M.SC. RESEARCH PROJECTS

MARTIN FRANKLAND

1. Simplicial enrichment of chain complexes

1.1. **Background.**

Notation 1.1. Let R be a ring and let $\mathrm{Ch}_{\geq 0}(R)$ denote the category of non-negatively graded chain complexes of (left) R-modules. Let $s\mathrm{Mod}_R$ denote the category of simplicial R-modules.

One would think that there is an "obvious" enrichment of $\mathrm{Ch}_{\geq 0}(R)$ in simplicial sets via the Dold–Kan correspondence, but there there are two such constructions.

(a) *Use Dold–Kan locally.* Use the mapping chain complexes in $\mathrm{Ch}_{\geq 0}(R)$ — i.e., the fact that $\mathrm{Ch}_{\geq 0}(R)$ is enriched in $\mathrm{Ch}(R)$ — to produce the mapping spaces via Dold–Kan, as explained in [Lur17, Construction 1.3.1.13].

(b) *Use Dold–Kan globally.* Transport the simplicial enrichment from $s\mathrm{Mod}_R$ via the equivalence of categories

$$N \colon s\mathrm{Mod}_R \rightleftarrows \mathrm{Ch}_{\geq 0}(R) \colon \Gamma,$$

as explained in [nLa17, Remark 2.8].

1.2. **Project.** The project encompasses three aspects.

(1) Show that the two simplicial enrichments on $\mathrm{Ch}_{\geq 0}(R)$ are *not* the same. Show that they are related by a natural map. Study the properties of that comparison map.

(2) Study how the two enrichments interact with the projective model structure on $\mathrm{Ch}_{\geq 0}(R)$. Understand why the enrichment (b) makes $\mathrm{Ch}_{\geq 0}(R)$ into a simplicial model category. Check whether the enrichment (a) does.

(3) Work out a formula for the tensoring of $\mathrm{Ch}_{\geq 0}(R)$ over simplicial sets based on the combinatorics of Δ^n viewed as a simplicial complex (as opposed to a simplicial set).

1.3. **References.** [Qui67, §II], [GJ09, §II], [Wei94, §8], [SS03, §2].

Date: May 18, 2021.

2. May's convergence theorem for the homology of DG-algebras

2.1. **Background.** Differential graded algebras (*DG-algebras* for short) are important both in algebra and in topology. Given a DG-algebra A, one often wants to compute its homology H_*A. A filtration on A yields a spectral sequence that computes H_*A starting from the homology of the filtration quotients. In seminal work, May analyzed the behavior of products and Massey products in the spectral sequence [May69]. Massey products provide the algebra H_*A with a richer algebraic structure.

2.2. **Project.** The goals of the project are the following.

(1) Compute examples of May's convergence theorem, in particular examples that illustrate why the technical assumptions are important.
(2) Compute the spectral sequence for the DG-algebra carrying the universal 3-fold Massey product in given degrees.
(3) Compute examples using different filtrations on the same DG-algebra, in particular Adams-style filtrations induced by projective classes. Find criteria for a filtration to have better convergence properties than others.

One of my motivations for the project is that May's convergence theorem should be an instance of a generalization of Moss' convergence theorem to triangulated categories. Understanding concrete examples with DG-algebras should shed light on the general statement.

2.3. **References.** [Car54], [GM03, §V.3], [May69], [GM74, §5], [BMR14, §10].

3. Relative derived category of the derived category of a ring

3.1. **Background.** Relative homological algebra goes back to the 1960s, in work of MacLane [ML63, §IX], Eilenberg, and Moore [EM65]. A projective class $\mathcal{P}$ specifies certain objects that will play the role of projective objects in an abelian category. This allows much more general resolutions than the usual projective resolutions.

Christensen and Hovey constructed model categories that refine certain relative derived categories, at least when starting from an abelian category [CH02]. Chachólski et al. have a different construction of certain relative derived categories starting from an abelian category with an injective class [CNPS18]. Their construction remains within the realm of triangulated categories without using model categories, but the input is an abelian category, not an arbitrary triangulated category.

A less studied construction is that of the $\mathcal{P}$-relative derived category $D_{\mathcal{P}}(\mathcal{T})$ starting from a *triangulated* category $\mathcal{T}$ equipped with a projective class $\mathcal{P}$. A motivation for studying $D_{\mathcal{P}}(\mathcal{T})$ is that it provides a natural home for $\mathcal{P}$-relative Ext groups, which appear as the E_2 term of the $\mathcal{P}$-relative Adams spectral sequence [Chr98].

3.2. **Project.**

Notation 3.1. Let R be a ring and let $\mathcal{T} = D^+(R)$ denote the bounded below derived category of R, viewed as a triangulated category. Let $\mathcal{P}$ denote the ghost projective class in $\mathcal{T}$, i.e., the stable projective class generated by $R[0]$ (the chain complex with R concentrated in degree 0).

The project consists of the following problems.

1. Describe the $\mathcal{P}$-relative bounded below derived category $D^+_{\mathcal{P}}(\mathcal{T})$. Check whether it is equivalent (as a triangulated category) to the homotopy category of some model category of bicomplexes of R-modules constructed by Muro and Roitzheim [MR19] or by Cirici et al. [CESLW20].
2. Show that $\mathrm{Ch}(\mathcal{T})$ admits Spaltenstein-style resolutions relative to $\mathcal{P}$, providing a construction of hom-sets in $D_{\mathcal{P}}(\mathcal{T})$.
3. Check what happens when we start with the triangulated category $\mathcal{T} = D(R)$, the unbounded derived category of R. Compare the triangulated categories $D^+_{\mathcal{P}}(D(R))$ and $D_{\mathcal{P}}(D^+(R))$.

3.3. **References.** [Chr98], [CH02], [CNPS18].

4. Near-rings up to homotopy

4.1. **Background.** Algebraic structures up to homotopy arise commonly in topology and algebra. Notably, loop spaces are associative up to coherent homotopies encoded by Stasheff's associahedra, which assemble into an A_∞ operad [Sta70]. Likewise, the structure of n-fold loop spaces is governed by an E_n operad [May72].

In the study of higher order cohomology operations, composition is bilinear up to homotopy—which is why the Steenrod algebra is an algebra—but not strictly. Linearity with respect to the right variable only holds up to coherent homotopies $a(x+y) \sim ax + ay$. This higher distributivity was described in [BF17]. Unlike higher associativity for loop spaces, higher distributivity for cohomology operations is not governed by an operad. The goal of this project is to show that higher distributivity is governed by a (topologically enriched) Lawvere theory.

4.2. **Project.** For our purposes here, let us introduce non-standard terminology.

Definition 4.1. A **near-ring** is defined similarly to a (unital) ring, except that multiplication is not assumed linear with respect to the right variable. That is, the distributivity axiom $a(x+y) = ax + ay$ is *not* assumed, though we do assume $x0 = 0$.

Notation 4.2. Let $\mathcal{T}_{\text{ring}}$ denote the (one-sorted Lawvere) theory of rings, and $\mathcal{T}_{\text{near}}$ the theory of near-rings.

There is a canonical map of theories $\pi\colon \mathcal{T}_{\text{near}} \to \mathcal{T}_{\text{ring}}$. Restriction along π induces on models the inclusion of the full subcategory of rings into near-rings.

The goals of the project are the following.

1. Construct a topologically enriched theory $\mathcal{T}_{D_\infty}$ whose models are the ∞-*distributive* topological near-rings in the sense of [BF17] (or rather the one-sorted analogue thereof). This means that multiplication is right linear up to coherent homotopy.
2. Exhibit maps of topologically enriched theories
$$\mathcal{T}_{\text{near}} \longrightarrow \mathcal{T}_{D_\infty} \xrightarrow{\sim} \mathcal{T}_{\text{ring}}$$
where the last step is a Dwyer–Kan equivalence. This would imply that every ∞-distributive topological near-ring is strictifiable (i.e., weakly equivalent to a topological ring).
3. Work out some consequences for the topological near-ring of self-maps of a generalized Eilenberg–MacLane space.

4.3. **References.** [BF17], [BV73, §II], [Sta70], [May72], [Bor94, §3.1–3.3].

5. Classifying rational homotopy types with a given homotopy Lie algebra

5.1. **Background.** Given a space X, its homotopy groups $\pi_* X$ admit primary homotopy operations, an algebraic structure known as a Π-*algebra*. One would like to classify the homotopy types with a given Π-algebra: find all spaces Y satisfying $\pi_* Y \cong \pi_* X$ (as Π-algebras). There is an obstruction theory building up the moduli space of all such spaces [BDG04]. The obstructions live in Quillen cohomology of the Π-algebra $\pi_* X$. Answering the problem in general is difficult. In special cases where most of the obstruction groups vanish, one can compute the moduli space explicitly [Fra11].

The problem becomes simpler if we want to classify all *rational* homotopy types with a given rational Π-algebra. For a simply-connected rational space X, its Π-algebra consists only of the graded $\mathbb{Q}$-vector space $\pi_* X$ together with Whitehead products, in other words, the homotopy Lie algebra $\pi_*(\Omega X)$. Quillen cohomology of a graded Lie algebra L corresponds to the usual Lie algebra cohomology. By the work of Quillen [Qui69], the problem becomes the following: Find all differential graded Lie algebras $\mathcal{L}$ over $\mathbb{Q}$ with homology $H_* \mathcal{L} \cong L$ (as graded Lie algebras). There is always at least one realization, namely, endowing L with the zero differential. Every realization of L can be obtained as a perturbation of the trivial one, and there is a deformation theory controlling the possible perturbations [Zaw16].

5.2. **Project.** The project consists of classifying the realizations of certain graded Lie algebras L over $\mathbb{Q}$. The approach is to use the obstruction theory where the obstructions live in the Lie algebra cohomology of L [Bla99, §4.22], [Bla04, §6].

The goals of the project are the following.

1. Check that Quillen cohomology of graded Lie algebras corresponds to Lie algebra cohomology in the sense of Chevalley–Eilenberg, in other words, the graded analogue of [Qui70, Proposition 3.7].
2. Find conditions on the graded Lie algebra L for all obstruction groups to vanish. Such an L admits only the trivial realization.
3. Find conditions on the graded Lie algebra L for most of the obstruction groups to vanish. Compute the moduli space of realizations for some of those examples.
4. Compare the results to those obtained using the deformation theory of Zawodniak.

5.3. **References.** [Bla99, §4.22], [Bla04, §6], [Zaw16], [FHT01, §IV].

6. Distinguishing rational homotopy types via secondary operations

6.1. **Background.** The cohomology algebra H^*X of a space rarely determines the homotopy type of X. Two inequivalent spaces $X \not\simeq Y$ can have isomorphic cohomology algebras $H^*X \cong H^*Y$. Likewise, the homotopy groups π_*X rarely determine the homotopy type of X. Two inequivalent spaces can have isomorphic homotopy groups $\pi_*X \cong \pi_*Y$ with the same Whitehead products, in other words, isomorphic homotopy Lie algebra $\pi_*(\Omega X) \cong \pi_*(\Omega Y)$.

This is also true in rational homotopy theory. In fact, the cohomology algebra H^*X and the homotopy Lie algebra $\pi_*(\Omega X)$ *together* still don't determine the rational homotopy type of X. Lemaire and Sigrist found an infinite family of distinct rational homotopy types that all share the same cohomology algebra *and* the same homotopy Lie algebra [LS78].

6.2. **Project.** Rationally, the cohomology algebra H^*X and the homotopy Lie algebra $\pi_*(\Omega X)$ contain all the primary algebraic structure. The idea is to look at the finer structure of secondary operations to distinguish the homotopy types in the Lemaire–Sigrist example.

The goals of the project are the following.

(1) For each homotopy type X in the Lemaire–Sigrist example, compute the secondary cohomology operations, namely the 3-fold Massey products in $H^*(X)$.
(2) Compute the secondary homotopy operations, namely the 3-fold Lie–Massey brackets in $\pi_*(\Omega X)$.
(3) Check to what extent the secondary information is enough to distinguish those rational homotopy types.

This project was inspired by discussions with David Blanc.

6.3. **References.** [LS78], [Fél80], [Tan83, §V], [FHT01].

7. Quillen cohomology of modules over groups

7.1. **Background.** André–Quillen cohomology is a cohomology theory for commutative rings, based on simplicial resolutions, due independently to André and Quillen [And67] [Qui70]. It was used to solve problems in commutative algebra. The construction of Quillen cohomology is available more generally in an algebraic category, recovering for instance group cohomology, Lie algebra cohomology, and Shukla cohomology of associative algebras. It has found various applications in algebra, deformation theory, and topology [GS07].

7.2. **Project.** The project consists of computing Quillen cohomology in the category ModGp of modules over groups, where an object (G, M) consists of a group G and a G-module M. The category ModGp is the same as the category of 2-truncated Π-algebras, which was my original motivation for studying it.

In my thesis, I carried out some ground work towards computing Quillen cohomology in ModGp [Fra10, §7]. I described Beck modules, abelianizations, pushforwards, and some reduction steps towards the computation.

The goals of the project are the following.

(1) Complete the calculations outlined in the reduction steps of [Fra10, §7.4].
(2) Specialize to the case where we start with a 3-truncated Π-algebra A, and the object of interest is its 2-truncation $P_2A = (A_1, A_2)$ with coefficient module $\Omega A = (A_2, A_3)$.
(3) Compute some explicit examples.

7.3. **References.** [Qui67, §II.5], [Qui70], [GS07], [Fra10, §7], [Fra15].

8. Pointed Reedy categories and model structures

8.1. **Background.** Given a model category $\mathcal{C}$, one often needs a homotopy theory of I-shaped diagrams in $\mathcal{C}$, for I a small category, where the weak equivalences are objectwise. If $\mathcal{C}$ is a nice enough model category, then the diagram category $\mathcal{C}^I$ admits the projective model structure, where the fibrations are objectwise and the cofibrations are cumbersome to work with. Dually, $\mathcal{C}^I$ admits the injective model structure, where the cofibrations are objectwise and the fibrations are cumbersome.

If the indexing category I is Reedy, then the diagram category $\mathcal{C}^I$ also admits the Reedy model structure, where both the cofibrations and fibrations are somewhat computable. For example, the simplex category Δ and its opposite Δ^{op} are Reedy, which yields a Reedy model structure on cosimplicial spaces and simplicial spaces, respectively. As another example, taking $I = \mathbb{N}$, the Reedy model structure on the category of filtered objects $\mathcal{C}^{\mathbb{N}}$ coincides with the projective model structure. Dually, the Reedy model structure on the category of towers $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ coincides with the injective model structure.

Enriched model categories often arise in homotopy theory. Given a symmetric monoidal (model) category $\mathcal{V}$, a $\mathcal{V}$-Reedy category I and a $\mathcal{V}$-model category $\mathcal{C}$, the enriched diagram category $\mathcal{C}^I$ admits the $\mathcal{V}$-Reedy model structure, by work of Angeltveit [Ang08]. Under suitable assumptions, the enriched diagram category $\mathcal{C}^I$ admits the projective and injective model structures, by work of Moser [Mos19].

8.2. **Project.** The project consists of specializing the enriched Reedy, projective, and injective model structures when the enrichment category is $\mathcal{V} = \mathrm{Set}_*$, the category of pointed sets (with the smash product $\wedge$ as monoidal structure). This case is simple enough to make all the structure explicit, yet useful since Set_*-enriched categories (a.k.a. pointed categories) abound in nature.

For instance, chain complexes in a pointed category do *not* form a diagram category, since the chain complex condition $\partial^2 = 0$ is not diagrammatic. However, chain complexes *do* form an enriched diagram category, enriched over Set_*. As an example of application, (co)chain complexes of spaces can be used to build certain (co)simplicial resolutions of spaces [BS18] [BJT19].

The goals of the project are the following.

1. Spell out the data of a Set_*-Reedy category.
2. Spell out the Set_*-Reedy, Set_*-projective, and Set_*-injective model structures.
3. Work out the examples of chain complexes, cochain complexes, and truncated (co)chain complexes in a nice enough pointed model category $\mathcal{C}$.

8.3. **References.** [Ang08], [Mos19], [GJ09, §VII], [Hir03, §15], [Rie14, §14].

9. Change of enrichment for enriched model categories

9.1. **Background.** A simplicial model category is a model category $\mathcal{C}$ that is enriched in simplicial sets and satisfies a compatibility condition between the enrichment and the model structure — Quillen's axiom SM7. One salient feature is that for cofibrant X and fibrant Y, the mapping space $\mathcal{C}(X,Y)$ is also the derived mapping space, which is harder to compute without the enrichment.

Many model categories in nature are enriched over a monoidal category $\mathcal{V}$ which is itself a model category, for example simplicial sets, topological spaces, chain complexes, or spectra. Given a lax monoidal functor $U\colon \mathcal{V} \to \mathcal{W}$ and a $\mathcal{V}$-enriched category $\mathcal{C}$, the **change of enrichment** along U produces a $\mathcal{W}$-enriched category given by the formula

$$(U\mathcal{C})(X,Y) := U\left(\mathcal{C}(X,Y)\right).$$

For enriched model categories, one would hope that the change of enrichment is compatible with the model structures. This happens if U is the right adjoint in a (strong) monoidal Quillen pair [Dug06, Lemma A.5] [GM20, Proposition 3.8], for example the singular set functor $\mathrm{Sing}\colon \mathrm{Top} \to s\mathrm{Set}$.

9.2. **Project.** The goals of the project are the following.

(1) Find (useful) sufficient conditions such that the change of enrichment along a lax monoidal functor $U\colon \mathcal{V} \to \mathcal{W}$ sends a $\mathcal{V}$-model category to a $\mathcal{W}$-model category.
(2) For the following lax monoidal functors $U\colon \mathcal{V} \to \mathcal{W}$, determine whether the change of enrichment along U sends a $\mathcal{V}$-model category to a $\mathcal{W}$-model category:
    (a) The normalized chain complex $N\colon s\mathrm{Ab} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$.
    (b) The denormalization $\Gamma\colon \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to s\mathrm{Ab}$.
    (c) The good truncation $\tau_{\geq 0}\colon \mathrm{Ch}(\mathbb{Z}) \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$.
    (d) The forgetful functor $U\colon s\mathrm{Ab} \to s\mathrm{Set}$.
    (e) Geometric realization $|\cdot|\colon s\mathrm{Set} \to \mathrm{Top}$.
    (f) Various models of the zeroth space functor $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{Top}$.

9.3. **References.** [Qui67, §II.2], [GJ09, §II.3], [Hov99, §4.2], [RSS01], [Dug06, §A], [MP12, §16.4], [GM20, §3.1].

References

[And67] M. André, *Méthode simpliciale en algèbre homologique et algèbre commutative*, Lecture Notes in Mathematics, Vol. 32, Springer-Verlag, Berlin-New York, 1967 (French). MR0214644 (35 #5493)

[Ang08] V. Angeltveit, *Enriched Reedy categories*, Proc. Amer. Math. Soc. **136** (2008), no. 7, 2323–2332, DOI 10.1090/S0002-9939-08-09185-5. MR2390498

[BMR14] T. Barthel, J. P. May, and E. Riehl, *Six model structures for DG-modules over DGAs: model category theory in homological action*, New York J. Math. **20** (2014), 1077–1159. MR3291613

[BF17] H.-J. Baues and M. Frankland, *Eilenberg–MacLane mapping algebras and higher distributivity up to homotopy*, New York J. Math. **23** (2017), 1539–1580.

[Bla99] D. Blanc, *Algebraic invariants for homotopy types*, Math. Proc. Cambridge Philos. Soc. **127** (1999), no. 3, 497 –523, DOI 10.1017/S030500419900393X. MR1713125 (2000h:55025)

[Bla04] ———, *Homotopy operations and rational homotopy type*, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., vol. 215, Birkhäuser, Basel, 2004, pp. 47–75. MR2039759

[BDG04] D. Blanc, W. G. Dwyer, and P. G. Goerss, *The realization space of a Π-algebra: a moduli problem in algebraic topology*, Topology **43** (2004), no. 4, 857 –892.

[BS18] D. Blanc and D. Sen, *Higher cohomology operations and R-completion*, Algebr. Geom. Topol. **18** (2018), no. 1, 247–312, DOI 10.2140/agt.2018.18.247. MR3748244

[BJT19] D. Blanc, M. W. Johnson, and J. M. Turner, *Higher homotopy invariants for spaces and maps* (2019), Preprint, available at `arXiv:1911.08259`.

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973. MR0420609

[Bor94] F. Borceux, *Handbook of Categorical Algebra 2: Categories and Structures*, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, 1994.

[Car54] H. Cartan, *Sur les groupes d'Eilenberg-Mac Lane $H(\Pi, n)$. I. Méthode des constructions*, Proc. Nat. Acad. Sci. U. S. A. **40** (1954), 467–471, DOI 10.1073/pnas.40.6.467 (French). MR0065160

[CNPS18] W. Chachólski, A. Neeman, W. Pitsch, and J. Scherer, *Relative homological algebra via truncations*, Doc. Math. **23** (2018), 895–937. MR3861041

[Chr98] J. D. Christensen, *Ideals in triangulated categories: phantoms, ghosts and skeleta*, Adv. Math. **136** (1998), no. 2, 284–339, DOI 10.1006/aima.1998.1735. MR1626856 (99g:18007)

[CH02] J. D. Christensen and M. Hovey, *Quillen model structures for relative homological algebra*, Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 2, 261–293, DOI 10.1017/S0305004102006126. MR1912401 (2003f:18012)

[CESLW20] J. Cirici, D. Egas Santander, M. Livernet, and S. Whitehouse, *Model category structures and spectral sequences*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 6, 2815–2848, DOI 10.1017/prm.2019.45. MR4190091

[Dug06] D. Dugger, *Spectral enrichments of model categories*, Homology Homotopy Appl. **8** (2006), no. 1, 1–30. MR2205213

[EM65] S. Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Mem. Amer. Math. Soc. No. **55** (1965), 39. MR0178036

[Fél80] Y. Félix, *Dénombrement des types de K-homotopie théorie de la déformation*, Mém. Soc. Math. France (N.S.) **3** (1980/81), ii+49 (French, with English summary). MR607618

[FHT01] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR1802847

[Fra10] M. Frankland, *Quillen Cohomology of Pi-Algebras and Application to their Realization*, ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–Massachusetts Institute of Technology. MR2801771

[Fra11] ———, *Moduli spaces of 2-stage Postnikov systems*, Topology Appl. **158** (2011), no. 11, 1296–1306, DOI 10.1016/j.topol.2011.05.002. MR2806362

[Fra15] ———, *Behavior of Quillen (co)homology with respect to adjunctions*, Homology Homotopy Appl. **17** (2015), no. 1, 67–109, DOI 10.4310/HHA.2015.v17.n1.a3. MR3338541

[GM03] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, 2nd ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR1950475 (2003m:18001)

[GS07] P. G. Goerss and K. Schemmerhorn, *Model categories and simplicial methods*, Contemporary Mathematics **436** (2007), 3 –49.

[GJ09] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612]. MR2840650

[GM20] B. J. Guillou and J. P. May, *Enriched model categories and presheaf categories*, New York J. Math. **26** (2020), 37–91. MR4047399

[GM74] V. K. A. M. Gugenheim and J. P. May, *On the theory and applications of differential torsion products*, American Mathematical Society, Providence, R.I., 1974. Memoirs of the American Mathematical Society, No. 142. MR0394720

[Hir03] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR1944041 (2003j:18018)

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR1650134 (99h:55031)

[LS78] J.-M. Lemaire and F. Sigrist, *Dénombrement de types d'homotopie rationnelle*, C. R. Acad. Sci. Paris Sér. A-B **287** (1978), no. 3, A109–A112 (French, with English summary). MR503736

[Lur17] J. Lurie, *Higher algebra*, Sep. 18, 2017, Preprint.

[ML63] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. MR0156879

[May69] J. P. May, *Matric Massey products*, J. Algebra **12** (1969), 533–568, DOI 10.1016/0021-8693(69)90027-1. MR0238929

[May72] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271. MR0420610

[MP12] J. P. May and K. Ponto, *More concise algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories. MR2884233

[Mos19] L. Moser, *Injective and projective model structures on enriched diagram categories*, Homology Homotopy Appl. **21** (2019), no. 2, 279–300, DOI 10.4310/HHA.2019.v21.n2.a15. MR3923784

[MR19] F. Muro and C. Roitzheim, *Homotopy theory of bicomplexes*, J. Pure Appl. Algebra **223** (2019), no. 5, 1913–1939, DOI 10.1016/j.jpaa.2018.08.007. MR3906533

[nLa17] nLab authors, *Dold–Kan correspondence* (Dec. 27, 2017), `http://ncatlab.org/nlab/show/Dold-Kan%20correspondence`. Accessed June 19, 2019. Revision 101.

[Qui67] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1967. MR0223432

[Qui69] D. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205 –295. MR0258031 (41 #2678)

[Qui70] ———, *On the (co-) homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 65–87. MR0257068

[RSS01] C. Rezk, S. Schwede, and B. Shipley, *Simplicial structures on model categories and functors*, Amer. J. Math. **123** (2001), no. 3, 551–575. MR1833153

[Rie14] E. Riehl, *Categorical homotopy theory*, New Mathematical Monographs, vol. 24, Cambridge University Press, Cambridge, 2014. MR3221774

[SS03] S. Schwede and B. Shipley, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. **3** (2003), 287–334, DOI 10.2140/agt.2003.3.287. MR1997322

[Sta70] J. Stasheff, *H-spaces from a homotopy point of view*, Lecture Notes in Mathematics, Vol. 161, Springer-Verlag, Berlin-New York, 1970. MR0270372

[Tan83] D. Tanré, *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, Lecture Notes in Mathematics, vol. 1025, Springer-Verlag, Berlin, 1983 (French). MR764769

[Wei94] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1994.

[Zaw16] M. Zawodniak, *A moduli space for rational homotopy types with the same homotopy Lie algebra*, 2016. Thesis (Ph.D.)–University of Georgia.

Email address: `Martin.Frankland@uregina.ca`