be a Hausdorff space and $A \subseteq X$ a subspace. Let $a, b \in A$ be distinct points. Since X is Hausdorff, there exist open subsets $U, V \subset X$ satisfying $a \in U$, $b \in V$, and $U \cap V = \emptyset$. Then $U \cap A$ is an open subset of A containing a, $V \cap A$ is an open subset of A containing b, and they are still disjoint:

$$(U \cap A) \cap (V \cap A) = U \cap V \cap A = \emptyset \cap A = \emptyset.$$

(2) Let $\{X_\alpha\}$ be a family of Hausdorff spaces. We want to show that the product $\prod_\alpha X_\alpha$ is Hausdorff. Let $x = (x_\alpha)$ and $y = (y_\alpha)$ be distinct points in $\prod_\alpha X_\alpha$. Then there exists some index α_0 for which x and y have distinct components:

$$x_{\alpha_0} \neq y_{\alpha_0} \in X_{\alpha_0}.$$

Since X_{α_0} was assumed Hausdorff, there exist disjoint open subsets $U_{\alpha_0}, V_{\alpha_0} \subset X_{\alpha_0}$ satisfying $x_{\alpha_0} \in U_{\alpha_0}$ and $y_{\alpha_0} \in V_{\alpha_0}$. Define the "large open box" $U = \prod_\alpha U_\alpha \subset \prod_\alpha X_\alpha$ by

$$U_\alpha = \begin{cases} U_{\alpha_0} & \text{if } \alpha = \alpha_0 \\ X_\alpha & \text{if } \alpha \neq \alpha_0. \end{cases}$$

Then U is open in the product topology and contains the point x. Similarly, define the "large open box" $V = \prod_\alpha V_\alpha \subset \prod_\alpha X_\alpha$ by

$$V_\alpha = \begin{cases} V_{\alpha_0} & \text{if } \alpha = \alpha_0 \\ X_\alpha & \text{if } \alpha \neq \alpha_0. \end{cases}$$

Then V is an open neighborhood of the point y. Moreover, U and V are disjoint:

$$\begin{aligned} U \cap V &= (\prod_\alpha U_\alpha) \cap (\prod_\alpha V_\alpha) \\ &= \prod_\alpha (U_\alpha \cap V_\alpha) \\ &= \emptyset \end{aligned}$$

because of the condition $U_{\alpha_0} \cap V_{\alpha_0} = \emptyset$.

(3) See Homework 6 Graduate Problem 3. □

Remark 6.7. Quotients of Hausdorff spaces need *not* be Hausdorff.

Example 6.8 (Line with two origins)**.** Consider the disjoint union of two lines $\mathbb{R} \times \{1, 2\} \cong \mathbb{R} \coprod \mathbb{R}$. Take the quotient space

$$X = (\mathbb{R} \times \{1, 2\})/\sim$$

where the equivalence relation is generated by $(t, 1) \sim (t, 2)$ for all $t \neq 0$. In other words, we glue together the two lines everywhere except at the origin.

This space X is not Hausdorff, because the two distinct origins $[(0, 1)]$ and $[(0, 2)]$ cannot be separated by neighborhoods. For any open neighborhoods U of $[(0, 1)]$ and V of $[(0, 2)]$ in X, we have $U \cap V \neq \emptyset$.

6.2 Properties of Hausdorff spaces

Definition 6.9. The **diagonal** of a topological space X is the set

$$\Delta := \{(x,x) \in X \times X \mid x \in X\} \subseteq X \times X.$$

Proposition 6.10. *A topological space X is Hausdorff if and only if the diagonal Δ is closed in $X \times X$.*

Proof. Consider the following equivalent statements.

The diagonal $\Delta \subseteq X \times X$ is closed.

$\iff$ The complement $X \times X \setminus \Delta \subseteq X \times X$ is open.

$\iff$ For every $(x,y) \notin \Delta$, there is a basic open neighborhood $U \times V$ of (x,y) satisfying $(U \times V) \cap \Delta = \emptyset$.

$\iff$ For every distinct points $x, y \in X$, there are open subsets $U, V \subset X$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

$\overset{\text{def}}{\iff}$ X is Hausdorff. □

Proposition 6.11 (Uniqueness of limits)**.** *Let X be a Hausdorff space, and $(x_n)_{n\in\mathbb{N}}$ a sequence in X with $x_n \to x$ and $x_n \to y$. Then $x = y$.*

In other words: Limits of sequences are unique (if they exist).

Proof. Assume $x_n \to x$ and let $y \in X$ be a point distinct from x. Since X is Hausdorff, there exist disjoint open subsets $U, V \subset X$ with $x \in U$ and $y \in V$. Since the sequence (x_n) converges to x, there exists an index N such that $x_n \in U$ holds for all $n \geq N$. Since U and V are disjoint, the condition $x_n \notin V$ holds for all $n \geq N$. In particular, (x_n) is not eventually in V, so that (x_n) cannot converge to y. □

Remark 6.12. In a Hausdorff space X, we can safely write the condition $x_n \to x$ as

$$\lim_{n\to\infty} x_n = x.$$

There is no ambiguity, since the sequence $(x_n)_{n\in\mathbb{N}}$ has at most one limit.

7 Countability axioms

7.1 First-countable spaces

Definition 7.1. Let X be a topological space. A **neighborhood base** at a point $x \in X$ is a collection $\mathcal{B}_x$ of neighborhoods of x such that for any neighborhood N of x, there is some $B \in \mathcal{B}$ satisfying $B \subseteq N$.

Remark 7.2. Given a base $\mathcal{B}$ of the topology on X, for any $x \in X$, the collection

$$\mathcal{B}_x := \{B \in \mathcal{B} \mid x \in B\}$$

is a neighborhood base at x. Conversely, given a neighborhood base $\mathcal{B}_x$ for each point $x \in X$, their union

$$\mathcal{B} := \bigcup_{x \in X} \mathcal{B}_x$$

forms a base of the topology on X.

Definition 7.3. A topological space X is **first-countable** if every point $x \in X$ has a countable neighborhood base.

Example 7.4. Every metric space is first-countable. For $x \in X$, consider the neighborhood base

$$\mathcal{B}_x = \{B(x, r) \mid r > 0, r \in \mathbb{Q}\}$$

consisting of open balls around x of rational radius.

The collection of open balls

$$\{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$$

is another countable neighborhood base at x.

Remark 7.5. Given a countable neighborhood base $\mathcal{B}_x$ at a point x, choose a labeling $\mathcal{B}_x = \{B_1, B_2, B_3, \ldots\}$. Without loss of generality, we may assume that the B_n are nested (decreasing), by replacing B_n with $\bigcap_{i=1}^n B_i$. The collection

$$\{\bigcap_{i=1}^n B_i \mid n \in \mathbb{N}\}$$

also forms a neighborhood base at x.

Proposition 7.6. *Let X be a first-countable topological space and $A \subseteq X$ a subset. Let $x \in \overline{A}$ be in the closure of A. Then there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in A satisfying $a_n \to x$.*

In other words, the sequential closure of A coincides with the closure: $\overline{A}^{\text{seq}} = \overline{A}$.

Corollary 7.7. *Let X be a first-countable topological space.*

1. *A subset $C \subseteq X$ is closed if and only if whenever a sequence $(x_n)_{n\in\mathbb{N}}$ in C satisfies $x_n \to x$, then we have $x \in C$.*

2. *A subset $U \subseteq X$ is open if and only if whenever a sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfies $x_n \to x \in U$, then the sequence is eventually in U.*

Example 7.8. The space $\mathbb{R}^{\mathbb{N}}$ with the box topology is *not* first-countable. On Homework 5 Problem 2, we found a subset $A = \{x \in \mathbb{R}^{\mathbb{N}} \mid x_n > 0 \text{ for all } n \in \mathbb{N}\}$ and a point $\underline{0} = (0, 0, 0, \ldots) \in \overline{A}$ which is not the limit of any sequence in A.

Example 7.9. Let X be an uncountable set equipped with the cocountable topology. Then X is *not* first-countable. In the notes from September 28 Example 3.4, we saw that for any uncountable subset $A \subset X$ and $x \in X \setminus A$, the point x is in $\overline{A}$ but is not the limit of any sequence in A.

Example 7.10. Let X be an uncountable set equipped with the cofinite topology. Then X is *not* first-countable. (See Homework 6 Problem 2.)

This can be shown directly. It *cannot* be shown using sequences, since in the cofinite topology, the sequential closure coincides with the closure:

$$\overline{A}^{\text{seq}} = \overline{A} = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite.} \end{cases}$$

Proposition 7.11. 1. *A subspace of a first-countable space is first-countable.*

2. *A* countable *product of first-countable spaces is first-countable.*

3. *An arbitrary coproduct of first-countable spaces is first-countable.*

Proof. (1) Let X be a first-countable space and $A \subseteq X$ a subset, endowed with the subspace topology. Let $a \in A$. Since X is first-countable, the point $a \in X$ admits a countable neighborhood base $\mathcal{B}_a$ in X. Then the collection

$$\{B \cap A \mid B \in \mathcal{B}_a\}$$

is a countable neighborhood base at a in A.

(2) Let $\{X_i\}_{i\in\mathbb{N}}$ be a countable family of first-countable spaces. We want to show that their product $\prod_{i\in\mathbb{N}} X_i$ (with the product topology) is also first-countable.

Let $x = (x_1, x_2, \ldots) \in \prod_{i\in\mathbb{N}} X_i$. We want to find a countable neighborhood base at x. Because each space X_i is first-countable, there is a countable neighborhood base $\mathcal{B}_{x_i}$ at $x_i \in X_i$. Without loss of generality, assume $X_i \in \mathcal{B}_{x_i}$ (which will simplify the notation). Consider the collection of subsets of $\prod_{i\in\mathbb{N}} X_i$

$$\mathcal{B}_x := \{\prod_{i\in\mathbb{N}} B_i \mid B_i \in \mathcal{B}_{x_i} \text{ and } B_i \neq X_i \text{ for at most finitely many } i\}.$$

Note that each $B \in \mathcal{B}$ is a neighborhood of x. We claim that $\mathcal{B}$ is a countable neighborhood base at x.

$\mathcal{B}_x$ **is a neighborhood base at** x**.** Any neighborhood of x contains a basic open neighborhood $\prod_i U_i$ of x, where $U_i \subseteq X_i$ is open, and $U_i \neq X_i$ for at most finitely many i.

Since $\mathcal{B}_{x_i}$ is a neighborhood base at $x_i \in X_i$, there is some $B_i \in \mathcal{B}_{x_i}$ satisfying $B_i \subseteq U_i$, where we pick $B_i = X_i$ for every index i satisfying $U_i = X_i$. By construction, we have $B := \prod_i B_i \in \mathcal{B}_x$ and $B \subseteq \prod_i U_i$.

$\mathcal{B}_x$ **is countable.** Rewrite the collection $\mathcal{B}_x$ as

$$\begin{aligned}\mathcal{B}_x &= \{\prod_{i\in\mathbb{N}} B_i \mid B_i \in \mathcal{B}_{x_i} \text{ and } B_i \neq X_i \text{ for at most finitely many } i\} \\ &= \bigcup_{n\in\mathbb{N}} \{\prod_{i\in\mathbb{N}} B_i \mid B_i \in \mathcal{B}_{x_i} \text{ and } B_i \neq X_i \text{ possibly for } i \leq n \text{ but } B_i = X_i \text{ for } i > n\} \\ &=: \bigcup_{n\in\mathbb{N}} \mathcal{B}_x^{(n)}.\end{aligned}$$

Each of those "finitely supported" subcollections $\mathcal{B}_x^{(n)}$ is in bijection with

$$\mathcal{B}_x^{(n)} \cong \prod_{i=1}^{n} \mathcal{B}_{x_i}$$

where the latter is a finite product of countable sets, hence countable. Therefore $\mathcal{B}_x$ is a countable union of countable sets, hence countable.

Part (3) is left as an exercise. □

7.2 Sequential continuity

Definition 7.12. Let X and Y be topological spaces. For $x \in X$, a function $f \colon X \to Y$ is **sequentially continuous at x** if whenever $x_n \to x$ holds, we have $f(x_n) \to f(x)$. The function f is **sequentially continuous** if it is sequentially continuous at x for every $x \in X$.

Proposition 7.13. *Let X and Y be topological spaces, where X is first-countable. A function $f \colon X \to Y$ is continuous at $x \in X$ if and only if whenever $x_n \to x$ holds, we have $f(x_n) \to f(x)$.*

In other words, continuity always implies sequential continuity, but if the domain X is first-countable, then continuity is equivalent to sequential continuity.

Proof. ($\Longrightarrow$) That implication holds for all topological spaces.

($\Longleftarrow$) Assume f is discontinuous at $x \in X$, which means there is a neighborhood N of $f(x)$ such that for any neighborhood M of x, we have $f(M) \nsubseteq N$. Since X is first-countable, there is a countable neighborhood base $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ of x. Because of the condition $f(M_n) \nsubseteq N$, we can pick $x_n \in M_n$ satisfying $f(x_n) \notin N$.

Then the sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfies $x_n \to x$ but $f(x_n)$ is never in N, so in particular we have $f(x_n) \nrightarrow f(x)$. □

Alternate proof. Recall that f is continuous if and only if for every subset $A \subseteq X$, the inclusion $f(\overline{A}) \subseteq \overline{f(A)}$ holds. The inclusion $\overline{A}^{\text{seq}} \subseteq \overline{A}$ holds in any space. Since X is first-countable, the sequential closure coincides with the closure: $\overline{A}^{\text{seq}} = \overline{A}$. The assumption that f is sequentially continuous guarantees $f(\overline{A}^{\text{seq}}) \subseteq \overline{f(A)}^{\text{seq}}$. Combining those inclusions yields

$$f(\overline{A}) = f(\overline{A}^{\text{seq}}) \subseteq \overline{f(A)}^{\text{seq}} \subseteq \overline{f(A)}.$$

□

7.3 Second-countable spaces

Definition 7.14. A topological space X is **second-countable** if its topology has a countable base.

Example 7.15. Euclidean space $\mathbb{R}^n$ is second-countable, because the collection

$$\mathcal{B} = \{B(x, r) \mid x \in \mathbb{Q}^n, r > 0, r \in \mathbb{Q}\}$$

consisting of open balls of rational radius around points with rational coordinates is a base for the topology, and $\mathcal{B}$ is a countable collection.

Proposition 7.16. *A second-countable space is always first-countable.*

Proof. Let $\mathcal{B}$ be a countable base for the topology of X, and let $x \in X$. Then the collection

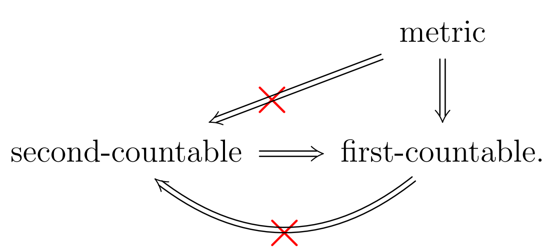
$$\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in B\}$$

is a neighborhood base for x, and it is countable. □

Remark 7.17. The converse does not hold. For example, consider X an uncountable set endowed with the discrete topology. Then X is first-countable but not second-countable.

Remark 7.18. We have seen that a metric space is always first-countable. However, it need not be second-countable. For example, consider again X an uncountable set endowed with the discrete topology. Then X is metrizable but not second-countable.

In diagrams:



Definition 7.19. A topological space X is called **separable** if it contains a countable dense subset.

Proposition 7.20. *A second-countable space is always separable.*

Proof. Let $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ be a countable base for the topology of X. Pick a point $b_i \in B_i$ in each basic open, and consider the set $A := \{b_i \mid i \in \mathbb{N}\}$. Then A is countable, and moreover it is dense in X.

Indeed, any non-empty open subset $U \subseteq X$ is a union of basic open subsets $U = \bigcup_{j \in J} B_j$ for some $J \subseteq \mathbb{N}$. Hence, U contains the points $b_j \in U \cap A$ for all $j \in J$. This shows that A is dense in X (by Proposition 1.8 of the notes from September 23). □

On Homework 6 Problem 2, you show that the converse fails: separable does *not* imply second-countable, in fact not even first-countable.

8 Compactness

8.1 Definitions and examples

Definition 8.1. Let X be a topological space.

- A **cover** of X is a collection $\{U_\alpha\}_{\alpha \in A}$ of subsets $U_\alpha \subseteq X$ satisfying $X = \bigcup_{\alpha \in A} U_\alpha$.
- An **open cover** of X is a cover $\{U_\alpha\}_{\alpha \in A}$ where each U_α is open in X.
- A **subcover** of $\{U_\alpha\}_{\alpha \in A}$ is a subcollection $\{U_\beta\}_{\beta \in B}$ (for some $B \subseteq A$) which is still a cover, that is, $X = \bigcup_{\beta \in B} U_\beta$.

Definition 8.2. A topological space X is **compact** if for every open cover $\{U_\alpha\}_{\alpha \in A}$ of X, there is a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$, that is, $X = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.

Example 8.3. A finite topological space is compact. Indeed, let $\{U_\alpha\}$ be an open cover of X. For each point $x \in X$, choose some $U_{\alpha(x)}$ containing x. Then we obtain a finite subcover

$$X = \bigcup_{x \in X} U_{\alpha(x)}.$$

Example 8.4. An indiscrete space is compact.

Example 8.5. More generally, a space endowed with a finite topology $\mathcal{T}$ is compact. Indeed, any collection of open subsets $\mathcal{U} \subseteq \mathcal{T}$ is then finite.

Exercise 8.6. Show that a space endowed with the cofinite topology is compact.

Example 8.7. A discrete space is compact if and only if it is finite.

Proof. ($\Longleftarrow$) Every finite space X is compact, by Example 8.3.

($\Longrightarrow$) Since the space X is discrete, each singleton $\{x\}$ is open in X. The equality

$$X = \bigcup_{x \in X} \{x\}$$

means that the collection $\{\{x\}\}_{x \in X}$ of all singletons is an open cover of X. Since X is compact, there is a finite subcover $\{\{x_1\}, \ldots, \{x_n\}\}$, which means

$$X = \{x_1\} \cup \ldots \cup \{x_n\} = \{x_1, \ldots, x_n\},$$

so that X is finite. □

Example 8.8. The real line $\mathbb{R}$ is not compact. Indeed, the open cover $\{(n-1, n+1) \mid n \in \mathbb{Z}\}$ admits no finite subcover.

Example 8.9. The interval $(0, 1]$ is not compact. Indeed, the open cover $\{(\frac{1}{n}, 1] \mid n \in \mathbb{N}, n \geq 2\}$ admits no finite subcover.

8.2 Facts about compactness

Proposition 8.10. *Let X be a topological space and $Y \subseteq X$ a subspace. Then Y is compact if and only if for every collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets $U_\alpha \subseteq X$ satisfying $Y \subseteq \bigcup_{\alpha \in A} U_\alpha$, there is a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ satisfying $Y \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.*

Proposition 8.11. *Let $K_1, \ldots, K_n$ be compact subspaces of X. Then their union $K_1 \cup \ldots \cup K_n$ is compact.*

Slogan: "finite union of compact is compact".

Proof. It suffices to prove the case $n = 2$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $K_1 \cup K_2$ in X, that is:

$$K_1 \cup K_2 \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

Since each K_i is compact, for $i = 1, 2$, there is a finite subcover

$$K_i \subseteq \bigcup_{\alpha \in B_i} U_\alpha$$

for some finite subset $B_i \subseteq A$. Taking the union of those finite indexing sets yields a finite set $B := B_1 \cup B_2 \subseteq A$ and a corresponding finite subcover

$$K_1 \cup K_2 \subseteq \bigcup_{\alpha \in B_1} U_\alpha \cup \bigcup_{\alpha \in B_2} U_\alpha = \bigcup_{\alpha \in B} U_\alpha.$$

□

Proposition 8.12. *Let $f \colon K \to Y$ be a continuous map between topological spaces, where K is compact. Then $f(K)$ is compact.*

Slogan: "continuous image of compact is compact".

Proof. Let $f(K) \subseteq \bigcup_{\alpha \in A} V_\alpha$ be an open cover of $f(K)$ in Y. Then their preimages form an open over of K:

$$\begin{aligned} K &= f^{-1}(f(K)) \\ &= f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha \right) \\ &= \bigcup_{\alpha \in A} f^{-1}(V_\alpha). \end{aligned}$$

Since K is compact, there is a finite subcover

$$\begin{aligned} K &= f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}) \\ &= f^{-1}(V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}). \end{aligned}$$

Applying f yields the inclusion

$$\begin{aligned} f(K) &\subseteq f\left(f^{-1}(V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}) \right) \\ &\subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n} \end{aligned}$$

which is a finite subcover of the given $\{V_\alpha\}_{\alpha \in A}$. □

Corollary 8.13. *A quotient of a compact space is always compact.*

Proposition 8.14. *Let K be a compact topological space and $C \subseteq K$ a closed subspace. Then C is compact.*

Slogan: "closed in compact is compact".

Proof. Let $C \subseteq \bigcup_{\alpha \in A} U_\alpha$ be an open cover of C in K. Then

$$K = C \cup C^c = \left(\bigcup_{\alpha \in A} U_\alpha \right) \cup C^c$$

is an open cover of K, since C^c is open in K. Since K is compact, there is a finite subcover

$$K = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cup C^c$$

where we included C^c without loss of generality (otherwise just add C^c to the finite subcover). This yields a finite subcover of C

$$C \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n},$$

showing that C is compact. □

Proposition 8.15. *Let X be a Hausdorff topological space and $K \subseteq X$ a compact subspace. Then K is closed in X.*

Slogan: "compact in Hausdorff is closed".

Proof. Let $x \in K^c = X \setminus K$. We want to show that x is an interior point of K^c. Since X is Hausdorff, for each $y \in K$, we can find U_y and V_y disjoint open neighborhoods of x and y respectively:

$$x \in U_y, \quad y \in V_y, \quad U_y \cap V_y = \emptyset.$$

Then $K \subseteq \bigcup_{y \in K} V_y$ is an open cover of K in X. Since K is compact, there is a finite subcover

$$K \subseteq V_{y_1} \cup \cdots \cup V_{y_n}.$$

The corresponding neighborhoods U_{y_i} of x yield an open neighborhood of x

$$U := U_{y_1} \cap \cdots \cap U_{y_n}$$

which is disjoint from K, since $U \cap V_{y_i} = \emptyset$ holds for each $i = 1, \ldots, n$. In other words, we have

$$x \in U \subseteq K^c,$$

which shows that x is an interior point of K^c. □

The assumption that X be Hausdorff cannot be dropped from the statement.

Example 8.16. Let X be an indiscrete space. Then every subspace $Y \subset X$ is compact, though most of them are not closed in X. Only the empty set $\emptyset$ and X itself are closed in X.

Example 8.17. Let X be a space with the cofinite topology. Then every subspace $Y \subset X$ is compact, though most of them are not closed in X. Only the finite subsets and X itself are closed in X.

Proposition 8.18. *Let $f\colon K \to Y$ be a continuous map between topological spaces, where K is compact and Y a Hausdorff. Then f is a closed map.*

In particular, if f is a continuous bijection, then f is a homeomorphism.

Proof. Let $C \subseteq K$ be a closed subset. Since K is compact and $C \subseteq K$ is closed, C is itself compact, by Proposition 8.14. Hence its image $f(C)$ under the continuous map f is also compact, by Proposition 8.12. Since Y is Hausdorff and $f(C)$ is compact, then $f(C) \subseteq Y$ is closed, by Proposition 8.15. □

8.3 An important example

A basic example of compact space, yet one of the most important, is provided by the following classic theorem.

Theorem 8.19 (Bolzano–Weierstrass)**.** *The interval* $[0,1]$ *is compact.*

Proof. Suppose $[0,1]$ is not compact, i.e., there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ which does not admit a finite subcover. Then either $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ (or both) cannot be covered by a finite subcover. Call this new interval $[a_1,b_1]$, where we write $[a_0,b_0] := [0,1]$.

Repeating the argument, for every $n \geq 0$, we obtain an interval $[a_n,b_n]$ which cannot be covered by a finite subcover, and each interval has length $b_n - a_n = \frac{1}{2^n}$. Moreover, the intervals are nested (decreasing):

$$[a_0,b_0] \supset [a_1,b_1] \supset [a_2,b_2] \supset \ldots$$

The sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are monotone and bounded, therefore they converge, say $a_n \to a$ and $b_n \to b$. We have

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty}\frac{1}{2^n} = b - a = 0$$

so that $a = b$. This point $a \in [0,1]$ lies in some U_{α_0}, which is open, so we can find some small radius $\epsilon > 0$ such that the open ball satisfies the inclusion $(a-\epsilon, a+\epsilon) \subseteq U_{\alpha_0}$. (To be nitpicky, we should instead write $(a-\epsilon,a+\epsilon)\cap[0,1]$, which is an open ball in $[0,1]$.)

By the convergence $a_n \to a$ and $b_n \to a$, for n large enough we have $[a_n,b_n] \subset (a-\epsilon,a+\epsilon) \subseteq U_{\alpha_0}$. These intervals $[a_n,b_n]$ can thus be covered by a finite subcover, namely the collection $\{U_{\alpha_0}\}$ consisting of only one member. This contradicts the construction of $[a_n,b_n]$. ☐

Remark 8.20. Any closed interval $[a,b] \subset \mathbb{R}$ is homeomorphic to $[0,1]$ and thus also compact.

Example 8.21. Consider the continuous map

$$f\colon [0,2\pi] \to S^1$$
$$t \mapsto (\cos t, \sin t)$$

which induces a continuous map on the quotient

$$\overline{f}\colon [0,2\pi]/{\sim} \to S^1$$

where the equivalence relation $\sim$ identifies the endpoints of the interval, i.e., is generated by $0 \sim 2\pi$. Then $\overline{f}$ is a continuous bijection, the domain $[0,2\pi]/{\sim}$ is compact (as a quotient of a compact space), and S^1 is Hausdorff. Therefore $\overline{f}$ is a homeomorphism.

9 The tube lemma and applications

9.1 The tube lemma

Lemma 9.1 (The tube lemma)**.** *Let X and Y be topological spaces with Y compact. Let $x \in X$ and let $O \subseteq X \times Y$ be an open subset containing the "slice" $\{x\} \times Y$. Then there exists a "tube" $U \times Y$ (where $U \subseteq X$ is open) satisfying*

$$\{x\} \times Y \subseteq U \times Y \subseteq O.$$

Proof. For each $y \in Y$, consider the point $(x, y) \in \{x\} \times Y \subseteq O$. Since O is open, there is a basic open neighborhood $B_y = U_{x,y} \times V_y$ of (x, y) satisfying

$$(x, y) \in U_{x,y} \times V_y \subset O.$$

As y varies, those open boxes cover the slice:

$$\{x\} \times Y \subseteq \bigcup_{y \in Y} B_y.$$

Note that the slice $\{x\} \times Y$ is homeomorphic to Y. Since Y is compact, there exists a finite subcover

$$\{x\} \times Y \subseteq B_{y_1} \cup \cdots B_{y_n}.$$

Take $U := U_{x,y_1} \cap \cdots U_{x,y_n}$, which is an open neighborhood of x in X. We claim that the "tube" $U \times Y$ is of the desired form. Indeed, the following inclusion holds:

$$\begin{aligned}
U \times Y &= U \times \left(\bigcup_{i=1}^{n} V_{y_i} \right) \\
&= \bigcup_{i=1}^{n} (U \times V_{y_i}) \\
&\subseteq \bigcup_{i=1}^{n} (U_{x,y_i} \times V_{y_i}) && \text{since } U \subseteq U_{x,y_i} \text{ for each } i \\
&= \bigcup_{i=1}^{n} B_{y_i} \\
&\subseteq \bigcup_{y \in Y} B_y \\
&\subseteq O && \text{since } B_y \subseteq O \text{ for each } y \in Y.
\end{aligned}$$

□

Theorem 9.2. *Let X and Y be compact spaces. Then their product $X \times Y$ is compact.*

Proof. We can show this using the tube lemma. See [Mun00, Theorem 26.7]. □

9.2 Closed graph theorem

Proposition 9.3. *Let X and Y be topological spaces with Y compact. Then the projection $p_X \colon X \times Y \to X$ is a closed map.*

Proof. Let $C \subseteq X \times Y$ be a closed subset. We want to show that its image $p_X(C) \subseteq X$ is closed. We will show that the complement $p_X(C)^c = X \setminus p_X(C)$ is open in X. Let $x \in p_X(C)^c$. This condition on x can be expressed as

$$\begin{aligned} x \in p_X(C)^c &\iff x \notin p_X(C) \\ &\iff \text{for all } (x', y') \in C,\ p_X(x', y') = x' \neq x \\ &\iff \text{for all } y \in Y,\ (x, y) \notin C \\ &\iff \{x\} \times Y \subseteq C^c = (X \times Y) \setminus C. \end{aligned}$$

Since $C^c \subseteq X \times Y$ is an open subset containing the slice $\{x\} \times Y$ and Y is compact, by the tube lemma, there is a tube $U \times Y$ satisfying

$$\{x\} \times Y \subseteq U \times Y \subseteq C^c.$$

Taking complements yields the inclusion

$$C \subseteq (U \times Y)^c = U^c \times Y.$$

Projecting onto X yields the inclusion

$$p_X(C) \subseteq p_X(U^c \times Y) = U^c.$$

Taking complements again yields

$$U \subseteq p_X(C)^c,$$

so that x is an interior point of $p_X(C)^c$. □

Remark 9.4. The assumption that Y be compact cannot be removed in general. For example, the projection $p_{[0,1]} \colon [0,1] \times \mathbb{R} \to [0,1]$ is not a closed map. See Homework 3 Problem 3(b).

Proposition 9.5. *Let $f \colon X \to Y$ be a function between topological spaces where Y is compact. If the graph $\Gamma_f \subseteq X \times Y$ is closed, then f is continuous.*

Proof. We will show that f is pointwise continuous at x for every $x \in X$. Let $x \in X$ and let $V \subseteq Y$ be an open neighborhood of $f(x)$. We want to find an open neighborhood U of x satisfying $f(U) \subseteq V$. Consider the slice $\{x\} \times Y \subseteq X \times Y$. For each $y \in Y$ with $y \neq f(x)$, the point (x, y) does not lie on the graph Γ_f. Since the graph $\Gamma_f \subseteq X \times Y$ was assumed closed, there is a basic open neighborhood $B_y = U_{x,y} \times V_y$ satisfying

$$(x, y) \in B_y \subseteq X \times Y \setminus \Gamma_f.$$

For $y = f(x)$, take $B_{f(x)} = X \times V$. As y varies, those open boxes cover the slice:

$$\{x\} \times Y \subseteq \bigcup_{y \in Y} B_y.$$

Since Y is compact, by the tube lemma, there is a tube $U \times Y$ satisfying

$$\{x\} \times Y \subseteq U \times Y \subseteq \bigcup_{y \in Y} B_y.$$

For all $x' \in U$, we have

$$(x', f(x')) \in U \times Y \subseteq \bigcup_{y \in Y} B_y.$$

But for $y \neq f(x)$, the open box B_y contains no point on the graph: $B_y \cap \Gamma_f = \emptyset$, hence $(x', f(x')) \notin B_y$. This forces $(x', f(x'))$ to be in the one remaining box:

$$(x', f(x')) \in B_{f(x)} = X \times V.$$

This shows $f(x') \in V$ and thus $f(U) \subseteq V$. □

Alternate proof. Let $C \subseteq Y$ be a closed subset. We want to show that its preimage $f^{-1}(C) \subseteq X$ is closed. Consider the equivalent conditions:

$$\begin{aligned} x \in f^{-1}(C) &\iff f(x) \in C \\ &\iff (x, f(x)) \in X \times C \\ &\iff \exists y \in Y \text{ with } (x, y) \in \Gamma_f \cap (X \times C) \\ &\iff x \in p_X(\Gamma_f \cap (X \times C)) \end{aligned}$$

yielding the equality

$$f^{-1}(C) = p_X(\Gamma_f \cap (X \times C)).$$

Since the graph $\Gamma_f \subseteq X \times Y$ was assumed closed, the intersection $\Gamma_f \cap (X \times C) \subseteq X \times Y$ is closed as well. Since Y is compact, the projection $p_X \colon X \times Y \to X$ is a closed map (by Proposition 9.3). Therefore, the subset

$$p_X(\Gamma_f \cap (X \times C)) = f^{-1}(C)$$

is closed in X. □

Corollary 9.6 (Closed graph theorem)**.** *Let $f \colon X \to Y$ be a function between topological spaces with Y compact Hausdorff. Then f is continuous if and only if its graph $\Gamma_f \subseteq X \times Y$ is closed.*

Proof. The implication ($\implies$) follows from Hausdorffness of Y and Homework 6 Problem 1(a). The implication ($\impliedby$) follows from compactness of Y and Proposition 9.5. □

Remark 9.7. The assumption that Y be compact cannot be removed in general. For example, consider the function $f \colon [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x > 0. \end{cases}$$

Then the graph $\Gamma_f \subseteq [0, 1] \times \mathbb{R}$ is closed, but f is not continuous.

9.3 Compactness in $\mathbb{R}^n$

Definition 9.8. Let (X, d) be a metric space. The **diameter** of a subset $A \subseteq X$ is

$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y).$$

A subset $A \subseteq X$ is called **bounded** if it has finite diameter. Equivalently, the distance between points in A is bounded above: there exists an $M \in \mathbb{R}$ such that the inequality $d(x, y) \leq M$ holds for all $x, y \in A$.

Example 9.9. A finite union of balls $\bigcup_{i=i}^{n} B(x_i, r_i)$ is bounded.

Example 9.10. A subset of a bounded set is also bounded.

Exercise 9.11. Show that a subset $A \subseteq X$ is bounded if and only if it is contained in some ball, i.e., there exists a point $x \in X$ and a radius $r > 0$ satisfying $A \subseteq B(x, r)$.

Remark 9.12. Boundedness is a metric property but not a topological invariant. For example, consider the real line $\mathbb{R}$ and the open interval $(0, 1)$ with their usual metrics. They are homeomorphic: $\mathbb{R} \cong (0, 1)$, but $\mathbb{R}$ is unbounded whereas $(0, 1)$ is bounded.

Lemma 9.13. *Let (X, d) be a metric space and $K \subseteq X$ a compact subspace. Then K is bounded.*

Proof. Pick any point $x \in X$ and consider the open cover by increasingly large open balls

$$K \subseteq \bigcup_{n \in \mathbb{N}} B(x, n).$$

Since K is compact, there is a finite subcover

$$K \subseteq B(x, n_1) \cup \ldots \cup B(x, n_k) = B(x, N)$$

with $N = \max\{n_1, \ldots, n_k\}$. Therefore K is bounded. □

Alternate proof. Consider the open cover by open balls

$$K \subseteq \bigcup_{x \in K} B(x, 1).$$

Since K is compact, there is a finite subcover

$$K \subseteq B(x_1, 1) \cup \ldots \cup B(x_k, 1).$$

Since a finite union of balls is bounded, so is K. □

Theorem 9.14 (Heine–Borel theorem)**.** *A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. A compact subset $A \subseteq \mathbb{R}^n$ is closed, since $\mathbb{R}^n$ is Hausdorff. Moreover, A is bounded, by Lemma 9.13.

Conversely, let $A \subseteq \mathbb{R}^n$ be closed and bounded. Since A is bounded, it is contained in some closed box

$$B = [a_1, b_1] \times \cdots \times [a_n, b_n].$$

By the Bolzano–Weierstrass theorem, the closed interval $[a_i, b_i]$ is compact. Since a finite product of compact spaces is compact (Theorem 9.2), the closed box B is compact. Since A is closed in $\mathbb{R}^n$, it is also closed in B. Hence A is compact, since a closed subset of a compact space is compact. □

Remark 9.15. We have seen that any compact subspace $K \subseteq X$ of a metric space is closed and bounded. The converse fails in general. For example, let X be an infinite set equipped with the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

which induces the discrete topology. Every subset $A \subseteq X$ is closed, since the topology is discrete. Moreover, every subset $A \subseteq X$ is bounded: $\operatorname{diam}(A) = 1$ (or 0 if A is a singleton). However, a subset $A \subseteq X$ is compact if and only if it is finite.

Example 9.16. The following spaces are compact.

(a) The n-cube $I^n = [0, 1]^n \subset \mathbb{R}^n$.

(b) The n-disk

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

(c) The n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

(d) The n-simplex

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0 \text{ for all } i\}.$$

In fact, the n-simplex is homeomorphic to the n-disk: $\Delta^n \cong D^n$.

Corollary 9.17 (Extreme value theorem)**.** *Let $f \colon D \to \mathbb{R}$ be a continuous function whose domain $D \subseteq \mathbb{R}^n$ is closed and bounded. Then f is bounded and achieves its bounds.*

Proof. By the Heine–Borel theorem, D is compact. Since f is continuous, the image $f(D) \subset \mathbb{R}$ is compact, in particular bounded. Denote $m := \inf f(D)$ and $M := \sup f(D)$. Then we have

$$m \in \overline{f(D)} = f(D)$$
$$M \in \overline{f(D)} = f(D)$$

since $f(D)$ is closed. Hence, there are points $x_{\min} \in D$ and $x_{\max} \in D$ satisfying $f(x_{min}) = m$ and $f(x_{\max}) = M$. That is, f achieves its lower bound at $x_{\min}$ and its upper bound at $x_{\max}$. □

Example 9.18. If $K \subset \mathbb{R}^n$ is compact, then for any $x \in \mathbb{R}^n$, there is a point $y \in K$ minimizing the distance to x:

$$d(x, y) = d(x, K) = \inf_{z \in K} d(x, z).$$

Remark 9.19.

1. The assumption that D be closed cannot be removed in general. For example, consider the bounded subset $D = (0, 1] \subset \mathbb{R}$. The function

$$f\colon (0, 1] \to \mathbb{R}$$
$$f(x) = \frac{1}{x}$$

is unbounded. The function

$$g\colon (0, 1] \to \mathbb{R}$$
$$g(x) = x$$

is bounded but does not achieve its infimum $m = 0$.

2. The assumption that D be bounded cannot be removed in general. For example, consider the closed subset $D = [0, +\infty) \subset \mathbb{R}$. The function

$$f\colon [0, +\infty) \to \mathbb{R}$$
$$f(x) = x$$

is unbounded. The function

$$g\colon [0, +\infty) \to \mathbb{R}$$
$$g(x) = \arctan(x)$$

is bounded but does not achieve its supremum $M = \frac{\pi}{2}$.

10 Sequential compactness

10.1 Definitions and examples

Definition 10.1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a topological space X. A **subsequence** of $(x_n)_{n\in\mathbb{N}}$ is a sequence $(x_{n_k})_{k\in\mathbb{N}}$ for some strictly increasing indices $n_1 < n_2 < n_3 < \cdots$.

Definition 10.2. A topological space X is **sequentially compact** if every sequence in X has a convergent subsequence.

Example 10.3. In the real line $\mathbb{R}$, consider the sequence $(x_n)_{n\in\mathbb{N}} = (\sqrt{7}, \pi, \sqrt{7}, \pi, \ldots)$, that is:

$$x_n = \begin{cases} \sqrt{7} & \text{if } n \text{ is odd} \\ \pi & \text{if } n \text{ is even.} \end{cases}$$

Although the sequence $(x_n)_{n\in\mathbb{N}}$ is not convergent, it has many convergent subsequences. For instance, the subsequence $(x_{n_k})_{k\in\mathbb{N}} = (x_1, x_3, x_5, \ldots)$ given by

$$n_k = 2k - 1$$

is the constant sequence $(\sqrt{7}, \sqrt{7}, \sqrt{7}, \ldots)$, which converges to $\sqrt{7}$. Likewise, the subsequence $(x_{n_k})_{k\in\mathbb{N}} = (x_2, x_4, x_6, \ldots)$ given by

$$n_k = 2k$$

is the constant sequence $(\pi, \pi, \pi, \ldots)$, which converges to π.

Example 10.4. In the real line $\mathbb{R}$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = n$ has *no* convergent subsequence. This shows in particular that $\mathbb{R}$ is not sequentially compact.

Example 10.5. In the open interval $(0, 2)$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = \frac{1}{n}$ has *no* convergent subsequence. This shows in particular that $(0, 2)$ is not sequentially compact.

10.2 Cluster points

Definition 10.6. In a topological space X, a point $x \in X$ is a **cluster point** of a sequence $(x_n)_{n\in\mathbb{N}}$ if for every neighborhood U of x, there exist infinitely many indices n satisfying $x_n \in U$. Equivalently: for every neighborhood U of x and every $N \in \mathbb{N}$, there is an index $n \geq N$ satisfying $x_n \in U$.

Lemma 10.7. *If a sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \xrightarrow{k\to\infty} x$, then x is a cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$.*

Proof. Let $U \subseteq X$ be a neighborhood of x and let $N \in \mathbb{N}$. We want to find some index $n \geq N$ such that $x_n \in U$ holds.

By the convergence $x_{n_k} \xrightarrow{k\to\infty} x$, there is an index $K_1 \in \mathbb{N}$ such that $x_{n_k} \in U$ holds for all $k \geq K_1$. Furthermore, since the indices n_k increase to infinity as $k \to \infty$, there is an index $K_2 \in \mathbb{N}$ such that $n_k \geq N$ holds for all $k \geq K_2$. Taking $K := \max\{K_1, K_2\}$, we find $n_K \geq N$ and $x_{n_K} \in U$. □

Proposition 10.8. *Let X be a first-countable topological space, $(x_n)_{n\in\mathbb{N}}$ a sequence in X, and $x \in X$ a cluster point of that sequence. Then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to x.*

Proof. See Homework 8 Problem 1. □

10.3 Countable compactness

Definition 10.9. A collection of subsets $\{A_\alpha\}$ of a set X has the **finite intersection property** if the intersection of any finite subcollection is non-empty:

$$A_{\alpha_1} \cap \cdots \cap A_{\alpha_n} \neq \emptyset.$$

Lemma 10.10. *A topological space X is compact if and only if every collection of closed subsets $\{C_\alpha\}$ that has the finite intersection property has a non-empty intersection:*

$$\bigcap_\alpha C_\alpha \neq \emptyset.$$

Proof. Consider the following chain of equivalent conditions.

X is compact.

$\overset{\text{def}}{\iff}$ For every collection of open subsets $\{U_\alpha\}$ satisfying $\bigcup_\alpha U_\alpha = X$, there exists a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ satisfying

$$U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} = X.$$

$\iff$ For every collection of open subsets $\{U_\alpha\}$ such that every finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ satisfies

$$U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \neq X,$$

the whole collection must also satisfy

$$\bigcup_\alpha U_\alpha \neq X.$$

Taking $C_\alpha := U_\alpha^c$ yields the next equivalence.

$\iff$ For every collection of closed subsets $\{C_\alpha\}$ such that every finite subcollection $\{C_{\alpha_1}, \ldots, C_{\alpha_n}\}$ satisfies

$$C_{\alpha_1}^c \cup \cdots \cup C_{\alpha_n}^c \neq X,$$

the whole collection must also satisfy

$$\bigcup_\alpha C_\alpha^c \neq X.$$

Using De Morgan's law and taking complements yields the next equivalence.

$\iff$ For every collection of closed subsets $\{C_\alpha\}$ such that every finite subcollection $\{C_{\alpha_1}, \ldots, C_{\alpha_n}\}$ satisfies

$$C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} \neq \emptyset,$$

the whole collection must also satisfy

$$\bigcap_\alpha C_\alpha \neq \emptyset.$$

□

Proposition 10.11 (Cantor's intersection theorem for compact spaces)**.** *Let X be a compact space and $C_1 \supseteq C_2 \supseteq \cdots$ a nested sequence of non-empty closed subsets. Then their intersection is non-empty:*

$$\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset.$$

Proof. The collection $\{C_n\}_{n \in \mathbb{N}}$ has the finite intersection property:

$$C_{n_1} \cap \cdots \cap C_{n_k} = C_m \neq \emptyset,$$

where m is the largest index $m = \max\{n_1, \ldots, n_k\}$. Since X is compact, the whole collection $\{C_n\}_{n \in \mathbb{N}}$ has a non-empty intersection. □

Note that we only used countable collections in the proof, so the result can be generalized.

Definition 10.12. A topological space X is **countably compact** if every countable open cover of X admits a finite subcover.

Remark 10.13. Compact always implies countably compact, but the converse does not hold in general. For instance, the first uncountable ordinal ω_1 endowed with the order topology is sequentially compact but not compact. See the discussion here:

`https://math.stackexchange.com/questions/489225/compact-and-countably-compact`

Proposition 10.14. *The following conditions on a space X are equivalent.*

1. *X is countably compact.*
2. *For every countable collection $\{C_n\}_{n \in \mathbb{N}}$ of closed subsets of X with the finite intersection property, their intersection is non-empty: $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.*
3. *For every nested sequence $C_1 \supseteq C_2 \supseteq \cdots$ of non-empty closed subsets of X, their intersection is non-empty: $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.*

Proof. ($1 \iff 2$) The proof of Lemma 10.10 also works here, using only countable collections instead of arbitrary collections.

($2 \implies 3$) As observed in the proof of Proposition 10.11, a nested sequence $C_1 \supseteq C_2 \supseteq \cdots$ of non-empty subsets satisfies the finite intersection property.

($3 \implies 2$) Let $\{C_n\}_{n \in \mathbb{N}}$ be a countable collection of closed subsets of X with the finite intersection property. Then the subsets

$$D_n := C_1 \cap \cdots \cap C_n = \bigcap_{i=1}^{n} C_i$$

are closed (as finite intersections of closed subsets), non-empty (by the finite intersection property), and they form a nested sequence

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots .$$

By assumption (3), their intersection is non-empty:

$$\bigcap_{n \in \mathbb{N}}^{n} D_n \neq \emptyset$$

and therefore the same is true of the original collection:

$$\bigcap_{n \in \mathbb{N}}^{n} C_n = \bigcap_{n \in \mathbb{N}}^{n} D_n \neq \emptyset. \qquad \square$$

The implication (1) $\implies$ (3) generalizes Cantor's intersection theorem from Proposition 10.11. Using Proposition 10.14, one can show the following.

Proposition 10.15. *A topological space X is countably compact if and only if every sequence in X has a cluster point.*

Proof. See Homework 8 Problem 2. $\square$

10.4 Relationship with compactness

Proposition 10.16. *Every sequentially compact space is countably compact.*

Proof. Let X be a sequentially compact space. By Proposition 10.15, it suffices to show that any sequence $(x_n)_{n\in\mathbb{N}}$ in X has a cluster point. Since X was assumed sequentially compact, there is a convergent subsequence $x_{n_k} \xrightarrow{k\to\infty} x$. By Lemma 10.7, x is a cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$. □

Remark 10.17. The converse does not hold in general. Homework 8 Problem 3 provides an example of a compact space that is *not* sequentially compact.

Proposition 10.18. *Let X be a first-countable countably compact space. Then X is sequentially compact.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Consider the ***n*-tail** of the sequence

$$T_n := \{x_i \mid i \geq n\} \subseteq X.$$

The tails form a nested sequence of non-empty subsets $T_1 \supseteq T_2 \supseteq \cdots$ and so do their closures

$$\overline{T_1} \supseteq \overline{T_2} \supseteq \cdots .$$

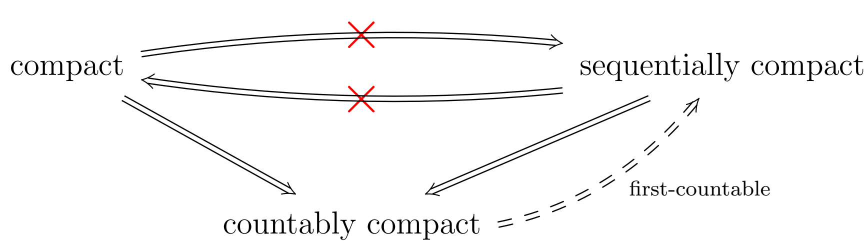
Since X is countably compact, the intersection $\bigcap_{n\in\mathbb{N}} \overline{T_n}$ is non-empty, by Proposition 10.14. Pick a point $x \in \bigcap_{n\in\mathbb{N}} \overline{T_n}$. Since X is first-countable, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to x, by Proposition 10.8. □

Theorem 10.19. *A metric space is compact if and only if it is sequentially compact.*

Proof. ($\Longrightarrow$) Since a metric space is first-countable, this implication is a special case of Proposition 10.18.

($\Longleftarrow$) See [Mun00, Theorem 28.2]. □

Summarizing the implications:



11 Complete metric spaces

11.1 Completeness

Definition 11.1. A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is a **Cauchy sequence** if for any $\epsilon > 0$, there is an index $N \in \mathbb{N}$ satisfying

$$d(x_m, x_n) < \epsilon$$

for all $m, n \geq N$.

Equivalently: $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} d(x_n, x_{n+k}) = 0$.

Lemma 11.2. *In a metric space (X, d), every convergent sequence is Cauchy.*

Proof. Let $x_n \to x$ be a convergent sequence in X. Let $\epsilon > 0$. Then there exists an index $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\epsilon}{2}$ holds for all $n \geq N$. For every $m, n \geq N$, we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

showing that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. □

Lemma 11.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space (X, d). If $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \to x$, then $x_n \to x$ also holds.*

Proof. Let $\epsilon > 0$. Since the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, there is an index $N \in \mathbb{N}$ such that $d(x_m, x_n) < \frac{\epsilon}{2}$ holds for all $m, n \geq N$. Since the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to the point x, there is an index $K \in \mathbb{N}$ such that $d(x_{n_k}, x) < \frac{\epsilon}{2}$ holds for all $k \geq K$. Pick an index $j \geq K$ that satisfies $n_j \geq N$. Then for every $n \geq N$, we have

$$d(x_n, x) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

showing the convergence $x_n \to x$. □

Definition 11.4. A metric space (X, d) is **complete** if every Cauchy sequence in X converges.

11.2 Properties and examples

Proposition 11.5. *Euclidean space $\mathbb{R}^n$ is complete.*

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$. Then the subset

$$A = \{x_i \mid i \in \mathbb{N}\}$$

is bounded. Its closure $\overline{A} \subseteq \mathbb{R}^n$ is closed and bounded, hence compact (by the Heine–Borel theorem), and therefore sequentially compact. The sequence $(x_i)_{i\in\mathbb{N}}$ in $\overline{A}$ has a convergent subsequence $x_{i_k} \to x \in \overline{A}$. By Lemma 11.3, the Cauchy sequence $(x_i)_{i\in\mathbb{N}}$ also converges to x. □

Example 11.6. The rational line $\mathbb{Q}$ is *not* complete. Indeed, take a sequence $(r_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ that converges to $\sqrt{2}$ in $\mathbb{R}$. Then that sequence is Cauchy but does *not* converge in $\mathbb{Q}$.

Example 11.7. The open interval $(0, 1)$ is not complete. Indeed, the sequence defined by $x_n = \frac{1}{n+1}$ is Cauchy but does *not* converge in $(0, 1)$.

Remark 11.8. The topological spaces $\mathbb{R}$ and $(0, 1)$ are homeomorphic, yet $\mathbb{R}$ is complete and $(0, 1)$ is not complete. This shows that completeness is not a topological invariant.

Lemma 11.9. *Let (X, d) be a complete metric space and $C \subseteq X$ a closed subset. Then C is complete.*

Slogan: "closed in complete is complete".

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in C. Since X is complete, the sequence converges to some point $x \in X$. But limits of sequences are in the closure: $x \in \overline{C} = C$. □

Lemma 11.10. *Let (X, d) be a metric space and $A \subseteq X$ a complete subspace. Then A is closed in X.*

Slogan: "complete is always closed".

Proof. Let $x \in \overline{A}$. Since X is first-countable, the closure point $x \in \overline{A}$ is the limit of some sequence $(a_n)_{n\in\mathbb{N}}$ in A. Since the sequence $(a_n)_{n\in\mathbb{N}}$ converges, it is Cauchy (by Lemma 11.2). Since A is complete, we have $a_n \to a$ for some point $a \in A$. Since X is Hausdorff, limits of sequences are unique, which ensures $x = a \in A$. This proves the inclusion $\overline{A} \subseteq A$. □

Lemma 11.11. *Any compact metric space (X, d) is complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Since X is compact (and first-countable), it is sequentially compact, so that the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \to x$. By Lemma 11.3, the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ also converges to x. □

The following theorem characterizes compact metric spaces.

Theorem 11.12. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Proof. See [Mun00, Theorem 45.1]. □

11.3 Cantor's intersection theorem

Proposition 11.13 (Cantor's intersection theorem for complete spaces)**.** *Let* (X, d) *be a complete metric space and* $C_1 \supseteq C_2 \supseteq \cdots$ *a nested sequence of non-empty closed subsets satisfying* $\operatorname{diam}(C_n) \to 0$ *as* $n \to +\infty$*. Then the intersection of the collection is a single point:*

$$\bigcap_{n \in \mathbb{N}} C_n = \{x\}.$$

Proof. The intersection $\bigcap_{n\in\mathbb{N}} C_n$ contains at most one point. Indeed, assume $x, y \in \bigcap_{n\in\mathbb{N}} C_n$. Then the distance between the points satisfies

$$\begin{aligned} & d(x,y) \leq \operatorname{diam}(C_n) \quad \text{for all } n \in \mathbb{N} \\ \implies & d(x,y) \leq \inf_{n\in\mathbb{N}} \operatorname{diam}(C_n) = 0 \\ \implies & d(x,y) = 0 \\ \implies & x = y. \end{aligned}$$

It remains to show that $\bigcap_{n\in\mathbb{N}} C_n$ is non-empty. For each $n \in \mathbb{N}$, pick a point $x_n \in C_n$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Indeed, let $\epsilon > 0$. There is an index $N \in \mathbb{N}$ such that $\operatorname{diam}(C_n) < \epsilon$ holds for all $n \geq N$. For all $m, n \geq N$, we have $x_m, x_n \in C_N$ and therefore

$$d(x_m, x_n) \leq \operatorname{diam}(C_N) < \epsilon.$$

Since X is complete, the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges to some point $x \in X$. For each index $i \in \mathbb{N}$, the tail of the sequence $(x_n)_{n\in\mathbb{N}}$ is contained in C_i, which implies that the point $x = \lim_{n\to+\infty} x_n$ is in the closure $\overline{C_i} = C_i$. Since this holds for all $i \in \mathbb{N}$, we conclude $x \in \bigcap_{i\in\mathbb{N}} C_i$. □

Remark 11.14. We cannot drop the assumption $\operatorname{diam}(C_n) \to 0$. For instance, consider the nested sequence of closed subsets $C_n = [n, +\infty)$, whose intersection is empty:

$$\bigcap_{n\in\mathbb{N}} [n, +\infty) = \emptyset.$$

Nor can we drop the assumption that the subsets be closed. For instance consider the nested sequence of subsets $A_n = (0, \frac{1}{n}]$, whose diameters tend to zero, but whose intersection is empty:

$$\bigcap_{n\in\mathbb{N}} (0, \frac{1}{n}] = \emptyset.$$

12 Banach fixed point theorems

12.1 For compact metric spaces

Definition 12.1. A map $f: X \to Y$ between metric spaces is called a **shrinking map** if it satisfies

$$d(f(x), f(y)) < d(x, y)$$

for all distinct points $x, y \in X$.

Theorem 12.2 (Banach fixed point theorem for compact spaces)**.** *Let (X, d) be a compact metric space and $f: X \to X$ a shrinking map. Then:*

1. *f has a unique fixed point.*
2. *For any starting point $x_0 \in X$, the sequence of iterates $f^n(x_0)$ converges to the fixed point of f.*

Proof. **(1)** The map f has at most one fixed point. Indeed, let $p, q \in X$ be fixed points of f. If they were distinct, then their distance would satisfy

$$d(p, q) = d(f(p), f(q)) < d(p, q),$$

which contradicts $d(p, q) > 0$.

To show existence of a fixed point, consider the continuous function $\varphi: X \to \mathbb{R}$ defined by

$$\varphi(x) = d(x, f(x)).$$

Note that $x \in X$ is a fixed point of f if and only if $\varphi(x) = 0$ holds:

$$f(x) = x \iff d(x, f(x)) = 0 = \varphi(x).$$

For any point $x \in X$ satisfying $x \neq f(x)$, applying f decreases the value of φ:

$$\varphi(f(x)) = d(f(x), f^2(x)) < d(x, f(x)) = \varphi(x).$$

Since X is compact, φ achieves its lower bound $m \geq 0$ at some point $x_{\min} \in X$. But the minimum $m = \varphi(x_{\min})$ must be 0. Otherwise, we would have $\varphi(f(x_{\min})) < \varphi(x_{\min}) = m$, contradicting the fact that m is the minimum value of φ. This proves $d(x_{\min}, f(x_{\min})) = 0$ and thus $f(x_{\min}) = x_{\min}$.

(2) Let $p \in X$ denote the (unique) fixed point of f and consider the sequence of iterates $x_n := f^n(x_0)$. We want to show the convergence $x_n \to p$. If $x_N = p$ holds for some index N, then the sequence becomes constant at p from that point on: $x_n = p$ holds for all $n \geq N$, which implies $x_n \to p$.

Now assume that $x_n \neq p$ holds for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, p) &= d(f^n(x_0), f(p)) \\ &< d(f^{n-1}(x_0), p) \\ &= d(x_{n-1}, p). \end{aligned}$$

This yields a strictly decreasing sequence of real numbers

$$d(x_0, p) > d(x_1, p) > d(x_2, p) > \cdots$$

which is bounded below by 0, hence converges to some number δ. We want to show $\delta = 0$.

Since X is compact, it is sequentially compact, so that the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \to y$. Continuity of the function $d(-, p)\colon X \to \mathbb{R}$ implies

$$d(y, p) = d(\lim_{k\to\infty} x_{n_k}, p) = \lim_{k\to\infty} d(x_{n_k}, p).$$

On the other hand, the sequence $d(x_n, p)$ in $\mathbb{R}$ converges to the number δ, hence all its subsequences also converge to δ, which shows

$$\lim_{k\to\infty} d(x_{n_k}, p) = \delta = d(y, p).$$

Likewise, we have

$$\begin{aligned} d(f(y), p) &= d(f(\lim_{k\to\infty} x_{n_k}), p) \\ &= d(\lim_{k\to\infty} f(x_{n_k}), p) \\ &= d(\lim_{k\to\infty} x_{n_k+1}, p) \\ &= \lim_{k\to\infty} d(x_{n_k+1}, p) \\ &= \delta \\ &= d(y, p). \end{aligned}$$

If $d(y, p)$ were positive, we would have

$$d(f(y), p) = d(f(y), f(p)) < d(y, p),$$

contradicting the equality $d(f(y), p) = d(y, p)$. This shows $d(y, p) = 0$, and therefore $y = p$ and $d(x_n, p) \to 0$, which shows $x_n \to p$. □

The following alternate proof is sketched in [Mun00, §28 Exercise 7].

Proof. For each $n \in \mathbb{N}$, consider the subset $A_n := f^n(X) \subseteq X$. Since X is compact and f is continuous, A_n is compact. Since X is Hausdorff, each $A_n \subseteq X$ is closed in X. Moreover, they form a nested sequence

$$X = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots .$$

Consider the intersection $A := \bigcap_{n\in\mathbb{N}} A_n$, which is closed in X. First, let us prove $f(A) = A$.

Claim: $f(A) \subseteq A$. An element of $f(X)$ is of the form $f(x)$ for some $x \in A$, that is, $x \in f^n(X)$ holds for all $n \in \mathbb{N}$. Then $f(x) \in f^{n+1}(X)$ holds for all $n \in \mathbb{N}$, proving $f(x) \in \bigcap_{n\in\mathbb{N}} A_n = A$.

Claim: $A \subseteq f(A)$. Let $x \in A$. For every $n \in \mathbb{N}$, there is a point $x_n \in X$ satisfying $x = f^n(x_n)$. Let us rewrite this equality as

$$x = f^n(x_n) = f(f^{n-1}(x_n)) = f(y_n),$$

with $y_n := f^{n-1}(x_n)$. Since X is compact, it is sequentially compact, and thus the sequence $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence $y_{n_k} \to y$ as $k \to +\infty$. Continuity of f then gives $f(y_{n_k}) \to f(y)$ as $k \to +\infty$. But the sequence $(f(y_{n_k}))_{k\in\mathbb{N}}$ is the constant sequence at x, which converges to x, proving $x = f(y)$. The condition $y_n \in A_{n-1}$ implies that for every $n \in \mathbb{N}$, we have $y \in \overline{A_n} = A_n$, and thus $y \in A$. This proves $f(y) \in f(A)$.

Claim: A is a singleton. By Cantor's intersection theorem (for compact spaces), A is non-empty: $A \neq \emptyset$.

Since $A \subseteq X$ is closed and X is compact, A is also compact. Therefore, the continuous function $d \colon A \times A \to \mathbb{R}$ (the metric on A) achieves its upper bound $\operatorname{diam}(A)$ at some pair of points $a, b \in A$. But the inclusion $A \subseteq f(A)$ guarantees that there exist points $a', b' \in A$ satisfying $f(a') = a$ and $f(b') = b$. If a and b are distinct, then so are a' and b', and we have

$$d(a, b) = d(f(a'), f(b')) < d(a', b') < d(a, b),$$

which is impossible. Therefore, we have $a = b$ and $\operatorname{diam}(A) = 0$. In other words, $A = \{p\}$ consists of a single point $p \in X$.

Claim: A is the fixed point set of f. The inclusion $f(A) \subseteq A$ shows $f(p) = p$, so that p is a fixed point of f. Moreover, any fixed point q of f satisfies $f^n(q) = q$ for all $n \in \mathbb{N}$, and hence $q \in A$. This shows that f has a unique fixed point p.

For the second statement, consider the sequence of iterates $x_n := f^n(x_0) \in A_n$. The sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $x_{n_k} \to x$, and as above we deduce $x \in A$, that is, $x = p$ is the fixed point of f. As in the first proof, this implies the convergence $d(x_n, p) \to 0$ and thus $x_n \to p$. □

Example 12.3. The assumption that f be shrinking cannot be dropped in general. For example, consider the unit circle $S^1 \subset \mathbb{R}^2$ and consider the rotation by an angle $\theta \in (0, 2\pi)$

$$\begin{aligned} R_\theta &: S^1 \to S^1 \\ R_\theta \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

Viewing S^1 as the unit circle in the complex plane $\mathbb{C}$, the rotation R_θ can be written as

$$R_\theta(z) = e^{i\theta} z.$$

The map R_θ is an isometry, in particular is non-expansive. However, R_θ has no fixed point.

Example 12.4. By the Brouwer fixed point theorem, every continuous map $f: [0,1] \to [0,1]$ has a fixed point. However, even for $X = [0,1]$, the other conclusions in the Banach fixed point theorem are not guaranteed without the assumptions.

Uniqueness of the fixed point may fail. For instance, the identity map

$$\mathrm{id}: [0,1] \to [0,1]$$

is an isometry and has infinitely many fixed points, since *every* point $x \in [0,1]$ satisfies $\mathrm{id}(x) = x$.

Even when $f: [0,1] \to [0,1]$ has a unique fixed point, convergence of the iterates $f^n(x_0)$ may fail. For instance, consider the map

$$\begin{aligned} f &: [0,1] \to [0,1] \\ f(x) &= 1 - x, \end{aligned}$$

which is an isometry. This map f has a unique fixed point $p = \frac{1}{2}$. However, for any starting point $x_0 \neq \frac{1}{2}$, the sequence of iterates $f^n(x_0)$ alternates between two points:

$$(x_0, 1 - x_0, x_0, 1 - x_0, \ldots)$$

and hence doesn't converge. For instance, taking the starting point $x_0 = 0.3$ yields the sequence of iterates

$$(0.3, 0.7, 0.3, 0.7, \ldots).$$

12.2 For complete metric spaces

Definition 12.5. A map $f\colon X \to Y$ between metric spaces is a **contraction** if there is a number $0 \le c < 1$ such that the inequality

$$d(f(x), f(y)) \le c\, d(x, y)$$

holds for all $x, y \in X$.

Theorem 12.6 (Banach fixed point theorem for complete spaces)**.** *Let (X, d) be a complete metric space and $f\colon X \to X$ a contraction. Then:*

1. *f has a unique fixed point.*
2. *For any starting point $x_0 \in X$, the sequence of iterates $f^n(x_0)$ converges to the fixed point of f.*

Proof. The map f has at most one fixed point, for the same reason as above. If $p, q \in X$ are two fixed points of f, then we have

$$d(p, q) = d(f(p), f(q)) \le c\, d(p, q)$$

which implies $d(p, q) = 0$ and thus $p = q$.

Now let us prove the existence of a fixed point and the second statement at the same time. Pick any $x_0 \in X$ and consider the sequence of iterates $x_n := f^n(x_0)$. Denote $\delta := d(x_0, f(x_0)) = d(x_0, x_1)$. For any indices $m < n$, we have

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{n-1}, x_n) \\ &\le c^m d(x_0, x_1) + c^{m+1} d(x_0, x_1) + \ldots + c^{n-1} d(x_0, x_1) \\ &= \delta c^m (1 + c + c^2 + \ldots + c^{n-m-1}) \\ &< \delta c^m \frac{1}{1-c} \\ &\to 0 \quad \text{as } m \to \infty. \end{aligned}$$

This shows that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, hence converges to some point $x \in X$ since X is complete. But then x is a fixed point of f:

$$\begin{aligned} f(x) &= f(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} f(x_n) \\ &= \lim_{n\to\infty} x_{n+1} \\ &= x. \end{aligned}$$

□

Example 12.7. The assumption that f be a contraction cannot be dropped in general. For example, consider the interval $X = [1, +\infty)$, which is complete, as a closed subset of the complete metric space $\mathbb{R}$. Consider the map

$$f\colon [1, +\infty) \to [1, +\infty)$$
$$f(x) = x + \frac{1}{x},$$

which is a shrinking map, but not a contraction. The map f has no fixed point. A similar example is discussed on Homework 9 Problem 2.

Example 12.8. The assumption that X be complete cannot be dropped in general. For example, consider the interval $X = [0, 1)$, which is not complete, since it is not closed in $\mathbb{R}$. Consider the map

$$f\colon [0, 1) \to [0, 1)$$
$$f(x) = \frac{1}{2}x + \frac{1}{2},$$

which is a contraction with Lipschitz constant $c = \frac{1}{2}$. However, the map f has no fixed point.

13 The evaluation map

Proposition 13.1. *Let X be a locally compact Hausdorff space and Y an arbitrary topological space. Then the evaluation map*

$$e\colon X \times C(X,Y) \to Y$$

given by $e(x,f) = f(x)$ is continuous.

Remark 13.2. The evaluation map is not always continuous.

Exercise 13.3. Let $\mathcal{T}$ be a topology on the set $C(X,Y)$ making the evaluation map

$$e\colon X \times C(X,Y) \to Y$$

continuous. Then $\mathcal{T}$ contains the compact-open topology [Mun00, Exercise 46.8].

Now let X, Y, and T be topological spaces, and let $F(X,Y)$ denote the *set of all functions* from X to Y. There is a natural bijection of sets:

$$\varphi\colon F(X \times T, Y) \xrightarrow{\cong} F(T, F(X,Y)) \tag{1}$$

sending a function $H\colon X \times T \to Y$ to the function $\varphi(H)\colon T \to F(X,Y)$ defined by

$$\varphi(H)(t) = H(-,t) =: h_t.$$

Proposition 13.4. ***(a)*** *If a function $H\colon X \times T \to Y$ is continuous, then the following two conditions hold:*

1. *$h_t\colon X \to Y$ is continuous for all $t \in T$;*
2. *The corresponding function $\varphi(H)\colon T \to C(X,Y)$ is continuous.*

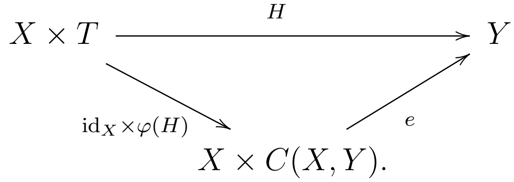
(b) *Assuming X is locally compact Hausdorff, the converse holds as well. In other words, if conditions* (1) *and* (2) *hold, then the corresponding function $H\colon X \times T \to Y$ is continuous.*

Proof. **(a)** Worksheet "The compact-open topology" Problem 2.

(b) Rewriting $H(x,t)$ as

$$\begin{aligned} H(x,t) &= H(-,t)(x) \\ &= (\varphi(H)(t))\,(x) \\ &= e\,(x, \varphi(H)(t)) \end{aligned}$$

we see that the function $H\colon X \times T \to Y$ corresponding to $\varphi(H)\colon T \to C(X,Y)$ is the composite



The map $\varphi(H)\colon T \to C(X,Y)$ is continuous by assumption, and so is $\mathrm{id}_X \times \varphi(H)$. Since X is locally compact Hausdorff, the evaluation map e is continuous by Proposition 13.1, and so is the composite $H = e \circ (\mathrm{id}_X \times \varphi(H))$. □

Interpretation. For any spaces X, Y, T, part (a) ensures that the bijection φ from (1) restricts to a map

$$\varphi\colon C(X \times T, Y) \to C\left(T, C(X,Y)\right) \tag{2}$$

sometimes called the **adjunction map**. This latter φ is always injective, since the original φ was injective.

However, the latter φ is not always surjective. In other words, a family $\{h_t\colon X \to Y\}_{t \in T}$ of continuous maps that vary continuously in the parameter $t \in T$ do not always yield a map $H\colon X \times T \to Y$ which is jointly continuous in both arguments.

Part (b) says that if X is nice enough (e.g. locally compact Hausdorff), then φ is indeed surjective.

Remark 13.5. Proposition 13.4 is useful when trying to show that a map $T \to C(X,Y)$ into a mapping space is continuous. By part (a), it suffices that the corresponding map $X \times T \to Y$ be continuous.

14 Nets

14.1 Directed sets and nets

Definition 14.1. A **preorder** on a set Λ is a relation $\leq$ which is:

1. reflexive: $\lambda \leq \lambda$ for all $\lambda \in \Lambda$;
2. transitive: $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$ implies $\lambda_1 \leq \lambda_3$.

Definition 14.2. A **directed set** $(\Lambda, \leq)$ is a set Λ equipped with a preorder $\leq$ such that for every $\lambda_1, \lambda_2 \in \Lambda$, there is some λ_3 satisfying $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

In other words, every finite subset of Λ has an upper bound.

Remark 14.3. A **partial order** is a preorder that moreover satisfies antisymmetry:

$$\lambda \leq \mu \quad \text{and} \quad \mu \leq \lambda \implies \lambda = \mu.$$

Some authors include that condition in their definition of directed set.

Example 14.4. Here are some examples of directed sets.

1. The natural numbers $\mathbb{N}$ with the usual order $\leq$.
2. The real numbers $\mathbb{R}$ with the usual order $\leq$.
3. More generally, any totally ordered set.
4. More generally still, any poset $(P, \leq)$ that has finite joins (i.e., suprema) $x \vee y$. For instance, the power set $\mathcal{P}(S)$ of a set S, ordered by inclusion.
5. If a poset $(P, \leq)$ has finite meets (i.e., infima) $x \wedge y$, then its opposite poset $(P, \leq^{\text{op}})$ (where the order is reversed) has finite joins, and is therefore a directed set. For instance, the power set $\mathcal{P}(S)$ of a set S, ordered by *reverse* inclusion.
6. $\mathbb{N} \times \mathbb{N}$ with the componentwise preorder.
7. More generally, given directed sets Λ and Γ, their product $\Lambda \times \Gamma$ with the componentwise preorder is a directed set. See Homework 10 Problem 1.

For our purposes, the following is one of the most important examples.

Example 14.5. Let X be a topological space, and $x \in X$. Then the set

$$\mathcal{N}_x := \{U \subseteq X \mid U \text{ is a neighborhood of } x\}$$

ordered by reverse inclusion (i.e., $U_1 \leq U_2$ if $U_2 \subseteq U_1$) is a directed set. Indeed, given neighborhoods U and V, their intersection serves as an upper bound:

$$U \leq U \cap V \quad \text{and} \quad V \leq U \cap V.$$

Definition 14.6. Let X be a topological space. A **net** in X is a function $x\colon \Lambda \to X$ from a directed set Λ into X.

We denote values of the net by $x_\lambda := x(\lambda)$ and denote the net by $(x_\lambda)_{\lambda\in\Lambda}$.

Example 14.7. A net in X indexed by $(\mathbb{N}, \leq)$ is a sequence in X.

Definition 14.8. A net $(x_\lambda)_{\lambda\in\Lambda}$ in a topological space X **converges** to a point $x \in X$ if for all neighborhood U of x, the net is eventually in U, i.e., there is an index $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ holds for all $\lambda \geq \lambda_0$.

Convergence will be denoted $x_\lambda \to x$.

14.2 Nets detect the topology

Proposition 14.9. *Let X be a topological space and $A \subseteq X$ a subset. Then $x \in \overline{A}$ holds if and only if there is a net $(a_\lambda)_{\lambda \in \Lambda}$ in A which converges to x, i.e., $a_\lambda \to x$.*

In words: The closure of A consists of all limits of nets in A.

Proof. ($\Longleftarrow$) Let U be a neighborhood of x. Since (a_λ) converges to x, there is an index $\lambda_0 \in \Lambda$ satisfying $a_\lambda \in U$ for all $\lambda \geq \lambda_0$. In particular, we have $a_{\lambda_0} \in U \cap A \neq \emptyset$. Since U was arbitrary, we conclude $x \in \overline{A}$.

($\Longrightarrow$) Let $x \in \overline{A}$. Consider the directed set $\mathcal{N}_x$ of all neighborhoods of x, ordered by reverse inclusion. For each $U \in \mathcal{N}_x$, we have $U \cap A \neq \emptyset$ so we can pick a point $a_U \in U \cap A$. This defines a net $(a_U)_{U \in \mathcal{N}_x}$ in A. We claim that it converges to x.

Given $V \geq U$, we have $V \subseteq U$ so that $a_V \in V \subseteq U$. In other words, "past the index $U \in \mathcal{N}_x$, the net is inside the neighborhood $U \subseteq X$", which proves the convergence $a_U \to x$. □

Remark 14.10. Consider $\mathbb{R}^{\mathbb{N}}$ with the box topology, the subset

$$A = \{x \in \mathbb{R}^{\mathbb{N}} \mid x_n > 0 \text{ for all } n \in \mathbb{N}\},$$

and the origin $\underline{0} = (0, 0, \ldots) \in \mathbb{R}^{\mathbb{N}}$. On Homework 5 Problem 2, you showed that $\underline{0}$ is a closure point $\underline{0} \in \overline{A}$, yet no sequence in A converges to $\underline{0}$. By Proposition 14.9, there must be a net in A converging to $\underline{0}$. Homework 10 Problem 2 provides an explicit such net.

Proposition 14.11. *Let $f \colon X \to Y$ be a map between topological spaces. Then f is continuous at $x \in X$ if and only if for every net $(x_\lambda)_{\lambda \in \Lambda}$ in X with $x_\lambda \to x$, we have $f(x_\lambda) \to f(x)$ in Y.*

In words: Continuity is equivalent to netwise continuity.

Proof. ($\Longrightarrow$) Assume $x_\lambda \to x$. We want to show $f(x_\lambda) \to f(x)$.

Let V be a neighborhood of $f(x)$. By continuity of f at x, there is a neighborhood U of x satisfying $f(U) \subseteq V$. By convergence of (x_λ), there is an index $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ holds for all $\lambda \geq \lambda_0$. Therefore we have $f(x_\lambda) \in f(U) \subseteq V$ whenever $\lambda \geq \lambda_0$, which proves $f(x_\lambda) \to f(x)$.

($\Longleftarrow$) Recall that f is continuous if and only if it satisfies $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset $A \subseteq X$. Let $x \in \overline{A}$. By Proposition 14.9, there is a net $(a_\lambda)_{\lambda \in \Lambda}$ in A converging to x, $a_\lambda \to x$. By the assumption on f, we have $\underline{f(x_\lambda)} \to f(x)$. By Proposition 14.9, this implies $f(x) \in \overline{f(A)}$, proving the inclusion $f(\overline{A}) \subseteq \overline{f(A)}$. □

Alternate proof. ($\Longleftarrow$) Assume f is discontinuous at x, which means there is a neighborhood V of $f(x)$ satisfying $f(U) \nsubseteq V$ for all neighborhoods U of x. For each such neighborhood U, pick a point x_U satisfying $f(x_U) \notin V$. This defines a net $(x_U)_{U \in \mathcal{N}_x}$ in X indexed by the directed set $\mathcal{N}_x$ of all neighborhoods of x. By construction, the net satisfies $x_U \to x$. However, the net $(f(x_U))_{U \in \mathcal{N}_x}$ in Y is *never* in V, so in particular $f(x_U) \not\to f(x)$. □

Proposition 14.12 (Uniqueness of limits of nets)**.** *A topological space X is Hausdorff if and only if every net in X has at most one limit. In other words: limits are unique, when they exist.*

Proof. ($\implies$) Assume X is Hausdorff and $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X with $x_\lambda \to x$ and $x_\lambda \to y$. We want to show $x = y$.

Let U be a neighborhood of x and V a neighborhood of y. By convergence to x, there is an index $\lambda_1 \in \Lambda$ such that $x_\lambda \in U$ holds whenever $\lambda \geq \lambda_1$ holds. By convergence to y, there is an index $\lambda_2 \in \Lambda$ such that $x_\lambda \in V$ holds whenever $\lambda \geq \lambda_2$ holds.

Let $\lambda_3 \in \Lambda$ be an upper bound for the two indices, i.e., $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$. Then we have

$$\begin{cases} \lambda_3 \geq \lambda_1 \implies x_{\lambda_3} \in U \\ \lambda_3 \geq \lambda_2 \implies x_{\lambda_3} \in V \end{cases}$$

which shows $x_{\lambda_3} \in U \cap V \neq \emptyset$, so that x and y cannot be separated by neighborhoods. Since X is Hausdorff, this proves $x = y$.

($\impliedby$) Assume X is not Hausdorff, which means there exist distinct points $x, y \in X$ which cannot be separated by neighborhoods. In other words, for any neighborhood U of x and neighborhood V of y, we have $U \cap V \neq \emptyset$. Pick a point in the intersection $x_{U,V} \in U \cap V$. This defines a net $(x_{U,V})_{(U,V) \in \Lambda}$ in X indexed by the directed set $\Lambda = \mathcal{N}_x \times \mathcal{N}_y$ of pairs of neighborhoods of x and y respectively.

We show that this net converges to both x and y. Let $\widetilde{U}$ be a neighborhood of x. For every index $(U, V) \geq (\widetilde{U}, X)$, we have

$$x_{U,V} \in U \cap V \subseteq U \subseteq \widetilde{U}$$

which proves $x_{U,V} \to x$. Likewise, we have $x_{U,V} \to y$. □

Remark 14.13. For sequences, the forward implication holds, but the backward implication does *not* hold. In other words, uniqueness of limits of sequences in X does not imply that X is Hausdorff.

For example, take X to be an uncountable set endowed with the cocountable topology. The only convergent sequences in X are the eventually constant sequences, and those have a unique limit since X is T_1. However, X is not Hausdorff.

Proposition 14.14. *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Then a net $(x^\lambda)_{\lambda \in \Lambda}$ in the product $\prod_{\alpha \in A} X_\alpha$ converges to a point $x = (x_\alpha)_{\alpha \in A}$ if and only if for each index $\alpha \in A$, the net $(x^\lambda_\alpha)_{\lambda \in \Lambda}$ in X_α converges to x_α.*

Here $x^\lambda_\alpha = p_\alpha(x^\lambda) \in X_\alpha$ denotes the α^{th} coordinate of the point $x^\lambda \in \prod_{\alpha \in A} X_\alpha$, and likewise $x_\alpha = p_\alpha(x)$, where

$$p_\alpha \colon \prod_{\alpha' \in A} X_{\alpha'} \to X_\alpha$$

denotes the projection onto the α^{th} factor.

Proof. See Homework 10 Problem 3. □

14.3 Subnets

If nets are meant to generalize sequences, what should be the generalization of subsequences to nets?

Definition 14.15. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. A **subnet** of $(x_\lambda)_{\lambda\in\Lambda}$ is a composite

$$M \xrightarrow{\varphi} \Lambda \xrightarrow{x} X$$

where M is a directed set and the function $\varphi\colon M \to \Lambda$ is *order-preserving* (also called *non-decreasing* or *monotone*) and *cofinal*.

Order-preserving means: $\mu_1 \leq \mu_2 \implies \varphi(\mu_1) \leq \varphi(\mu_2)$.

Cofinal means that the function will eventually "pass" any index: for all $\lambda \in \Lambda$, there is a $\mu \in M$ satisfying $\varphi(\mu) \geq \lambda$.

We write $\lambda_\mu := \varphi(\mu)$. Note that a subnet $(x_{\lambda_\mu})_{\mu\in M}$ is itself a net, since M is a directed set.

Example 14.16. The function $\varphi\colon \mathbb{N} \to \mathbb{N}$ defined by $\varphi(k) = 5k$ is order-preserving and cofinal. Given a sequence $(x_n)_{n\in\mathbb{N}}$, this function φ yields the subnet

$$(x_{n_k})_{k\in\mathbb{N}} = (x_5, x_{10}, x_{15}, \ldots)$$

where we write $n_k := \varphi(k)$. Note that this is a subsequence.

Example 14.17. The function $\varphi\colon \mathbb{N} \to \mathbb{N}$ defined by $\varphi(k) = \lceil \frac{k}{2} \rceil$ is order-preserving and cofinal. (Here the brackets denote the ceiling function, which rounds up to the next integer.) Given a sequence $(x_n)_{n\in\mathbb{N}}$, this function φ yields the subnet

$$(x_{n_k})_{k\in\mathbb{N}} = (x_1, x_1, x_2, x_2, x_3, x_3, \ldots).$$

Note that this is *not* a subsequence.

Example 14.18. A function $\varphi\colon \mathbb{N} \to \mathbb{N}$ is cofinal if and only if it is unbounded. Thus, a subnet $(x_{n_k})_{k\in\mathbb{N}}$ of a sequence which is still indexed by $\mathbb{N}$ is almost a subsequence, except that indices n_k are allowed to be repeated finitely many times, as in Example 14.17.

In contrast, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is defined as having strictly increasing indices: $k_1 < k_2$ implies $n_{k_1} < n_{k_2}$.

A subnet of a sequence can also be indexed by any directed set, not just $\mathbb{N}$.

Example 14.19. The function $\varphi\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by $\varphi(k,\ell) = 2k+4\ell$ is order-preserving and cofinal. Given a sequence $(x_n)_{n\in\mathbb{N}}$, this function φ yields the subnet $(x_{k,\ell})_{(k,\ell)\in\mathbb{N}\times\mathbb{N}}$ with values $x_{k,\ell} := x_{\varphi(k,\ell)} = x_{2k+4\ell}$.

15 Compactness via nets

15.1 Cluster points

Definition 15.1. Let X be a topological space and $(x_\lambda)_{\lambda\in\Lambda}$ a net in X. A point $x \in X$ is a **cluster point** of the net $(x_\lambda)_{\lambda\in\Lambda}$ if for any neighborhood U of x, the net is *frequently* in U, i.e., for all index $\lambda_0 \in \Lambda$, there is a $\lambda \in \Lambda$ satisfying $\lambda \geq \lambda_0$ and $x_\lambda \in U$.

Proposition 15.2. *Let X be a topological space and $(x_\lambda)_{\lambda\in\Lambda}$ a net in X. Then $x \in X$ is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$ if and only if there is a subnet $(x_{\lambda_\mu})_{\mu\in M}$ converging to x.*

Proof. ($\Longleftarrow$) Assume $x_{\lambda_\mu} \to x$. Let U be a neighborhood of x and $\lambda_0 \in \Lambda$ an arbitrary index. By cofinality of the indices λ_μ, there is a $\mu_0 \in M$ satisfying $\lambda_{\mu_0} \geq \lambda_0$. By convergence of the subnet $x_{\lambda_\mu} \to x$, there is a $\mu_1 \in M$ such that $x_{\lambda_\mu} \in U$ holds for all $\mu \geq \mu_1$. Let $\mu_2 \in M$ be an upper bound for μ_0 and μ_1, i.e., satisfying

$$\begin{cases} \mu_0 \leq \mu_2 \\ \mu_1 \leq \mu_2. \end{cases}$$

The first inequality yields $\lambda_{\mu_2} \geq \lambda_{\mu_0} \geq \lambda_0$. The second inequality guarantees $x_{\lambda_{\mu_2}} \in U$, showing that the net $(x_\lambda)_{\lambda\in\Lambda}$ is frequently in U.

($\Longrightarrow$) Assume that $x \in X$ is a cluster point of the net $(x_\lambda)_{\lambda\in\Lambda}$. Consider the preordered set

$$M := \{(\lambda, U) \in \Lambda \times \mathcal{N}_x \mid x_\lambda \in U\}$$

with the componentwise preorder. Here, $\mathcal{N}_x$ denotes the directed set of all neighborhoods of x (as in the notes from November 16, Example 1.5).

Claim: M is a directed set. Let $(\lambda_1, U_1), (\lambda_2, U_2) \in M$. Let $\lambda_3 \in \Lambda$ be an upper bound for λ_1 and λ_2, i.e., satisfying $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$. Since x is a cluster point of the net $(x_\lambda)_{\lambda\in\Lambda}$, the net is frequently in the neighborhood $U_1 \cap U_2$. In particular, there is an index $\lambda_3' \geq \lambda_3$ satisfying $x_{\lambda_3'} \in U_1 \cap U_2$. This provides the element $(\lambda_3', U_1 \cap U_2)$ in M which is an upper bound for the two elements we started with:

$$\begin{cases} (\lambda_1, U_1) \leq (\lambda_3', U_1 \cap U_2) \\ (\lambda_2, U_2) \leq (\lambda_3', U_1 \cap U_2). \end{cases}$$

Claim: The projection $p_\Lambda \colon M \to \Lambda$ is order-preserving and cofinal. The projection is order-preserving since the preorder on M is componentwise:

$$\begin{aligned} (\lambda, U) \leq (\lambda', U') &\implies \lambda \leq \lambda' \\ &\iff p_\Lambda(\lambda, U) \leq p_\Lambda(\lambda', U'). \end{aligned}$$

For cofinality, let $\lambda_0 \in \Lambda$. Then we have $x_{\lambda_0} \in X$, yielding an element $(\lambda_0, X) \in M$ which satisfies

$$p_\Lambda(\lambda_0, X) = \lambda_0 \geq \lambda_0.$$

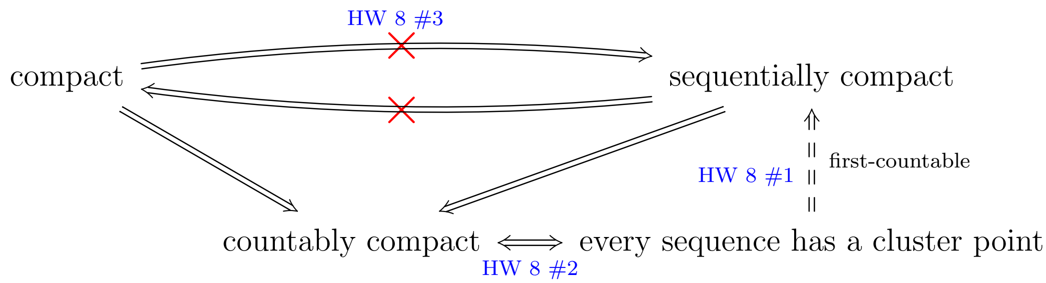
Claim: The subnet $x_{(\lambda,U)}$ converges to x. Let V be a neighborhood of x. Since x is a cluster point of the net $(x_\lambda)_{\lambda\in\Lambda}$, there is an index $\lambda_0 \in \Lambda$ satisfying $x_{\lambda_0} \in V$, yielding the element $(\lambda_0, V) \in M$. But then for every index $(\lambda, U) \geq (\lambda_0, V)$ in M, we have

$$x_{(\lambda,U)} = x_\lambda \in U \subseteq V,$$

showing that the subnet $(x_{(\lambda,U)})_{(\lambda,U)\in M}$ is eventually in V. □

15.2 Compactness via nets

Recall what happens when we try to describe compactness via sequences. The following diagram summarizes relationships between some conditions on a topological space X:



When using nets instead of sequences, we obtain the following statement.

Theorem 15.3. *The following conditions on a topological space X are equivalent:*

1. *X is compact.*
2. *Every net in X has a convergent subnet.*
3. *Every net in X has a cluster point.*

Proof. The equivalence (2) $\iff$ (3) follows from Proposition 15.2.

(1) $\implies$ (3) Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X. For each index $\lambda \in \Lambda$, consider the **$\boldsymbol{\lambda}$-tail** of the net

$$T_\lambda := \{x_{\lambda'} \in X \mid \lambda' \geq \lambda\} \subseteq X.$$

The collection of all tails $\{T_\lambda\}_{\lambda \in \Lambda}$ satisfies the finite intersection property. Indeed, let $\lambda_1, \ldots, \lambda_k \in \Lambda$, and let $\lambda' \in \Lambda$ be an upper bound for $\{\lambda_1, \ldots, \lambda_k\}$, i.e., satisfying $\lambda_i \leq \lambda'$ for all $1 \leq i \leq k$. Then we have $x_{\lambda'} \in T_{\lambda_i}$ for all $1 \leq i \leq k$, so that the finite intersection is non-empty:

$$x_{\lambda'} \in T_{\lambda_1} \cap \cdots \cap T_{\lambda_k} \neq \emptyset.$$

Consequently, the closures of the tails $\{\overline{T_\lambda}\}_{\lambda \in \Lambda}$ also satisfy the finite intersection property. Since X was assumed compact, the intersection of the entire collection is non-empty:

$$\bigcap_{\lambda \in \Lambda} \overline{T_\lambda} \neq \emptyset.$$

Pick a point $x \in \bigcap_{\lambda \in \Lambda} \overline{T_\lambda}$.

Claim: x is a cluster point of the net $(x_\lambda)_{\lambda \in \Lambda}$. Let U be a neighborhood of x and $\lambda_0 \in \Lambda$ an arbitrary index. The condition $x \in \overline{T_{\lambda_0}}$ guarantees $U \cap T_{\lambda_0} \neq \emptyset$. A point $x' \in U \cap T_{\lambda_0}$ is of the form $x' = x_\lambda$ for some $\lambda \geq \lambda_0$ and satisfies $x_\lambda \in U$, showing that the net $(x_\lambda)_{\lambda \in \Lambda}$ is frequently in U.

(3) $\implies$ (1) Let $\{C_i\}_{i \in I}$ be a collection of closed subsets of X satisfying the finite intersection property. We want to show that their intersection is non-empty: $\bigcap_{i \in I} C_i \neq \emptyset$. Take the poset

$$\mathcal{P}_{\text{fin}}(I) := \{J \subseteq I \mid J \text{ is finite}\}$$

ordered by inclusion. Since $\mathcal{P}_{\text{fin}}(I)$ has (finite) joins, given by the union $J \vee J' = J \cup J'$, $\mathcal{P}_{\text{fin}}(I)$ is a directed set. For each finite subset $J \subset I$, the intersection $\bigcap_{i \in J} C_i$ is non-empty; pick a point

$$x_J \in \bigcap_{i \in J} C_i.$$

By the assumption on X, the net $(x_J)_{J \in \mathcal{P}_{\text{fin}}(I)}$ has a cluster point $x \in X$.

Claim: $x \in \bigcap_{i \in I} C_i$**.** Since the C_i are closed, it suffices to show $x \in \overline{C_i} = C_i$ for all $i \in I$.

Fix an index $i_0 \in I$, viewed as the singleton $\{i_0\} \subseteq I$. Let U be a neighborhood of x. Since x is a cluster point of the net $(x_J)_{J \in \mathcal{P}_{\text{fin}}(I)}$, the net is frequently in U. In particular, there is a $J \in \mathcal{P}_{\text{fin}}(I)$ satisfying $J \geq \{i_0\}$ and $x_J \in U$. But we have

$$x_J \in \bigcap_{i \in J} C_i \subseteq C_{i_0},$$

showing $x_J \in U \cap C_{i_0} \neq \emptyset$. Since the neighborhood U was arbitrary, this shows $x \in \overline{C_{i_0}}$. $\square$

16 Tychonoff's theorem

16.1 Zorn's lemma

In order to prove Tychonoff's theorem, we will make use of Zorn's lemma. It is often used to prove the existence of objects with certain maximality properties, e.g., maximal interval of existence of solutions to certain differential equations, maximal ideals in a ring, etc.

Definition 16.1. A **partial order** on a set P is a relation $\leq$ which is:

1. reflexive: $x \leq x$ for all $x \in P$;
2. transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$;
3. antisymmetric: $x \leq y$ and $y \leq x$ implies $x = y$.

Note that a relation satisfying (1) and (2) is what we previously called a *preorder*.

A **partially ordered set** or **poset** $(P, \leq)$ is a set P equipped with a partial order $\leq$.

Example 16.2. Let S be a set and consider the poset $\mathcal{P}(S)$ of all subsets of S, ordered by inclusion.

Remark 16.3. Note that reverse inclusion also defines a partial order on $\mathcal{P}(S)$. More generally, given any partial order, its reverse is also a partial order.

Definition 16.4. A **chain** in a poset P is a subset $C \subseteq P$ which is totally ordered. In other words, any two elements of C are comparable: for all $x, y \in C$, we have either $x \leq y$ or $y \leq x$.

Example 16.5. Let $S = \{a, b, c, d, e\}$ and consider the collection

$$\mathcal{C} = \{\{a, c, d\}, \{a, c, d, e\}, \{d\}\} \subset \mathcal{P}(S).$$

Then $\mathcal{C}$ is a chain in $\mathcal{P}(S)$, i.e., consists of nested subsets of S:

$$\{d\} \subset \{a, c, d\} \subset \{a, c, d, e\}.$$

In contrast, the collection $\{\{a, c, d\}, \{a, c, e\}, \{c\}\} \subset \mathcal{P}(S)$ is *not* a chain, since $\{a, c, d\}$ and $\{a, c, e\}$ are not comparable:

$$\{a, c, d\} \not\subseteq \{a, c, e\}$$
$$\{a, c, e\} \not\subseteq \{a, c, d\}.$$

Example 16.6. In the poset $\mathbb{N} \times \mathbb{N}$ with componentwise order, the subset

$$C = \{(8, 5), (3, 1), (4, 3)\}$$

is a chain, because of the ordering:

$$(3,1) \leq (4,3) \leq (8,5).$$

In contrast, the subset

$$\{(2,4),(3,7),(5,6)\}$$

is *not* a chain, since $(3,7)$ and $(5,6)$ are not comparable:

$$(3,7) \not\leq (5,6)$$
$$(5,6) \not\leq (3,7).$$

Definition 16.7. An element $m \in P$ is a poset P is **maximal** if no element is greater than m. In other words, the inequality $x \geq m$ implies $x = m$.

Example 16.8.

1. In the poset $\mathcal{P}(S)$, the entire set $S \in \mathcal{P}(S)$ is maximal, and in fact is the only maximal element.
2. In the poset $\mathcal{P}(S) \setminus \{S\}$, a subset of S is maximal if and only if it is of the form $S \setminus \{s\}$ for some element $s \in S$.
3. In the totally ordered set $\mathbb{N}$ (which is in particular a poset), there is no maximal element.
4. In the poset of proper ideals of a commutative ring R, the maximal elements are the maximal ideals $M \subset R$. For example, the maximal ideals of $\mathbb{Z}$ are those of the form $(p) \subset \mathbb{Z}$ for some prime number $p \in \mathbb{Z}$.

Definition 16.9. Let $A \subseteq P$ be a subset of a poset P. An **upper bound** for A is an element $b \in P$ satisfying $a \leq b$ for all $a \in A$. Note that b need not be in A.

Example 16.10. In the totally ordered set $(\mathbb{R}, \leq)$, consider the subset $A = (5,7)$. Then 7 is an upper bound for A, as are 20 and 99. The subset A has the least upper bound $\sup A = 7$, which is not in A.

Now take the subset $B = (5,7]$. Then 7, 20, and 99 are still upper bounds for B, with 7 being in B.

Remark 16.11. When an upper bound is in the subset $A \subseteq P$, it is automatically unique and is the least upper bound. In other words, if $m, m' \in A$ are upper bounds for A, then we have $m = m' = \sup A$.

Example 16.12. Consider the poset $\mathcal{P}(S)$ and a collection $\mathcal{C} = \{S_i\}_{i \in I}$ of subsets of S. Then the union $\bigcup_{i \in I} S_i$ is an upper bound for $\mathcal{C} \subseteq \mathcal{P}(S)$, in fact the least upper bound for $\mathcal{C}$.

Theorem 16.13 (Zorn's lemma)**.** *Let P be a non-empty poset such that every chain in P has an upper bound (in P). Then P has a maximal element (i.e., at least one).*

The proof of Zorn's lemma relies on the axiom of choice. In fact, it turns out that Zorn's lemma is equivalent to the axiom of choice.

16.2 Tychonoff's theorem

Theorem 16.14 (Tychonoff's theorem). *Let $\{X_\alpha\}_{\alpha \in A}$ be a family of compact spaces. Then their product $\prod_{\alpha \in A} X_\alpha$ is compact.*

Without loss of generality, each space X_α is non-empty. Before proving the theorem, recall a few relevant facts from the notes from November 18.

- A space is compact if and only if every net has a cluster point.
- A point x is a cluster point of a net if and only if the net has a subnet converging to x.
- A net in a product space $\prod_{\alpha \in A} X_\alpha$ converges to a point $x = (x_\alpha)_{\alpha \in A}$ if and only if it converges componentwise to x.

To show that the product $\prod_{\alpha \in A} X_\alpha$ is compact, it suffices to show that every net $(x^\lambda)_{\lambda \in \Lambda}$ in $\prod_{\alpha \in A} X_\alpha$ has a cluster point. Looking at the second and third facts above, a naive attempt would go as follows.

For each coordinate $\alpha \in A$, the space X_α is compact, so that the net $(x^\lambda_\alpha)_{\lambda \in \Lambda}$ in X_α has a cluster point $x_\alpha \in X_\alpha$. One might hope that the point $x = (x_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha$ is then a cluster point of the original net $(x^\lambda)_{\lambda \in \Lambda}$, but this is **not** true, even in a finite product!

Example 16.15. Consider the compact spaces $X = Y = [0, 1]$ and their product $X \times Y = [0, 1]^2$. Consider the sequence in $X \times Y$ given by

$$(x_n, y_n) = \left(\frac{1 + (-1)^n}{2}, \frac{1 + (-1)^n}{2} \right),$$

whose first few terms are

$$(0, 0), (1, 1), (0, 0), (1, 1), \cdots .$$

Then the sequence in $X = [0, 1]$

$$(x_n)_{n \in \mathbb{N}} = (p_X(x_n, y_n))_{n \in N} = (\frac{1 + (-1)^n}{2})_{n \in \mathbb{N}}$$

has exactly two cluster points: 0 and 1. Likewise for the sequence $(y_n)_{n \in \mathbb{N}}$ in Y. However, the pairs $(0, 1)$ and $(1, 0)$ are *not* cluster points of the original sequence $(x_n, y_n)_{n \in \mathbb{N}}$ in $X \times Y$, whose only cluster points are $(0, 0)$ and $(1, 1)$.

Even if the point $x = (x_\alpha)_{\alpha \in A}$ happened to be a cluster point of the net $(x^\lambda)_{\lambda \in \Lambda}$, one might not be able to build a subnet of $(x^\lambda)_{\lambda \in \Lambda}$ converging to x by picking a subnet of $(x^\lambda_\alpha)_{\lambda \in \Lambda}$ converging to x_α for each coordinate $\alpha \in A$.

Example 16.16. Consider $X = Y = [0, 1]$ as in Example 16.15. Consider the sequence in $X \times Y$ given by

$$(x_n, y_n) = \left(\frac{1 + (-1)^{\lceil \frac{n}{2} \rceil}}{2}, \frac{1 + (-1)^n}{2} \right)$$

whose first few terms are

$$(0,0),(0,1),(1,0),(1,1),(0,0),(0,1),(1,0),(1,1),\cdots.$$

That sequence in $X \times Y$ has exactly four cluster points: $(0,0)$, $(0,1)$, $(1,0)$, and $(1,0)$. Working one component at a time, first consider the sequence $(x_n)_{n\in\mathbb{N}}$ in X and pick its cluster point $0 \in X$, which is the limit of the subsequence

$$(x_{4k-3})_{k\in\mathbb{N}} = (x_1, x_5, x_9, x_{13}, \cdots) = (0,0,0,0,\cdots).$$

The corresponding subsequence in $X \times Y$

$$(x_{4k-3}, y_{4k-3})_{k\in\mathbb{N}} = ((x_1,y_1),(x_5,y_5),(x_9,y_9),\cdots) = ((0,0),(0,0),(0,0),\cdots)$$

has a unique cluster point $(0,0)$. In particular, it does *not* have a subsequence converging to $(0,1)$, which was a cluster point of the original sequence (x_n, y_n).

To prove that a finite product of compact spaces is compact, we used the tube lemma. Let us reprove that statement using nets, as a warm-up for arbitrary products.

Proposition 16.17. *Let X and Y be compact spaces. Then the product $X \times Y$ is compact.*

Alternate proof using nets. Let $(x_\lambda, y_\lambda)_{\lambda\in\Lambda}$ be a net in $X \times Y$. Since X is compact, the net $(x_\lambda)_{\lambda\in\Lambda}$ has a convergent subnet $(x_{\lambda_\mu})_{\mu\in M}$, with $x_{\lambda_\mu} \to x$.

Now consider the corresponding subnet $(x_{\lambda_\mu}, y_{\lambda_\mu})_{\mu\in M}$ in $X \times Y$. Since Y is compact, the net $(y_{\lambda_\mu})_{\mu\in M}$ has a convergent subnet $(y_{\lambda_{\mu_\nu}})_{\nu\in N}$, with $y_{\lambda_{\mu_\nu}} \to y$. But a subnet of a convergent net is also convergent (to the same limit points), so we have $x_{\lambda_{\mu_\nu}} \to x$. Therefore, the subnet $(x_{\lambda_{\mu_\nu}}, y_{\lambda_{\mu_\nu}})_{\nu\in N}$ of $(x_\lambda, y_\lambda)_{\lambda\in\Lambda}$ converges to $(x,y) \in X \times Y$. □

16.3 The proof

We present a proof of Tychonoff's theorem due to Chernoff [Che92]. A different proof is presented in [Mun00, §37]. All the proofs rely on some version of the axiom of choice, because it turns out that Tychonoff's theorem is equivalent to the axiom of choice.

Proof. We want to show that the given net $(x^\lambda)_{\lambda\in\Lambda}$ in $\prod_\alpha X_\alpha$ has a cluster point. For any subset $I \subseteq A$, denote by

$$p_I\colon \prod_{\alpha\in A} X_\alpha \to \prod_{\alpha\in I} X_\alpha$$

the projection map, which we also denote as a restriction $x|_I := p_I(x)$. Define a **partial cluster point** with domain I as a pair (I, x) where $x \in \prod_{\alpha\in I}$ is a cluster point of the net $(x^\lambda|_I)_{\lambda\in\Lambda}$. Order the partial cluster points by extension of the domain, that is:

$$(I, x) \leq (J, y) \iff I \subseteq J \text{ and } y|_I = x.$$

This relation defines a partial order on the set P of all partial cluster points.

Claim: P is non-empty. For every $\alpha \in A$, the space X_α is compact. Therefore, the net $(x^\lambda_\alpha)_{\lambda\in\Lambda}$ in X_α has a cluster point $x_\alpha \in X_\alpha$. This provides a partial cluster point $(\{\alpha\}, x_\alpha) \in P$. ✓

Claim: Every chain in P has an upper bound. Let $\{(I_{(t)}, x_{(t)})\}_{t\in T}$ be a chain in P. Take the union of the domains

$$I := \bigcup_{t\in T} I_{(t)} \subseteq A$$

and take $x \in \prod_{\alpha\in I} X_\alpha$ to be the unique point restricting to the given ones in the chain, that is, satisfying

$$x|_{I_{(t)}} = x_{(t)} \in \prod_{\alpha\in I_{(t)}} X_\alpha$$

for all $t \in T$. Then for all $t \in T$, we have $(I_{(t)}, x_{(t)}) \leq (I, x)$. We still need to make sure that (I, x) is an element of the poset P, in other words:

Subclaim: $x \in \prod_{\alpha\in I} X_\alpha$ is a cluster point of the net $(x^\lambda|_I)_{\lambda\in\Lambda}$.

Let $\lambda_0 \in \Lambda$ and let $U = \prod_{\alpha\in I} U_\alpha$ be a basic open neighborhood of $x \in \prod_{\alpha\in I} X_\alpha$. Explicitly, each $U_\alpha \subseteq X_\alpha$ is open, and $U_\alpha \neq X_\alpha$ holds for at most finitely many coordinates $\alpha \in I$, say, $\alpha_1, \ldots, \alpha_n$. For each $1 \leq k \leq n$, the coordinate α_k must appear in a domain $I_{(t_k)}$ for some $t_k \in T$.

Since the collection $\{(I_{(t)}, x_{(t)})\}_{t\in T}$ is a chain in P, the coordinates α_k must in fact appear in some common domain $I_{(t')}$. For instance, take $I_{(t')}$ to be the maximal element among $I_{t_1}, \ldots, I_{t_n}$ with respect to inclusion. But then we have the partial cluster point $(I_{(t')}, x_{(t')})$, that is, $x_{(t')} \in \prod_{\alpha \in I_{(t')}} X_\alpha$ is a cluster point of the net $(x^\lambda|_{I_{(t')}})_{\lambda \in \Lambda}$. Hence, there is an index $\lambda \geq \lambda_0$ satisfying

$$x^\lambda|_{I_{(t')}} \in \prod_{\alpha \in I_{(t')}} U_\alpha.$$

This implies $x^\lambda|_I \in \prod_{\alpha \in I} U_\alpha$, since the remaining coordinates add no constraint: for $\alpha \in I \setminus I_{(t')}$, we have $U_\alpha = X_\alpha$. Hence, the net $(x^\lambda|_I)_{\lambda \in \Lambda}$ is frequently in U. ✓

By Zorn's lemma, the poset P has a maximal element (J, y).

Claim: The domain of a maximal element must be $J = A$. Assume that there is a coordinate $\alpha_0 \notin J$. Since $y \in \prod_{\alpha \in J} X_\alpha$ is a cluster point of the net $(x^\lambda|_J)_{\lambda \in \Lambda}$ there is a convergent subnet $x^{\lambda_\mu}|_J \to y$. Moreover, the space X_{α_0} is compact, so the net $(x^{\lambda_\mu}_{\alpha_0})_{\mu \in M}$ has a cluster point $x_{\alpha_0} \in X_{\alpha_0}$.

The extension $\widetilde{y} \in \prod_{\alpha \in J \cup \{\alpha_0\}} X_\alpha$ defined by

$$\begin{cases} \widetilde{y}|_J = y \\ \widetilde{y}_{\alpha_0} = x_{\alpha_0} \end{cases}$$

is therefore a cluster point of the net $(x^\lambda|_{J \cup \{\alpha_0\}})_{\lambda \in \Lambda}$. This yields a partial cluster point $(J \cup \{\alpha_0\}, \widetilde{y}) \in P$ which is strictly greater than (J, y):

$$(J, y) < (J \cup \{\alpha_0\}, \widetilde{y}),$$

so that (J, y) is not maximal. ✓

In conclusion, the partial cluster point $(J, y) = (A, y) \in P$ is in fact a cluster point $y \in \prod_{\alpha \in A} X_\alpha$ of the given net $(x^\lambda)_{\lambda \in \Lambda}$. □

17 Separation axioms

17.1 Definitions

Definition 17.1. A topological space X is called:

- $\boldsymbol{T_0}$ or **Kolmogorov** if any distinct points are topologically distinguishable: For $x, y \in X$ with $x \neq y$, there is an open subset $U \subset X$ containing one of the two points but not the other.
- $\boldsymbol{T_1}$ or **Fréchet** if any distinct points are separated (i.e., not in the closure of the other): For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$ but $y \notin U_x$, whereas $y \in U_y$ but $x \notin U_y$.
- $\boldsymbol{T_2}$ or **Hausdorff** if any distinct points can be separated by neighborhoods: For $x, y \in X$ with $x \neq y$, there are open subsets $U_x, U_y \subset X$ satisfying $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.
- **regular** if points and closed sets can be separated by neighborhoods: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there are open subsets $U_x, U_C \subset X$ satisfying $x \in U_x$, $C \subset U_C$, and $U_x \cap U_C = \emptyset$.
- $\boldsymbol{T_3}$ if it is and T_1 and regular.
- **completely regular** if points and closed sets can be separated by functions: For $x \in X$ and $C \subset X$ closed with $x \notin C$, there is a continuous function $f \colon X \to [0, 1]$ satisfying $f(x) = 0$ and $f|_C \equiv 1$.
- $\boldsymbol{T_{3\frac{1}{2}}}$ or **Tychonoff** if it is T_1 and completely regular.
- **normal** if closed sets can be separated by neighborhoods: For $A, B \subset X$ closed and disjoint, there are open subsets $U, V \subset X$ satisfying $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.
- $\boldsymbol{T_4}$ if it is T_1 and normal.

Warning 17.2. Munkres includes the T_1 condition in the definition of *regular* and *normal* [Mun00, §31]. In other words, what Munkres calls *regular* is what we call T_3, and what Munkres calls *normal* is what we call T_4.

17.2 Examples and non-examples

The implications $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$ hold, as well as $T_{3\frac{1}{2}} \implies T_3$. By Urysohn's lemma, the implication $T_4 \implies T_{3\frac{1}{2}}$ also holds, so that the chain can be written as

$$T_4 \implies T_{3\frac{1}{2}} \implies T_3 \implies T_2 \implies T_1 \implies T_0.$$

Moreover, each implication is strict, i.e., there are counterexamples to the reverse direction.

Example 17.3. An indiscrete space with at least two points is not T_0.

Example 17.4. The Sierpinski space is T_0 but not T_1. Recall that the Sierpinski space is a two-point space $X = \{a, b\}$ with the topology $\mathcal{T} = \{\emptyset, \{a\}, X\}$.

Example 17.5. Let X be a set equipped with the cofinite topology. Then X is T_1. If X infinite, then X is not T_2.

Example 17.6. Let X be a set equipped with the cocountable topology. Then X is T_1. If X uncountable, then X is not T_2.

Example 17.7. An indiscrete space X is regular and normal (vacuously). Indeed, the only non-empty closed subset $C \subseteq X$ is $C = X$, for which there is no point $x \notin C$.

Proposition 17.8. *Any compact Hausdorff space is T_4.*

Proof. See Homework 7 Problem 2. □

Lemma 17.9. *Let (X, d) be a metric space, $A \subseteq X$ a non-empty subset, and consider the function $f_A \colon X \to \mathbb{R}$ defined by*

$$f_A(x) = d(x, A).$$

Then f_A is Lipschitz continuous with Lipschitz constant 1, that is:

$$|f_A(x) - f_A(y)| \leq d(x, y) \quad \text{for all } x, y \in X.$$

In particular, f_A is continuous.

Proof. For every $x, y \in X$ and $a \in A$, we have

$$d(x, a) \leq d(x, y) + d(y, a).$$

Taking the infimum over $a \in A$ yields

$$d(x, A) \leq d(x, y) + d(y, A),$$

which can be rewritten as

$$d(x, A) - d(y, A) \leq d(x, y).$$

Interchanging the role of x and y, we also obtain

$$d(y, A) - d(x, A) \leq d(x, y)$$

and therefore

$$|f_A(x) - f_A(y)| = |d(x, S) - d(y, S)| \leq d(x, y).$$

□

Proposition 17.10. *Any metric space is T_4.*

Proof. Let $A, B \subset X$ be disjoint (non-empty) closed subsets of a metric space (X, d). Consider the function $f\colon X \to [0,1]$ defined by

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}.$$

This function satisfies the following properties.

- f is well-defined since the denominator is strictly positive on X:

$$\begin{aligned} f_A(x) + f_B(x) = 0 &\iff f_A(x) = 0 \text{ and } f_B(x) = 0 \\ &\iff x \in \overline{A} = A \text{ and } x \in \overline{B} = B \\ &\iff x \in A \cap B = \emptyset. \end{aligned}$$

- f is continuous, since the sum $f_A + f_B$ is continuous, so that the quotient $f = \frac{f_A}{f_A + f_B}$ is continuous on X.
- f takes values in $[0,1]$, by the inequalities $0 \leq f_A(x) \leq f_A(x) + f_B(x)$ for all $x \in X$.
- f takes the value 0 precisely on A:

$$\begin{aligned} f(x) = 0 &\iff \frac{f_A(x)}{f_A(x) + f_B(x)} = 0 \\ &\iff f_A(x) = 0 \\ &\iff x \in \overline{A} = A. \end{aligned}$$

- f takes the value 1 precisely on B:

$$\begin{aligned} f(x) = 1 &\iff \frac{f_A(x)}{f_A(x) + f_B(x)} = 1 \\ &\iff f_A(x) = f_A(x) + f_B(x) \\ &\iff f_B(x) = 0 \\ &\iff x \in \overline{B} = B. \end{aligned}$$

Then the subsets $U := f^{-1}([0, \frac{1}{3}))$ and $V := f^{-1}((\frac{2}{3}, 1])$ are open in X, disjoint, and satisfy $A \subseteq U$ and $B \subseteq V$. □

Remark 17.11. A function $f\colon X \to [0,1]$ satisfying the properties in the proof is said to **precisely separate** the subsets A and B. A space is called **perfectly normal** if any disjoint closed subsets can be precisely separated by a function. A space is called $\boldsymbol{T_6}$ if it is T_1 and perfectly normal.

The previous proof (with a slight adjustment in the case $B = \emptyset$) shows that every metric space is T_6.

17.3 Equivalent characterizations

Proposition 17.12. *The following are equivalent.*

1. *X is T_1.*
2. *Every singleton $\{x\}$ is closed in X.*
3. *For every $x \in X$, we have*

$$\{x\} = \bigcap_{\substack{\text{all neighborhoods}\\ U \text{ of } x}} U.$$

Proposition 17.13. *The following are equivalent.*

1. *X is T_2.*
2. *The diagonal $\Delta \subseteq X \times X$ is closed in $X \times X$.*
3. *For every $x \in X$, we have*

$$\{x\} = \bigcap_{\substack{\text{closed neighborhoods}\\ C \text{ of } x}} C.$$

Proposition 17.14. *The following are equivalent.*

1. *X is regular.*
2. *For every $x \in X$, any neighborhood of x contains a closed neighborhood of x. In other words, the closed neighborhoods form a neighborhood base at x.*
3. *Given $x \in U$ where U is open, there exists an open $V \subseteq X$ satisfying*

$$x \in V \subseteq \overline{V} \subseteq U.$$

Proposition 17.15. *The following are equivalent.*

1. *X is T_2 and regular.*
2. *X is T_1 and regular.*
3. *X is T_0 and regular.*

Recall that X is called T_3 if it satisfies those equivalent conditions.

Proof. In light of the implications $T_2 \implies T_1 \implies T_0$, it suffices to show that a regular T_0 space is also T_2.

Let $x, y \in X$ be distinct points. Since X is T_0, there exists an open subset $U \subset X$ containing one of the two points but not the other. Without loss of generality, U contains x but not y. Since X is regular, there is an open subset $V \subset X$ satisfying

$$x \in V \subseteq \overline{V} \subseteq U,$$

by Proposition 17.14. Then V and $\overline{V}^c$ are separating neighborhoods for x and y: we have $x \in V$, $y \in \overline{V}^c$, and $V \cap \overline{V}^c = \emptyset$. □

Proposition 17.16. *The following are equivalent.*

1. *X is normal.*
2. *For every $A \subseteq X$ closed, any neighborhood of A contains a closed neighborhood of A.*
3. *Given $A \subseteq U$ where A is closed and U is open, there exists an open $V \subseteq X$ satisfying*

$$A \subseteq V \subseteq \overline{V} \subseteq U.$$

Remark 17.17. A normal T_0 space need **not** be T_1. For example, the Sierpinski space as in Example 17.4 is normal (vacuously). Indeed, the only non-empty closed subsets of X are the closed point $\{b\}$ and X, so that there are no disjoint (non-empty) closed subsets.

17.4 A few properties

Proposition 17.18. *Behavior of subspaces.*

1. *A subspace of a T_0 space is T_0.*
2. *A subspace of a T_1 space is T_1.*
3. *A subspace of a T_2 space is T_2.*
4. *A subspace of a regular (resp. T_3) space is regular (resp. T_3).*
5. *A subspace of a completely regular (resp. $T_{3\frac{1}{2}}$) space is completely regular (resp. $T_{3\frac{1}{2}}$).*
6. *A **closed** subspace of a normal (resp. T_4) space is normal (resp. T_4).*

Remark 17.19. A subspace of a normal space need **not** be normal in general.

Proposition 17.20. *Behavior of (arbitrary) products.*

1. *A product of T_0 spaces is T_0.*
2. *A product of T_1 spaces is T_1.*
3. *A product of T_2 spaces is T_2.*
4. *A product of regular (resp. T_3) spaces is regular (resp. T_3).*
5. *A product of completely regular (resp. $T_{3\frac{1}{2}}$) spaces is completely regular (resp. $T_{3\frac{1}{2}}$).*

Remark 17.21. A product of normal spaces need **not** be normal in general, even a finite product. See [Mun00, §31, Example 3].

17.5 The Kolmogorov quotient

Definition 17.22. Two points $x, y \in X$ in a topological space X are called **topologically distinguishable** if there exists an open subset $U \subset X$ that contains one of the points but not the other.

Recall that X is called T_0 if any distinct points are topologically distinguishable.

The points x, y are called **topologically indistinguishable** if they are not topologically distinguishable, which amounts to x and y having exactly the same neighborhoods. More explicitly: For every open subset $U \subseteq X$, we have

$$x \in U \iff y \in U.$$

Exercise 17.23. Show that topological indistinguishability is an equivalence relation on X.

Definition 17.24. The **Kolmogorov quotient** of X is the quotient space $KQ(X) := X/\!\sim$ where topologically indistinguisable points become identified.

In particular, X is T_0 if and only if the quotient map $\pi \colon X \twoheadrightarrow KQ(X)$ is a homeomorphism.

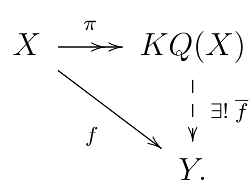
Lemma 17.25. *The Kolmogorov quotient $KQ(X)$ is T_0.*

Proof. Let $a, b \in KQ(X)$ be distinct points. Pick representatives $x, y \in X$ of a and b respectively. Since a and b are distinct, x and y are topologically distinguishable. Let $U \subset X$ be an open subset that distinguishes x and y, without loss of generality $x \in U$ and $y \notin U$.

By definition of $\sim$, U is a union of equivalence classes (i.e., $z \in U$ implies that any $z' \sim z$ is also in U), which is saying $\pi^{-1}\pi(U) = U$. Therefore $\pi(U)$ is open in $KQ(X)$ and contains $\pi(x) = a$. However $\pi(U)$ does not contain b, since every $u \in U$ is distinguishable from y, so that $\pi(u) \neq \pi(y) = b$. □

Corollary 17.26. *X is T_0 if and only if X is homeomorphic to its Kolmogorov quotient.*

Proposition 17.27. *The Kolmogorov quotient satisfies the following universal property. For any T_0 space Y and continuous map $f \colon X \to Y$, there is a unique continuous map $\overline{f} \colon KQ(X) \to Y$ satisfying $f = \overline{f} \circ \pi$, i.e., making the following diagram commute:*



In other words, $KQ(X)$ is the "closest T_0 space which X maps into".

Proof. By the universal property of the quotient topology, if suffices to show that such a map $f \colon X \to Y$ is constant on equivalence classes, that is, $x \sim x'$ implies $f(x) = f(x')$.

Assuming $f(x) \neq f(x')$, there is an open $U \subset Y$ that distinguishes $f(x)$ and $f(x')$ since Y is T_0. Without loss of generality $f(x) \in U$ and $f(x') \notin U$. Then $f^{-1}(U) \subset X$ is an open that distinguishes x and x', given $x \in f^{-1}(U)$ and $x' \notin f^{-1}(U)$. We conclude $x \not\sim x'$. □

18 Urysohn's lemma

18.1 The statement

Theorem 18.1 (Urysohn's lemma). *Let X be a normal space. Then closed subsets of X can be separated by functions: For $A, B \subseteq X$ closed and disjoint, there is a continuous function $f\colon X \to [0,1]$ satisfying $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

*Such a function is called an **Urysohn function** for A and B.*

Proof. **Step 1: Construction.**

Since A and B are disjoint, the inclusion $A \subseteq B^c =: U_1$ holds, and note that A is closed and U_1 is open.

Since X is normal, there is an open $U_{\frac{1}{2}}$ satisfying

$$A \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_1.$$

Consider the inclusion $A \subseteq U_{\frac{1}{2}}$ where A is closed and $U_{\frac{1}{2}}$ is open. There is an open $U_{\frac{1}{4}}$ satisfying

$$A \subseteq U_{\frac{1}{4}} \subseteq \overline{U_{\frac{1}{4}}} \subseteq U_{\frac{1}{2}}.$$

Likewise, consider $\overline{U_{\frac{1}{2}}} \subseteq U_1$ where $\overline{U_{\frac{1}{2}}}$ is closed and U_1 is open. There is an open $U_{\frac{3}{4}}$ satisfying

$$\overline{U_{\frac{1}{2}}} \subseteq U_{\frac{3}{4}} \subseteq \overline{U_{\frac{3}{4}}} \subseteq U_1.$$

Repeating the process, we obtain for every "dyadic rational" $r = \frac{k}{2^n}$ for some $n \geq 0$ and $0 < k \leq 2^n$ an open subset U_r satisfying

- $A \subseteq U_r$ for all r;
- $\overline{U_r} \subseteq U_s$ whenever $r < s$.

In particular we have $U_r \subseteq U_1 = B^c$ for all r, i.e., every U_r is disjoint from B.

Define the function $f\colon X \to [0,1]$ by the formula

$$f(x) = \begin{cases} 1 & \text{if } x \text{ belongs to no } U_r \\ \inf\{r \mid x \in U_r\} & \text{otherwise.} \end{cases}$$

Claim: f is an Urysohn function for A and B.

Step 2: Verification.

First, note that the dyadic rationals in $(0,1]$ are dense in $[0,1]$.

The condition $A \subseteq U_r$ for all r implies $f|_A \equiv 0$.

The condition $B \cap U_r = \emptyset$ for all r implies $f|_B \equiv 1$.

It remains to show that f is continuous. This follows from two facts.

Fact A: $x \in \overline{U_r} \implies f(x) \leq r$. Indeed, the inclusion $\overline{U_r} \subseteq U_s$ holds for all $s > r$, and s can be made arbitrarily close to r.

Fact B: $x \notin U_r \implies f(x) \geq r$. This is because the set $\{s \mid x \in U_s\}$ is upward closed, and thus cannot contain numbers $q < r$ if r is not in the set. This implies $r \leq \inf\{s \mid x \in U_s\} = f(x)$.

Continuity where $f = 0$.

Assume $f(x) = 0$, and let $\epsilon > 0$. Let r be a dyadic rational in $(0, \epsilon)$. Then we have $x \in U_r$ (by fact B) and $f(y) \leq r < \epsilon$ for all $y \in U_r$ (by fact A). Since U_r is a neighborhood of x, f is continuous at x.

Continuity where $f = 1$.

Assume $f(x) = 1$, and let $\epsilon > 0$. Let r be a dyadic rational in $(1-\epsilon, 1)$. Then we have $x \in \overline{U_r}^c$ (by fact A) and $f(y) \geq r > 1 - \epsilon$ for all $y \in \overline{U_r}^c$ (by fact B). Since $\overline{U_r}^c$ is a neighborhood of x, f is continuous at x.

Continuity where $0 < f < 1$.

Assume $0 < f(x) < 1$, and let $\epsilon > 0$. Take r, s dyadic rationals satisfying

$$f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon.$$

This implies $x \in U_s$ (by fact B) and $x \in \overline{U_r}^c$ (by fact A), in other words $x \in U_s \setminus \overline{U_r}$, which is a neighborhood of x.

Every $y \in U_s$ satisfies $f(y) \leq s$ (by fact A), whereas every $y \in \overline{U_r}^c$ satisfies $f(y) \geq r$ (by fact B), so that the inequality

$$f(x) - \epsilon < r \leq f(y) \leq s < f(x) + \epsilon$$

holds for all $y \in U_s \setminus \overline{U_r}$. This proves continuity of f at x. □

Alternate proof of continuity. Since intervals of the form $[0, \alpha)$ or $(\alpha, 1]$ form a subbase for the topology of $[0, 1]$, it suffices to show that their preimages $f^{-1}[0, \alpha)$ and $f^{-1}(\alpha, 1]$ are open in X.

Consider the equivalent statements:

$$\begin{aligned} x \in f^{-1}[0, \alpha) &\iff f(x) < \alpha \\ &\iff \text{There is a dyadic rational } r < \alpha \text{ satisfying } x \in U_r \\ &\iff x \in \bigcup_{r<\alpha} U_r. \end{aligned}$$

This proves the equality

$$f^{-1}[0, \alpha) = \bigcup_{r<\alpha} U_r$$

which is open in X since each U_r is open.

Likewise, consider the equivalent statements:

$$\begin{aligned} x \in f^{-1}(\alpha, 1] &\iff f(x) > \alpha \\ &\iff \text{ There is a dyadic rational } s > \alpha \text{ satisfying } x \notin U_s \\ &\iff \text{ There is a dyadic rational } r > \alpha \text{ satisfying } x \notin \overline{U_r} \\ &\iff x \in \bigcup_{r>\alpha} \overline{U_r}^c. \end{aligned}$$

This proves the equality

$$f^{-1}(\alpha, 1] = \bigcup_{r>\alpha} \overline{U_r}^c$$

which is open in X since each $\overline{U_r}^c$ is open. □

Remark 18.2. The result is trivially true if either A or B is empty, but the proof still works!

Remark 18.3. The Urysohn function need not separate A and B *precisely*. In other words, there can be points $x \notin A$ with $f(x) = 0$ and points $y \notin B$ with $f(y) = 1$.

18.2 Tietze extension theorem

One important application of Urysohn's lemma is in proving the following.

Theorem 18.4 (Tietze extension theorem). *Let X be a normal space and $A \subseteq X$ a closed subset.*

1. *Any continuous function $f\colon A \to [a, b]$ admits a continuous extension $\widetilde{f}\colon X \to [a, b]$.*
2. *Any continuous function $f\colon A \to \mathbb{R}$ admits a continuous extension $\widetilde{f}\colon X \to \mathbb{R}$.*

Proof. See [Mun00, Theorem 35.1]. □

References

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