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A Compilation of the Foundations of Algebraic K-Theory and Their Connections to Higher K-Theory

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Abstract

This text provides a comprehensive introduction to algebraic K-theory and its extension to higher K-theory. Starting with foundational topics in group theory and module theory, the work proceeds to discuss the classical K-groups K_0, K_1, and K_2, which capture fundamental invariants of rings and modules. The text then extends into higher K-theory, highlighting its topological foundation and connections to homotopy theory. By bridging algebraic structures with topological spaces, higher K-theory offers a powerful framework for understanding both algebraic and geometric objects. This work serves as a foundation for further exploration into the applications and advancements of K-theory across mathematics.

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Partial List of Notations

Let A be a set, and $+ : A \times A \to A$ and $\times : A \times A \to A$ be binary operations defined on A (called addition and multiplication respectively) such that the distributivity law holds when A is equipped with both operations. The algebraic structure A receives the following names based on what laws the binary operations satisfy (NE stands for Not Equipped):

A	$+$				$\times$			
	Associativity	Identity	Inverses	Commutativity	Associativity	Identity	Inverses	Commutativity
Semigroup	✓				NE	NE	NE	NE
Monoid	✓	✓			NE	NE	NE	NE
Commutative Monoid	✓	✓		✓	NE	NE	NE	NE
Group	✓	✓	✓		NE	NE	NE	NE
Abelian Group	✓	✓	✓	✓	NE	NE	NE	NE
Semiring (Rig)	✓	✓		✓	✓	✓		
Rng	✓	✓	✓	✓	✓			
Ring	✓	✓	✓	✓	✓	✓		
Commutative Ring	✓	✓	✓	✓	✓	✓		✓
Skew Field	✓	✓	✓	✓	✓	✓	✓	
Field	✓	✓	✓	✓	✓	✓	✓	✓

$\mathcal{C}(X, Y)$	set of morphisms from X to Y in the category $\mathcal{C}$
Mon	category of monoids
CMon	category of commutative monoids
Grp	category of groups
Ab	category of abelian groups
Ring	category of rings
CRing	category of commutative rings
R-**Mod**	category of R-modules
$\mathrm{Ho}(\mathbf{CW})$	homotopy category of CW-complexes
$\mathrm{Ho}(\mathbf{CW})_*$	homotopy category of pointed CW-complexes
$\mathrm{Hom}(A, B)$	set of homomorphism from A to B
$\oplus$	direct sum
$\otimes$	tensor product
$\mathbb{Z}/n$	group of integers modulo n
$M(R)$	infinite matrix group of the ring R
$GL(R)$	infinite general linear group of the ring R
$E(R)$	infinite elementary matrix group of the ring R
$H_n(G, M)$	n-th homology group of G with coefficients M
$H^n(G, M)$	n-th cohomology group of G with coefficients M
$\mathrm{St}(R)$	Steinberg group of the ring R
$K_n(R)$	n-th K-group of the ring R
$K_n(R, I)$	n-th relative K-group of the ring R with respect to its two-sided ideal I
$\mathbb{Z}[G]$	group ring of G over $\mathbb{Z}$
IBN	invariant basis number
BG	classifying space of the group G
$\star$	join of topological spaces
X^+	plus construction of the topological space X
$K(R)$	K-theory space of the ring R

Introduction

Algebraic K-theory and its extension to higher K-theory form a crucial part of modern mathematics, bridging algebraic topology, category theory, and homotopy theory. At the heart of this theory lies the study of algebraic invariants that reflect the structure of rings, modules, and groups in sophisticated and often subtle ways. The development of these invariants has profound implications across many areas of mathematics, providing a toolset for analyzing everything from the structure of algebraic varieties to the topology of manifolds.

This text is a comprehensive introduction to the core aspects of classical and higher K-theory. The goal is to build an understanding that starts from basic concepts in the theory of groups and modules, progresses through the classical constructions of algebraic K-theory, and culminates at the start of the modern framework of higher K-theory.

Chapter 1: Foundations of Algebraic Structures. The journey begins with the foundational concepts that form the basis of much of algebraic K-theory: group theory, modules, and their relationships. Although these topics are not the primary focus of the text, they serve as the essential building blocks for the more advanced constructions in subsequent chapters.

In particular, four main areas are explored: *projective and stably free modules*, which play a central role in defining the basic K-theory groups. The discussion then turns to the concept of *group completion*, which provides a method for constructing abelian groups from abelian monoids. The study continues with *universal central extensions*, an important tool in both group theory and phonological algebra. Finally *group homology and cohomology* also make an appearance, helping to lay the groundwork for understanding the interactions between algebraic structures and their topological properties.

While these topics are essential in their own right, they form part of the larger landscape that motivates the study of higher K-theory in later chapters.

Chapter 2: Classical Algebraic K-Theory. The second chapter delves into the classical algebraic K-theory groups: K_0, K_1, and K_2. These groups serve as the algebraic invariants that capture essential structural properties of rings and modules. Algebraic K-theory initially arose from the need to study and classify projective modules, and to capture information about invertible matrices. The groups K_0, K_1, and K_2 reflect various levels of complexity in this study.

The *Grothendieck group* K_0 is the foundation of classical K-theory. It is constructed from the isomorphism classes of projective modules over a ring R, and provides an algebraic tool to classify these modules up to isomorphism. More than a mere construction, K_0 reflects deeper questions about the failure of modules to be free, and the structure of the ring itself.

Building on K_0, the *Whitehead group* K_1 introduces a more refined measure of algebraic structure by considering the invertible elements of the ring. Specifically, K_1 is related to the general linear group $GL(R)$ of invertible matrices over a ring, and encodes information about the elementary operations needed to reduce a matrix to its identity form. This leads to insights into the topology of the space of invertible matrices and links the algebraic structure of $GL(R)$ with its topological properties.

Moving from K_1 to K_2, we encounter a more complex group that studies further the matrix identities and their extensions. The *Milnor group*, or the second K-group, addresses higher-order relations among matrices and connects with important algebraic constructions such as Steinberg groups. Although the computations for K_2 become increasingly intricate, they offer valuable insight into the deeper structure of algebraic objects.

Each of these groups comes with a rich theory of exact sequences, and their relationships give rise to advanced constructions, such as the extension of the Mayer-Vietoris sequence, which relates K_0, K_1, and K_2 to each other in intricate ways. These sequences are critical in the study of K-theory and serve as a cornerstone for the development of the higher K-groups.

Chapter 3: Higher K-Theory The concept of higher K-theory, as developed by Quillen and others, significantly extends the classical K-theory by incorporating techniques from algebraic topology. Rather than focusing solely on algebraic objects, higher K-theory introduces homotopy-theoretic constructions that encode algebraic invariants as homotopy groups of certain topological spaces. This shift marks a profound deepening of the theory, where the structure of rings is viewed through the lens of the topology of associated spaces.

The chapter begins with the construction of the *classifying space* of a topological group, which serves as a key example of how algebraic objects can be understood in a topological context. This leads naturally to the *plus construction*, a powerful tool used in algebraic topology to simplify the structure of spaces by eliminating perfect normal subgroups from their fundamental groups.

From this point, the discussion shifts toward the *higher K-groups* themselves, which are derived from the work of Quillen. These groups generalize the classical K-groups K_0, K_1, and K_2, and their homotopy-theoretic formulation enables them to encode richer information about the structure of rings and their associated spaces. Higher K-theory gives rise to powerful techniques for computing and analyzing these groups, bridging algebra and topology in novel ways.

Chapter I

Foundations of Algebraic Structures

This chapter addresses fundamental topics in the theory of groups and modules. While these topics are of great importance in their own right, the chapter focuses on presenting the most relevant results for the development of the central theme of the text, including examples that illustrate the key concepts and their applicability.

I.1 Projective and Stably Free Modules

Definition I.1.1 (Projective Module)**.** An R-module P is called *projective* if one of the following equivalent statements holds:

- *Lifting Property.* For every surjective R-module homomorphism $g : M \twoheadrightarrow N$ and R-module homomorphism $f : P \to N$ there exists an R-module homomorphism $f' : P \to M$ satisfying $g \circ f' = f$, i.e., the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists f'"'] \arrow[d, "f"] & \\ M \arrow[r, two heads, "g"'] & N \arrow[r] & 0. \end{tikzcd}$$

- *Direct Summand of Free.* There exists an R-module Q such that $P \oplus Q$ is a free R-module.

- *Splits Exact Sequences.* Every exact sequence of R-modules of the form

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow P \longrightarrow 0$$

splits.

- *Exactness of* $\mathrm{Hom}_R(P,-)$. The functor $\mathrm{Hom}_R(P,-) : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is exact.

From the previous definition, it follows directly that all free R-modules are projective R-modules.

Proposition I.1.2. *The statements from the definition of projective module are equivalent.*

Proof. See [Lan02, III §4]. □

Proposition I.1.3. *A direct sum of R-modules $P = \bigoplus_{i\in I} P_i$ is projective if and only if each summand P_i is projective.*

Proof. Suppose that P is projective. Let Q be an R-module such that $P \oplus Q$ is free. Then, for each $j \in I$ one has that

$$P \oplus Q \cong P_j \oplus \left(\bigoplus_{\substack{i\in I \\ i\neq j}} P_i \oplus Q \right),$$

so P_j is projective. On the other hand, if each summand P_i is projective, for each $i \in I$ there exists an R-module Q_i such that $P_i \oplus Q_i$ is free. Therefore

$$\bigoplus_{i\in I} P_i \oplus \bigoplus_{i\in I} Q \cong \bigoplus_{i\in I} (P_i \oplus Q_i),$$

which is free, so P is projective. □

Definition I.1.4 (Stably Free Module)**.** An R-module P is called *stably free of rank* $n-m$ if there exist non-negative integers m and n such that $P \oplus R^m \cong R^n$. A module is simply called *stably free* if it is stably free of rank $n-m$ for some m and n.

Notice that all stably free modules are projective by definition. If the previous definition allowed m to be an infinite cardinal, then all projective modules would be stably free. This is due to an algebra trick usually referred to as the "Eilenberg swindle":

Proposition I.1.5 (Eilenberg Swindle)**.** *For all projective R-modules P there exists a free R-module F such that $P \oplus F$ is free.*

Proof. Let P be a projective R-module, so that there exists an R-module Q such that $P \oplus Q$ is free. It is clear that $\bigoplus_{i=0}^{\infty}(P \oplus Q)$ is free. Then

$$\begin{aligned} P \oplus \bigoplus_{i=0}^{\infty}(P \oplus Q) &\cong P \oplus (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\ &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\ &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\ &\cong \bigoplus_{i=0}^{\infty}(P \oplus Q) \end{aligned}$$

is free. □

On the other hand, if the definition of a stably free module allowed n to be an infinite cardinal then all stably free modules would be free. To prove this, notice that if n were infinite then the stably free module would necessarily be not finitely generated, and using the following proposition yields the desired statement.

Proposition I.1.6. *If P is stably free but not finitely generated, then P is free.*

Proof. Let $P \oplus R^m \cong \bigoplus_{i \in I} R = F$ with P not finitely generated. Since the standard projection from F to P is surjective, and P cannot be finitely generated, it follows that the indexing set I must be infinite. Let $f : F \twoheadrightarrow R^m$ be the standard projection. Because m is finite, there exists a finite subset $I_0 \subset I$ and a collection $\{e_i\}_{i \in I_0}$ of the standard basis elements in F such that if $F_0 = \sum_{i \in I_0} e_i \cdot R \subset F$ then $f|_{F_0} : F_0 \twoheadrightarrow R^m$ is a surjection. Such a collection can be defined by all the non-zero entries in F of every inverse image with respect to f of the elements in the standard basis of R^m.

Let $Q = P \cap F_0$ and observe that $\bigoplus_{i \in I} R = P + F_0$. Then, by the Second Isomorphism Theorem, it follows that

$$F/F_0 \cong (P + F_0)/F_0 \cong P/(P \cap F_0) \cong P/Q,$$

where $F/F_0 \cong \bigoplus_{i \in I - I_0} R$ is free of infinite rank since I_0 is finite, so $P/Q \cong R^m \oplus F_1$ for some free module F_1. Now, using the fact that free modules are projective and the fact that projective modules split exact sequences, it follows that the sequences

$$0 \longrightarrow Q \xhookrightarrow{\text{inc}} P \twoheadrightarrow P/Q \cong R^m \oplus F_1 \longrightarrow 0$$

$$0 \longrightarrow Q \cong \ker(f|_{F_0}) \hookrightarrow F_0 \xrightarrow{f|_{F_0}} R^m \longrightarrow 0$$

split, so

$$P \cong Q \oplus (R^m \oplus F_1) \cong (Q \oplus R^m) \oplus F_1 \cong F_0 \oplus F_1.$$

Therefore, P is free. □

Proposition I.1.7. *An R-module P is stably free if and only if it is the kernel of a surjection of finitely generated free R-modules.*

Proof. First, suppose that P is stably free, so there exist non-negative integers m, n such that $P \oplus R^m \cong R^n$. Let $\varphi : R^n \to P \oplus R^m$ be an isomorphism. Observe that the first row of

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, hook, "\text{inc}_P"] & P \oplus R^m \arrow[r, two heads, "\text{proj}_{R^m}"] & R^m \arrow[r] & 0 \\ & & R^n \arrow[u, "\varphi"'] & & \end{tikzcd}$$

is a split short exact sequence. Then, $P \cong \varphi^{-1}(\ker(\text{proj}_{R^m})) \cong \ker(R^n \xrightarrow{\text{proj}_{R^m} \circ \varphi} R^m)$.

Now suppose that $P \cong \ker(R^n \xrightarrow{f} R^m)$. Since R^m is free, and therefore projective, the exact sequence

$$0 \longrightarrow P \cong \ker(f) \xhookrightarrow{\text{inc}} R^n \xrightarrow{f} R^m \longrightarrow 0$$

splits, so $P \oplus R^m \cong R^n$. □

Proposition I.1.8. *The stably free R-module $P = \ker(R^n \xrightarrow{f} R^m)$ is free if and only if f can be lifted to an isomorphism $\tilde{f} : R^n \xrightarrow{\cong} R^m \oplus R^r$ (for some r) such that* $\text{proj}_{R^m} \circ \tilde{f} = f$.

$$\begin{tikzcd} & R^m \oplus R^r \arrow[d, "\text{proj}_{R^m}"] & \\ R^n \arrow[ur, "\tilde{f}"] \arrow[r, two heads, "f"'] & R^m \arrow[r] & 0. \end{tikzcd}$$

Proof. See [Lam06, I.4.3]. □

Definition I.1.9 (Finite Free Resolution)**.** An R-module M has a *finite free resolution* if there exists an exact sequence of R-modules

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

such that all F_i are free and finitely generated. Such a resolution has *finite length* if there exists an integer N such that $F_i = 0$ for all $i > N$.

Proposition I.1.10. *Let P be a projective module. Then P is stably free if and only if P admits a finite free resolution of finite length.*

Proof. See [Lan02, XXI.2.1]. □

The following diagram summarizes the information in this section for any ring R:

$$\begin{tikzcd}
M \text{ is a free } R\text{-module} \arrow[r, Rightarrow] & M \text{ is a stably free } R\text{-module} \arrow[r, Rightarrow] \arrow[l, Rightarrow, bend left, "\substack{\text{iff exists a lift to an isomorphism}\\ \text{as in Proposition I.1.8}}"] & M \text{ is a projective } R\text{-module.} \arrow[l, Rightarrow, bend left, "\substack{\text{iff } M \text{ admits a finite free}\\ \text{resolution of finite length}}"]
\end{tikzcd}$$

Definition I.1.11 (Stably Isomorphic)**.** A pair of R-modules M and N are said to be *stably isomorphic* if there exists a non-negative integer n such that $M \oplus R^n \cong N \oplus R^n$.

Proposition I.1.12. *If an R-module M is stably isomorphic to a finitely generated free module, then it is stably free.*

Proof. Follows directly from the definition. □

I.2 The Group Completion

Definition I.2.1 (Monoid)**.** A set M equipped with a binary operation

$$\mu : \ M \times M \to M$$

$$(m_1, m_2) \mapsto m_1 m_2$$

is called a *monoid* if it satisfies the following properties:

- *Associativity*: for all $m_1, m_2, m_3 \in M$, $m_1(m_2 m_3) = (m_1 m_2)m_3$;
- *Existence of identity*: there exists an element $e \in M$ such that for each $m \in M$, $em = m = me$.

Furthermore, a monoid is called a *commutative monoid* if the operation μ is commutative, i.e.,

- *Commutativity*: for all $m_1, m_2 \in M$, $m_1 m_2 = m_2 m_1$.

In a commutative monoid, it is common for the binary operation μ to be written as $\mu(m_1, m_2) = m_1 + m_2$, and the identity to be written as 0.

Definition I.2.2 (Monoid Homomorphism)**.** Let M, N be two monoids. A function $f : M \to N$ is called a *monoid homomorphism* or *monoid map* if it satisfies the following properties:

- *Linearity*: for all $m_1, m_2 \in M$, $f(m_1 m_2) = f(m_1)f(m_2)$;
- *Maps identity to identity*: if e_M and e_N are the identities of M and N respectively, then $f(e_M) = e_N$.

Remark I.2.3. A monoid is a semigroup that satisfies the existence of an identity. A group is a monoid that satisfies the existence of inverses. Notice that a group can be thought of as a monoid if one 'forgets' that every element of the monoid has an inverse. This idea can be further developed into a functor.

Proposition I.2.4. *The mapping* $(-)_{\mathrm{md}} : \mathbf{Grp} \to \mathbf{Mon}$ *that forgets the unary operation that maps an element of a group to its inverse is a functor.*

Definition I.2.5 (Group Completion)**.** Let M be a commutative monoid. The *group completion* of M is the pair $(M^{\mathrm{gp}}, \varphi)$, where M^{gp} is an abelian group and $\varphi : M \to (M^{\mathrm{gp}})_{\mathrm{md}}$ is a monoid homomorphism, subject the following universal property: for every abelian group G and monoid homomorphism $\psi : M \to G_{\mathrm{md}}$, there exists a unique group homomorphism $\tilde{\psi} : M^{\mathrm{gp}} \to G$ such that $\tilde{\psi}_{\mathrm{md}} \circ \varphi = \psi$. This is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & (M^{\mathrm{gp}})_{\mathrm{md}} \arrow[d, dashed, "\exists!\, \tilde{\psi}_{\mathrm{md}}"] \\ & G_{\mathrm{md}}, \end{tikzcd} \qquad \begin{tikzcd} M^{\mathrm{gp}} \arrow[d, dashed, "\exists!\, \tilde{\psi}"] \\ G. \end{tikzcd}$$

Remark I.2.6. By abuse of notation, the universal property of a group completion may be stated as follows:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & M^{\mathrm{gp}} \arrow[d, dashed, "\exists!\,\tilde{\psi}"] \\ & G. \end{tikzcd}$$

In fact, throughout this exposition, similar abuses of notation will be made when objects or morphisms belong to a bigger category, and being explicit about it does not help to comprehend the general idea. In this case, $\mathbf{Grp} \subset \mathbf{Mon}$.

Proposition I.2.7. *For every monoid M, its group completion $(M^{\mathrm{gp}}, \varphi)$ exists and is unique up to isomorphism in the sense that, if $((M^{\mathrm{gp}})', \varphi')$ is any other such pair, then there exists a unique group isomorphism $\alpha : M^{\mathrm{gp}} \xrightarrow{\cong} (M^{\mathrm{gp}})'$ satisfying $\alpha \circ \varphi = \varphi'$.*

Proof. Consider the free functor $F : \mathbf{Set} \to \mathbf{Ab}$ that constructs the free abelian group on a given set, and the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ that forgets the algebraic structure of a monoid. The group $F(U(M))$ is the free abelian group on symbols (m) for each $m \in M$. Let R be the subgroup of $F(U(M))$ generated by the relations $(m_1) + (m_2) - (m_1 m_2)$, i.e.,

$$R = \langle \{(m_1) + (m_2) - (m_1 m_2) \mid m_1, m_2 \in M\} \rangle \subset F(U(M)).$$

Let $M^{\mathrm{gp}} = F(U(M))/R$, and let $[m]$ be the image of (m) in this quotient. Define $\varphi : M \to M^{\mathrm{gp}}$ by $\varphi(m) = [m]$. To prove that $(M^{\mathrm{gp}}, \varphi)$ is a group completion of M, let G be a group and $\psi : M \to G$ be a group homomorphism. If a group homomorphism $\tilde{\psi} : M^{\mathrm{gp}} \to G$ were to satisfy $\tilde{\psi} \circ \varphi = \psi$, then for every $m \in M$, $\tilde{\psi}([m]) = \psi(m)$.

By the universal property of a free group, there exists a unique group homomorphism $\tilde{\tilde{\psi}}$: $F(U(M)) \to G$ given by $\tilde{\tilde{\psi}}((m)) = \psi(m)$ and extended by linearity. Next, notice that $\tilde{\tilde{\psi}}(R) = 0$ because of the linearity of the homomorphism, so by the universal property of a quotient group there exists a unique $\tilde{\psi}$ satisfying the definition of a group completion of M.

Now, let $((M^{\mathrm{gp}})', \varphi')$ be another group completion of M. By the property proved previously, there exists group homomorphisms $\alpha : M^{\mathrm{gp}} \to (M^{\mathrm{gp}})'$ and $\beta : (M^{\mathrm{gp}})' \to M^{\mathrm{gp}}$ satisfying

$\alpha \circ \varphi = \varphi'$ and $\beta \circ \varphi' = \varphi$ respectively. Using this in the commutative diagrams

$$\begin{tikzcd} & M^{\mathrm{gp}} \arrow[d, "\alpha"] \\ M \arrow[ur, "\varphi"] \arrow[r, dashed, "\varphi'"] \arrow[dr, "\varphi"'] & (M^{\mathrm{gp}})' \arrow[d, "\beta"] \\ & M^{\mathrm{gp}} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & (M^{\mathrm{gp}})' \arrow[d, "\beta"] \\ M \arrow[ur, "\varphi'"] \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & M^{\mathrm{gp}} \arrow[d, "\alpha"] \\ & (M^{\mathrm{gp}})', \end{tikzcd}$$

it follows that

$$\varphi = (\beta \circ \alpha) \circ \varphi \qquad \text{and} \qquad \varphi' = (\alpha \circ \beta) \circ \varphi'.$$

By construction, $\varphi(M)$ generates M^{gp}, so $\beta \circ \alpha = \mathrm{id}_{M^{\mathrm{gp}}}$. It remains to prove that $\alpha \circ \beta = \mathrm{id}_{(M^{\mathrm{gp}})'}$. To show this, let H be the subgroup of $(M^{\mathrm{gp}})'$ generated by $\varphi'(M)$. Consider the group $(M^{\mathrm{gp}})'/H$, and the zero homomorphism $0 : M \to (M^{\mathrm{gp}})'/H$. Notice that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\varphi'"] \arrow[dr, "0"'] & (M^{\mathrm{gp}})' \arrow[d, shift right, "0"'] \arrow[d, shift left, "q"] \\ & (M^{\mathrm{gp}})'/H, \end{tikzcd}$$

where $0 : (M^{\mathrm{gp}})' \to (M^{\mathrm{gp}})'/H$ is the zero homomorphism, and $q : (M^{\mathrm{gp}})' \to (M^{\mathrm{gp}})'/H$ is the quotient homomorphism. Therefore, by the previously proved part of the universal property of a group completion, it follows that $q = 0$, i.e., $H = (M^{\mathrm{gp}})'$, which implies that $\varphi'(M)$ generates $(M^{\mathrm{gp}})'$, so $\alpha \circ \beta = \mathrm{id}_{(M^{\mathrm{gp}})'}$. Hence, $\alpha : M^{\mathrm{gp}} \to (M^{\mathrm{gp}})'$ is an isomorphism satisfying $\alpha \circ \varphi = \varphi'$. □

Proposition I.2.8. *Group completion can be further developed to a functor* $(-)^{\mathrm{gp}} : \mathbf{CMon} \to \mathbf{Ab}$

Proof. Let M, N and L be monoids, and $f : M \to N$, $g : N \to L$ be monoid homomorphisms. Let $(M^{\mathrm{gp}}, \varphi_M)$, $(N^{\mathrm{gp}}, \varphi_N)$ and $(L^{\mathrm{gp}}, \varphi_L)$ be the group completions of M, N and L respectively. Applying the universal property of a group completion to $\psi = \varphi_N \circ f$ it follows that there exists a group homomorphism $\tilde{\psi} : M^{\mathrm{gp}} \to N^{\mathrm{gp}}$ such that $\tilde{\psi} \circ \varphi_M = \varphi_N \circ f$. Define $f^{\mathrm{gp}} = \tilde{\psi}$.

$$\begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[d, "f"'] \arrow[dr, "\psi"] & M^{\mathrm{gp}} \arrow[d, "\tilde{\psi}"] \\ N \arrow[r, "\varphi_N"'] & N^{\mathrm{gp}} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} M \arrow[d, "f"'] & M^{\mathrm{gp}} \arrow[d, "f^{\mathrm{gp}}"'] \\ N & N^{\mathrm{gp}}. \end{tikzcd}$$

It remains to prove the functoriality of $(-)^{\mathrm{gp}}$. First, consider the identity homomorphism $\mathrm{id}_M : M \to M$. It is clear that the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[d, "\mathrm{id}_M"'] & M^{\mathrm{gp}} \arrow[d, "\mathrm{id}_{M^{\mathrm{gp}}}"] \\ M \arrow[r, "\varphi_M"] & M^{\mathrm{gp}} \end{tikzcd}$$

commutes, so $(\mathrm{id}_M)^{\mathrm{gp}} = \mathrm{id}_{M^{\mathrm{gp}}}$. Now, observe that the following diagram commutes since each square commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "f"'] & M^{\mathrm{gp}} \arrow[d, "f^{\mathrm{gp}}"] \\ N \arrow[r, "\varphi"] \arrow[d, "g"'] & N^{\mathrm{gp}} \arrow[d, "g^{\mathrm{gp}}"] \\ L \arrow[r, "\varphi_L"'] & L^{\mathrm{gp}}. \end{tikzcd}$$

By the uniqueness of the Universal Property of a group completion applied to M and the monoid homomorphism $\varphi_L \circ g \circ f : M \to L^{\mathrm{gp}}$ it follows that $g^{\mathrm{gp}} \circ f^{\mathrm{gp}} = (g \circ f)^{\mathrm{gp}}$. □

Corollary I.2.9. *The functors $(-)^{\mathrm{gp}}$ and $(-)_{\mathrm{md}}$ form an adjoint pair of functors. This means that for every monoid M and group G, there exists a natural bijection*

$$\varphi_{M,G} : \mathbf{Ab}(M^{\mathrm{gp}}, G) \xrightarrow{\cong} \mathbf{CMon}(M, G_{\mathrm{md}})$$

between the two bifunctors

$$\mathbf{Ab}((-)^{\mathrm{gp}}, -), \mathbf{CMon}(-, (-)_{\mathrm{md}}) : \mathbf{CMon}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Set}.$$

Proposition I.2.10. *Let M be a commutative monoid. Then:*

(a) *Every element of M^{gp} is of the form $[m_1] - [m_2]$ for some $m_1, m_2 \in M$.*

(b) *The monoid map $M \times M \to M^{\mathrm{gp}}$ sending (m_1, m_2) to $[m_1] - [m_2]$ is surjective.*

(c) *If $m_1, m_2 \in M$ then $[m_1] = [m_2]$ in M^{gp} if and only if $m_1 + p = m_2 + p$ for some $p \in M$.*

Proof. See [Wei13, II.1.1]. □

Corollary I.2.11. *Let M be a commutative monoid. The abelian group M^{gp} is isomorphic to the quotient $(M \times M)/\sim$, where $\sim$ denotes the equivalence relation on $M \times M$ given by*

$$(m_1, m_2) \sim (m_1 + p, m_2 + p) \quad \textit{for all} \quad m_1, m_2, p \in M.$$

From the previous corollary, if $[m_1, m_2]$ denotes the equivalence class of the pair (m_1, m_2), one can see that the element $[m_1, m_2] \in (M \times M)/\sim$ corresponds to the element $[m_1] - [m_2] \in M^{\mathrm{gp}}$.

Example I.2.12.

(a) It is well-known that the integers are constructed from the natural numbers as in Corollary I.2.11, therefore $(\mathbb{N}, +)^{\mathrm{gp}} \cong (\mathbb{Z}, +)$.

(b) Let $(\mathbb{N}, \times)$ denote the monoid of the natural numbers under the usual multiplication as a binary operation. Then its group completion corresponds to the group of rational numbers under the usual multiplication: $(\mathbb{N}, \times)^{\mathrm{gp}} \cong (\mathbb{Q}^+, \times)$.

(c) If M is a commutative monoid such that for each $m \in M$ there exists $m' \in M$ such that $m + m' = 0$, i.e., M already satisfies the definition of a group, then $M^{\mathrm{gp}} \cong M$. Notice that M is seen as a monoid and a group indistinguishably.

Proposition I.2.13. *Let M be a commutative monoid, and $(M^{\mathrm{gp}}, \varphi)$ its group completion. Then φ is an injection if and only if M is a cancellation monoid, i.e., for all $m_1, m_2, p \in M$, if $m_1 + p = m_2 + p$ then $m_1 = m_2$.*

Proof. Let $m_1, m_2 \in M$.

$$\begin{aligned} M \text{ is a cancellation monoid} &\iff \text{if } m_1 + p = m_2 + p \text{ then } m_1 = m_2 \\ &\iff \text{if } \varphi(m_1) = \varphi(m_2) \text{ then } m_1 = m_2 \qquad \text{(by Prop. I.2.10 (c))} \\ &\iff \varphi \text{ is an injection,} \end{aligned}$$

where $\varphi(m) = [m] \in M^{\mathrm{gp}}$ for all $m \in M$. □

Definition I.2.14 (Cofinal Submonoid)**.** Let M be a monoid. A subset $N \subseteq M$ is called a *submonoid* of M if N is itself a monoid under the same binary operation as M. If M is abelian, a submonoid L of M is called *cofinal* if for every $m \in M$ there exists $m' \in M$ such that $m + m' \in L$.

Proposition I.2.15. *If L is cofinal in an commutative monoid M, then*

(a) *The abelian group L^{gp} is a subgroup of M^{gp}.*

(b) *Every element of M^{gp} is of the form $[m] - [\ell]$, for some $m \in M$ and $\ell \in L$.*

(c) *If $[m_1] = [m_2]$ in M^{gp}, then $m_1 + \ell = m_2 + \ell$ for some $\ell \in L$.*

Proof.

(a) This follows from the construction of L^{gp}. In fact, this statement holds for any submonoid N of M.

(b) Let $[m_1] - [m_2]$ be an arbitrary element in M^{gp}. Since L is cofinal in M, there exists $m_2' \in M$ such that $m_2 + m_2' = \ell \in L$. Set $m = m_1 + m_2' \in M$. Therefore

$$[m_1] - [m_2] = [m_1] + ([m_2'] - [m_2']) - [m_2] = [m_1 + m_2'] - [m_2 + m_2'] = [m] - [\ell].$$

(c) Suppose that $[m_1] = [m_2]$ in M^{gp}. By Proposition I.2.10, there exists $p \in M$ such that $m_1 + p = m_2 + p$. Since L is cofinal in M there exists $p' \in M$ such that $p + p' = \ell \in L$. Hence,

$$m_1 + p = m_2 + p \implies m_1 + p + p' = m_2 + p + p' \implies m_1 + \ell = m_2 + \ell. \qquad \square$$

A monoid does not have anything special about it in the sense that a group completion can also be performed from a semigroup applying the same construction as before. One can also define a *Ring Completion*[1] if the word monoid is substituted by semiring.

This will only be stated as a proposition without proof, but for reference, one can see Chapter 1 of [Ros94], in which a similar procedure is done with semigroups. For the case of semirings, just observe that those are commutative monoids equipped with another compatible operation.

[1] Do not confuse the ring completion as defined here with the completion of a ring as typically defined in commutative algebra.

Proposition I.2.16. *Applying an analogous construction as Proposition I.2.7 to an abelian semigroup or a semiring creates an abelian group or a ring respectively.*

Proposition I.2.17. *Let $\{M_i\}_{i\in I}$ be a collection of commutative monoids. Then $(\bigoplus_{i\in I} M_i)^{\mathrm{gp}} \cong \bigoplus_{i\in I}(M_i^{\mathrm{gp}})$.*

Proof. Follows directly from Corollary I.2.9 and the fact that left adjoints preserve all colimits. Recall that the direct sum is the coproduct in the category **CMon**. □

I.3 Universal Central Extensions

Definition I.3.1 (Central Extension)**.** Let G be a group, and A be an abelian group. A *central extension* of G by A is a pair (E, φ) consisting of a group E containing A as a central subgroup, and a surjective homomorphism $\varphi : E \twoheadrightarrow G$ such that the following sequence is exact:

$$1 \longrightarrow A \xhookrightarrow{\mathrm{inc}} E \overset{\varphi}{\twoheadrightarrow} G \longrightarrow 1.$$

Definition I.3.2 (Category of Central Extensions)**.** The collection of central extensions of G by any abelian group forms a category. Given (E, φ) and (E', φ') central extensions of G, then a morphism from (E, φ) to (E', φ') corresponds to a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "f"'] & G \arrow[d, equal] \\ E' \arrow[r, "\varphi'"'] & G, \end{tikzcd}$$

where $f : E \to E'$ is a group homomorphism.

Proposition I.3.3. *The category of central extensions of a group G is well-defined.*

Proof. It suffices to define the composition of morphisms. Let (E, φ), (E', φ') and (E'', φ'') be central extensions of G. Let

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "f"'] & G \arrow[d, equal] \\ E' \arrow[r, "\varphi'"'] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E' \arrow[r, "\varphi'"] \arrow[d, "g"'] & G \arrow[d, equal] \\ E'' \arrow[r, "\varphi''"'] & G \end{tikzcd}$$

be morphisms from E to E' and E' to E'' respectively. The composition of these morphisms is defined as the commutative diagram

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "g\circ f"'] & G \arrow[d, equal] \\ E'' \arrow[r, "\varphi''"'] & G. \end{tikzcd}$$

The axioms of associativity and existence of identities follow directly from the commutativity of the diagrams when thinking of them as diagrams in **Grp**. □

Definition I.3.4 (Trivial and Universal Extensions)**.** Let G be a group, and A be an abelian group.

(a) A central extension (E, φ) of G by A is called *trivial* if it is isomorphic to the central extension $(G \times A, \mathrm{proj}_G)$ in the category of central extensions of G.

(b) A central extension (U, ν) is called *universal* if for any central extension (E, φ) of G by any abelian group, there exists a unique group homomorphism $h : U \to E$ making the following diagram commute:

$$\begin{tikzcd} U \arrow[r, "\nu"] \arrow[d, dashed, "\exists!\, h"'] & G \\ E. \arrow[ur, "\varphi"'] & \end{tikzcd}$$

The property that h verifies is usually referred to as a homomorphism from U to E over G.

Lemma I.3.5. *Let G be a group. A central extension (E, φ) of G is trivial if and only if $\varphi : E \twoheadrightarrow G$ admits a section.*

Proof. The forward implication is clear. For the converse, suppose that $\varphi : E \to G$ admits a section $s : G \to E$. Define a homomorphism $f : A \times G \to E$ by $f = s \circ \mathrm{proj}_G + \mathrm{inc} \circ \mathrm{proj}_A$. Use the following non-commutative diagram for reference:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook, "\mathrm{inc}"] & E \arrow[r, two heads, "\varphi"'] & G \arrow[r] \arrow[l, bend right, "s"'] & 0 \\
 & & A \times G. \arrow[ul, "\mathrm{proj}_A"] \arrow[u, dashed, "f"] \arrow[ur, "\mathrm{proj}_G"'] & &
\end{tikzcd}$$

Let $e \in E$. Observe that e is of the form $a_e + e_0$ for unique $a_e \in A$ and $e_0 \in s(G)$ by the exactness of the top sequence. Let g_e be the unique element of G such that $s(g_e) = e_0$. Then $f(a_e, g_e) = e$, so f is surjective.

Now suppose that $f(a, g) = 0$ for some $(a, g) \in A \times G$. Note that $\varphi(f(a, g)) = 0$ and

$$\varphi((s \circ \mathrm{proj}_G)(a, g) + (\mathrm{inc} \circ \mathrm{proj}_A)(a, g)) = \varphi(s(g) + a) = (\varphi \circ s)(g) + \varphi(a) = g + 0,$$

so $g = 0$. Then $0 = f(a, 0) = (\mathrm{inc} \circ \mathrm{proj}_A)(a, g) = a$. Because $(a, g) = (0, 0)$, it follows that f is injective. Hence, f is an isomorphism. By commutativity of the diagram

$$\begin{tikzcd}
E \arrow[r, two heads, "\varphi"] & G \arrow[d, equal] \\
A \times G \arrow[u, "f"] \arrow[r, two heads, "\mathrm{proj}_G"] & G,
\end{tikzcd}$$

the claim follows. □

Proposition I.3.6. *When they exist, universal central extensions of a group G are unique up to isomorphism.*

Proof. Universal central extensions of a group G are the initial object in the category of central extensions of G. The claim follows since initial objects are unique up to isomorphism in any category. □

Definition I.3.7 (Perfect Group)**.** A group X is called *perfect* if its commutator subgroup $[X, X]$ is equal to X itself.

Lemma I.3.8. *Let G be a group and (X, φ), (Y, ψ) be central extensions of G. If Y is perfect, then there exists at most one homomorphism from Y to X over G.*

Proof. Suppose that there exists two homomorphisms f_1, f_2 from Y to X over G.

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] & G \arrow[d, equal] \\ X \arrow[r, "\varphi"'] & G. \end{tikzcd}$$

For each $y \in Y$ one has that $\varphi(f_1(y)) = \varphi(f_2(y))$, so there exists $z_y \in \ker(\varphi) \subseteq Z(Y)$ such that $f_1(y) = f_2(y)z_y$. Therefore, for $a, b \in Y$,

$$\begin{aligned} f_1(aba^{-1}b^{-1}) &= f_1(a)f_1(b)f_1(a)^{-1}f_1(b)^{-1} \\ &= f_2(a)z_a f_2(b)z_b(f_2(a)z_a)^{-1}(f_2(b)z_b)^{-1} \\ &= f_2(a)f_2(b)f_2(a)^{-1}f_2(b)^{-1}z_a z_a^{-1}z_b z_b^{-1} \\ &= f_2(a)f_2(b)f_2(a)^{-1}f_2(b)^{-1} \\ &= f_2(aba^{-1}b^{-1}). \end{aligned}$$

Since $Y = [Y, Y]$, then $f_1 = f_2$. □

Lemma I.3.9. *Let G be a group, and (Y, ψ) be a central extension of G. If Y is not perfect, then for suitably chosen (X, φ) there exists more than one homomorphism from Y to X over G.*

Proof. Since Y is not perfect, the abelianization of Y, $Y/[Y, Y]$, is not the zero group. Let $(G \times Y/[Y, Y], \mathrm{proj}_G)$ be the trivial extension of G by $Y/[Y, Y]$. Setting

$$f_1(y) = (\psi(y), 1_{Y/[Y,Y]}) \quad \text{and} \quad f_2(y) = (\psi(y), q(y)),$$

where $q : Y \to Y/[Y, Y]$ denotes the quotient homomorphism, one has that f_1 and f_2 are distinct homomorphisms from Y to $G \times Y/[Y, Y]$ over G. □

Lemma I.3.10. *If (X, φ) is a central extension of a perfect group G, then $[X, X]$ is perfect and maps onto G. In particular, X is generated by $[X, X]$ and* $\ker(\varphi)$.

Proof. Since G is perfect and φ is surjective, the restriction $\varphi|_{[X,X]}$ is still surjective. Therefore, for each $x \in X$ there exists $x' \in [X,X]$ satisfying $\varphi(x) = \varphi(x')$, hence, there is a $z_x \in \ker(\varphi) \subseteq Z(X)$ such that $x = x' z_x$. Consider a commutator $[x_1, x_2] \in [X,X]$. Hence,

$$[x_1, x_2] = [x_1' z_{x_1}, x_2 z_{x_2}] = [x_1', x_2'],$$

so $[X,X]$ is perfect. □

Theorem I.3.11. *Let G be a group.*

(a) *The group G has a universal central extension if and only if G is perfect.*

(b) *A central extension (U,ν) of G is universal if and only if U is perfect and all central extensions of U are trivial.*

Proof. (b) Suppose that U is perfect and all central extensions of U are trivial. Let (E,φ) be a central extension of G. Consider the fibered product $U \times_G E$ and the homomorphism $\pi : U \times_G E \to U$ given by the universal property of a fibered product.

$$\begin{tikzcd}
U \times_G E \arrow[r, "\pi"] \arrow[d] \arrow[d, dashed, bend right, "h"'] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d, "\nu"] \arrow[l, dashed, bend right, "s"'] \\
E \arrow[r, "\varphi"'] & G.
\end{tikzcd}$$

Notice that $(U \times_G E, \pi)$ is a central extension of U, so it must be trivial. Because of this, there exists a section $s : U \to U \times_G E$ of π. Setting $s(u) = (u, h(u))$ for each $u \in U$ produces a homomorphism $h : U \to E$. By Lemma I.3.8, $h \circ s$ is the unique homomorphism from U to E over G; hence, (U,ν) is the universal central extension of G.

Now, suppose that (U,ν) is the universal central extension of G. By the contrapositive of Lemma I.3.9, it follows that U is perfect. Let (E,φ) be a central extension of U. Suppose that $\nu(\varphi(x_0)) = 1_G$ for some $x_0 \in E$, then $\varphi(x_0) \in \ker(\nu) \subseteq Z(U)$. Therefore, defining $h : E \to E$ by $h(x) = x_0 x x_0^{-1}$ creates an homomorphism from E to E over U. Observe that $h([E,E]) \subseteq [E,E]$ because for $aba^{-1}b^{-1} \in [E,E]$,

$$x_0aba^{-1}b^{-1}x_0^{-1} = (x_0aba^{-1}x_0^{-1}b^{-1})(bx_0b^{-1}x_0^{-1}) = [x_0a, b][b, x_0].$$

So $h|_{[E,E]}$ is a homomorphism from $[E,E]$ to $[E,E]$ over U. By Lemma I.3.10, $[E,E]$ is perfect, and by Lemma I.3.9, $h|_{[E,E]}$ must be the identity. Thus, $x_0 \in Z([E,E])$. In fact, by Lemma I.3.10, E is generated by $[E,E]$ and $\ker(\varphi) \subseteq Z(E)$, $x_0 \in Z(E)$.

Hence, $(E, \varphi \circ \nu)$ is a central extension of G since $Z(E) \subseteq \ker(\varphi \circ \nu)$. Because (U,ν) is the universal central extension, there exists a homomorphism s from U to E over G. Therefore, $\varphi \circ s$ equals the identity of U because of the definition of a universal central extension. By Lemma I.3.5, the central extension (E,φ) of U is trivial.

(a) suppose that G admits a universal central extension (U,ν). Then, by part (b), U is perfect, so the image of $[U,U]$ under ν must be perfect, i.e., G is perfect.

Now, suppose that G is perfect. Consider a presentation of G:

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1.$$

Here, F is a free group. Notice that $[R,F]$ is a normal subgroup of F since

$$x(rfr^{-1}f^{-1})x^{-1} = (xrx^{-1})(xfx^{-1})(xr^{-1}x^{-1})(xf^{-1}x^{-1})$$

for every $x \in F$ and $rfr^{-1}f^{-1} \in [R,F]$, where $(xrx^{-1}), (xr^{-1}x^{-1}) \in R$ because R is a normal subgroup of F, and $(xfx^{-1}), (xf^{-1}x^{-1}) \in F$. Observe that $[R,F] \subseteq R$, so there is a natural surjection

$$\nu : F/[R,F] \twoheadrightarrow F/R \cong G,$$

with $\ker(\nu) \cong R/[R,F] \cong (R/[R,R])/([R,F]/[R,R])$, which is abelian.

Let $U = [F/[R,F], F/[R,F]] \cong [F,F]/[R,F]$, so that $(U, \nu|_U)$ is a central extension of G.

By Lemma I.3.10, it follows that U is perfect. Suppose that (E, φ) is a central extension of G. By the universal property of a free group, there exists a homomorphism h' from F to E over G. Notice that $h'([R, F]) = 1$ since $G \cong F/R$ and $[R, F] \subseteq R$, so h' induces an homomorphism h'' from $F/[R, F]$ to E over G. By restricting h'' to U, one obtains a homomorphism h from U to E over G, which is unique by Lemma I.3.8. Hence, (U, ν) is the universal central extension of G. □

Example I.3.12. Consider the alternating group A_5. It is known that A_5 is perfect, so by Theorem I.3.11 it has a universal central extension. Its computation requires some work which can be found in [Suz82, Chapter 2 §9]. To perform this calculation, one relies on the fact that A_5 is isomorphic to $PSL_2(\mathbb{F}_5)$, which is also known as the icosahedral rotation group.

Consider the group $SL_2(\mathbb{F}_5)$, which is known to be the full icosahedral group, and the quotient map $q : SL_2(\mathbb{F}_5) \twoheadrightarrow PSL_2(\mathbb{F}_5)$. The claim is that $(SL_2(\mathbb{F}_5), q)$ is the universal central extension of $PSL_2(\mathbb{F}_5) \cong A_5$.

$$1 \longrightarrow \mathbb{Z}/2 \cong \{1, -1\} \xrightarrow{\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)} SL_2(\mathbb{F}_5) \xrightarrow{q} PSL_2(\mathbb{F}_5) \longrightarrow 1.$$

Definition I.3.13 (Parametrized Central Extensions)**.** Let G be a group, and A be an abelian group. A *parametrized central extension* of G by A is a triple (E, φ, ι), where the pair (E, φ) is a central extension of G by A, and $\iota : A \to E$ is an isomorphism between A and $\ker(\varphi)$.

Definition I.3.14 ($\mathrm{Ext}(G, A)$)**.** Let G be a group, and A be an abelian group. The set of isomorphism classes of parametrized central extension of G by A is denoted by $\mathrm{Ext}(G, A)$.

Proposition I.3.15. *The set* $\mathrm{Ext}(G, A)$ *forms an abelian group, in which the trivial extension is the zero element.*

Proof. Let $(E_1, \varphi_1, \iota_1)$ and $(E_2, \varphi_2, \iota_2)$ be parametrized central extensions of G by A. Define the sum $[(E_1, \varphi_1, \iota_1)] + [(E_2, \varphi_2, \iota_2)]$ as follows:

Consider the fibered product $E_1 \times_G E_2$, and its quotient $(E_1 \times_G E_2)/\{(\iota_1(a), -\iota_2(a)) \mid a \in A\}$,

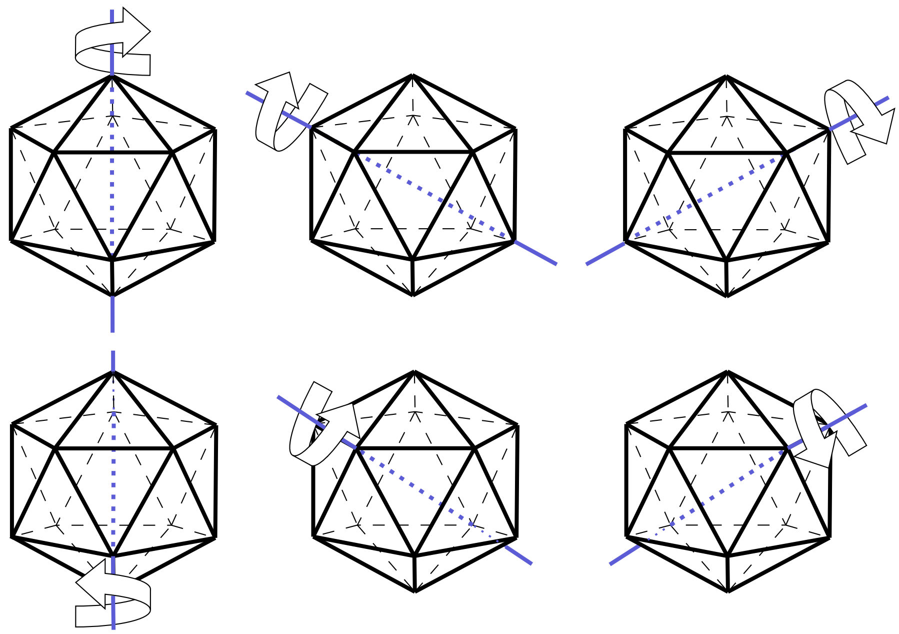

Figure I.1. Representations of icosahedrons being rotated in different axes. The group $SL_2(\mathbb{F}_5)$ describes the full symmetry group of the icosahedron, including both rotations and reflections, while its quotient $PSL_2(\mathbb{F}_5)$ corresponds to the rotational symmetries only.

denoted by E.

$$\begin{tikzcd}
E_1 \times_G E_2 \arrow[rr] \arrow[dd] \arrow[dr, "q"] & & E_2 \arrow[dd, "\varphi_2"] \\
& E \arrow[dr, "\varphi"] & \\
E_1 \arrow[rr, "\varphi_1"'] & & G.
\end{tikzcd}$$

By construction of the fibered product, the maps $E_1 \times_G E_2 \to E_1 \xrightarrow{\varphi_1} G$ and $E_1 \times_G E_2 \to E_2 \xrightarrow{\varphi_2} G$ are equal. Notice that such a map factors through E by the universal property of the quotient. Denote this quotient map by φ.

The kernel of φ is central and is given by

$$\ker(\varphi) = \{(\iota_1(a_1), \iota_2(a_2) \mid a_1, a_2 \in A)\}/\{(\iota_1(a), -\iota_2(a)) \mid a \in A\}$$

Furthermore, $\ker(\varphi)$ is isomorphic to A via the homomorphism $\iota : A \to \ker(\varphi)$ given by

$$\iota(a) = [(\iota_1(a), 0)].$$

Define $[(E_1, \varphi_1, \iota_1)] + [(E_2, \varphi_2, \iota_2)] = [(E, \varphi, \iota)]$. This operation, often referred to as the *Baer sum*, is well-defined since different representatives of the isomorphism classes of $[(E_1, \varphi_1, \iota_1)]$ or $[(E_2, \varphi_2, \iota_2)]$ yield a commutative diagram of the form

$$\begin{tikzcd}
E_1' \times_G E_2' \arrow[rrr] \arrow[drr, "q'"] \arrow[ddr] \arrow[ddd] & & & E_2' \arrow[ddr, "\varphi_2'"] \arrow[ddd] & \\
& & E' \arrow[drr, "\varphi'"] \arrow[ddd] & & \\
& E_1' \arrow[rrr, "\varphi_1'"'] \arrow[ddd] & & & G \arrow[ddd, equal] \\
E_1 \times_G E_2 \arrow[rrr] \arrow[drr, "q"] \arrow[ddr] & & & E_2 \arrow[ddr, "\varphi_2"] & \\
& & E \arrow[drr, "\varphi"] & & \\
& E_1 \arrow[rrr, "\varphi_1"'] & & & G.
\end{tikzcd}$$

This diagram has isomorphisms in all of the vertical arrows, which implies that the produced central extension is isomorphic to (E, φ, ι).

Associativity and commutativity follow from the properties of fibered products. For an arbitrary class $[(E_1, \varphi_1, \iota_1)]$, if $[(E_1, \varphi_1, \iota_1)] + [G \times A, \mathrm{proj}_G, \mathrm{inc}_A] = [(E, \varphi, \iota)]$, one has that

$$\begin{aligned}
E &\cong (E_1 \times_G (G \times A))/\{(\iota_1(a), -\mathrm{inc}_A(a)) \mid a \in A\} \\
&\cong \big\{(e, (\varphi(e), a)) \mid e \in E_1, a \in A\big\}/\{(\iota_1(a), -\mathrm{inc}_A(a)) \mid a \in A\} \\
&\cong \{(e, a) \mid e \in E_1, a \in A\}/\{(\iota_1(a), -a) \mid a \in A\} \\
&\cong E_1
\end{aligned}$$

and $\varphi = \varphi_1$, so $[((G \times A), \mathrm{proj}_G, \mathrm{inc}_A)] = 0$ in $\mathrm{Ext}(G, A)$. Now, observe that if $[(E_1, \varphi_1, \iota_1)] +$

$[(E_1, \varphi_1, -\iota_1)]$ is given by $[(E, \varphi, \iota)]$, then

$$\begin{aligned} E &\cong (E_1 \times_G E_1)/\{(\iota_1(a), \iota_A(a)) \mid a \in A\} \\ &\cong \{(e, e+a) \mid e \in E_1, a \in A\}/\{(\iota_1(a), \iota_A(a)) \mid a \in A\} \\ &\cong \{(g, g+a) \mid g \in G, a \in A\} \\ &\cong G \times A, \end{aligned}$$

$\varphi = \mathrm{proj}_G$, and $\iota = \mathrm{inc}_A$. Hence, $[(E_1, \varphi_1, -\iota_1)]$ is the inverse of $[(E_1, \varphi_1, \iota_1)]$. □

I.4 Group (Co)homology

Definition I.4.1 (Group ring of G over $\mathbb{Z}$)**.** Let G be a multiplicative group, and let I be an indexing set for the elements of G. The group ring of G over $\mathbb{Z}$, denoted by $\mathbb{Z}[G]$, is the set of formal sums

$$\mathbb{Z}[G] = \left\{ \sum_{i \in I} \lambda_i g_i \mid n_i \in \mathbb{Z}, g_i \in G, \text{ and } g_i = 0 \text{ except for finitely many } i \in I \right\},$$

together with the binary operations

$$+ : \mathbb{Z}[G] \times \mathbb{Z}[G] \to \mathbb{Z}[G] \quad \text{and} \quad \cdot : \mathbb{Z}[G] \times \mathbb{Z}[G] \to \mathbb{Z}[G]$$

given by

$$\sum_{i \in I} \lambda_i g_i + \sum_{i \in I} \mu_i g_i = \sum_{i \in I} (\lambda_i + \mu_i) g_i$$

and

$$\sum_{i \in I} \lambda_i g_i \cdot \sum_{i \in I} \mu_i g_i = \sum_{j,k \in I} (\lambda_j \mu_k) g_j g_k = \sum_{i \in I} \left(\sum_{\substack{j,k \in I \\ g_j g_k = g_i}} \lambda_j \mu_k \right) g_i.$$

It can be immediately verified that $(\mathbb{Z}[G], +, \cdot)$ is a ring after noticing that both $\lambda_i + \mu_i$ and $\sum_{\substack{j,k \in I \\ g_j g_k = g_i}} \lambda_j \mu_k$ are equal to zero except for finitely many $i \in I$.

Observe that both G and $\mathbb{Z}$ are embedded in $\mathbb{Z}[G]$ via

$$\begin{aligned} G &\longrightarrow \mathbb{Z}[G] \\ g &\longmapsto 1g \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z}[G] \\ n &\longmapsto n1_G. \end{aligned}$$

Proposition I.4.2 (Universal Property of a Group Ring)**.** *Let G be a multiplicative group, R be a ring, and $\varphi : G \to R$ be a monoid homomorphism from G to the multiplicative structure of R. Then there exists a unique ring homomorphism $\tilde{\varphi} : \mathbb{Z}[G] \to R$ such that the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\mathrm{inc}"] \arrow[dr, "\varphi"'] & \mathbb{Z}[G] \arrow[d, dashed, "\exists!\,\tilde{\varphi}"] \\ & R. \end{tikzcd}$$

Proof. If such a ring homomorphism $\tilde{\varphi}$ exists, it must verify that $\tilde{\varphi}(1g) = \varphi(g)$. By linearity, it follows that

$$\tilde{\varphi}\left(\sum_{i\in I} \lambda_i g_i\right) = \sum_{i\in I} \lambda_i \varphi(g_i).$$

It is straightforward to check that such a function defines a ring homomorphism. □

Definition I.4.3 (Augmentation ideal)**.** Let G be a group. Consider the unique ring homomorphism $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ obtained via the universal property of the group ring from the trivial multiplicative monoid homomorphism $f : G \to \mathbb{Z}$, where $f(g) = 1$ for all $g \in G$. The homomorphism ε is called the *augmentation map of G*. Specifically, ε satisfies $\varepsilon(1g) = 1$ for all $g \in G$. Thus, for any element $\sum_{i\in I} n_i g_i \in \mathbb{Z}[G]$,

$$\varepsilon\left(\sum_{i\in I} n_i g_i\right) = \sum_{i\in I} n_i.$$

The *augmentation ideal of G*, denoted by IG, is defined by

$$IG = \ker\left(\mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z}\right).$$

Proposition I.4.4. *The ideal IG, when seen as a $\mathbb{Z}$-module, is free with basis $\{g - 1 \mid g \in G - \{1\}\}$.*

Proof. Let $\sum_{i \in I} \lambda_i g_i \in IG$. Let $k \in I$ be the index such that $g_k = 1$. Observe that

$$0 = \varepsilon\left(\sum_{i \in I} \lambda_i g_i\right) = \sum_{i \in I} \lambda_i,$$

so $\lambda_k = -\sum_{i \in I - \{k\}} \lambda_i$. From this, it follows that

$$\sum_{i \in I} \lambda_i g_i = \sum_{i \in I - \{k\}} \lambda_i (g_i - 1).$$

Even more, the coefficients λ_i are unique since $\mathbb{Z}[G]$ can be considered to be a free $\mathbb{Z}$-module with basis G. $\square$

Definition I.4.5 (G-module)**.** Let M be an abelian group. A left G-*module* consists of the pair $(M, \kappa : G \times M \to M)$ where M is an abelian group and κ is a left action of G on M compatible with the group structure of M, i.e. for all $m_1, m_2 \in M$ and $g \in G$,

$$\kappa(g, m_1 + m_2) = \kappa(g, m_1) + \kappa(g, m_2).$$

A right G-module is defined similarly by considering a right action of G on M compatible with the group structure of M.

Example I.4.6.

(a) Let G be a group. Any abelian group M can be viewed as a G-module by considering the trivial group action, where the action of every element of G on M is the identity. In this case, M is called a *trivial* G*-module.*

In particular, unless otherwise specified, the structure of $\mathbb{Z}$ as a G-module always refers to the trivial G-module structure.

(b) Let R be a ring. Any subgroup G of the multiplicative group of R acts on R by left (right) multiplication. This implies that R can be seen as a left (right) G-module.

In particular, unless otherwise specified, the structure of $\mathbb{Z}[G]$ as a G-module always refers to the left multiplication by an element of G.

Proposition I.4.7. *Every G-module is naturally a $\mathbb{Z}[G]$-module, and every $\mathbb{Z}[G]$-module is naturally a G-module. Furthermore, the categories G-**Mod** and $\mathbb{Z}[G]$-**Mod** are isomorphic.*

Proof. Given a left G-module (M, κ), Consider the left $\mathbb{Z}[G]$-module $(M, \cdot)$ with scalar multiplication given by

$$\sum_{i\in I} \lambda_i g_i \cdot m = \sum_{i\in I} \kappa(g, \underbrace{m + \cdots + m}_{\lambda_i \text{ times}}) = \sum_{i\in I} \lambda_i \kappa(g, m).$$

Now, given a left $\mathbb{Z}[G]$-module $(M, \cdot)$, consider the left G-module (M, κ) with group action given by

$$\kappa(g, m) = g \cdot m.$$

Observe that these correspondences are well-defined and that a similar correspondence can be made by now considering right G-modules and right $\mathbb{Z}[G]$-modules.

Let $F : G\textbf{-Mod} \to \mathbb{Z}[G]\textbf{-Mod}$ be the functor that associates the G-module (M, κ) with the $\mathbb{Z}[G]$-module $(M, \cdot)$ as described previously and leaves the homomorphisms unchanged. Observe that a homomorphism of left G-modules $f : (M, \kappa_1) \to (N, \kappa_2)$ is $\mathbb{Z}[G]$-linear:

$$\begin{aligned} f\left(\sum_{i\in I} \lambda_i g_i \cdot_1 m\right) &= f\left(\sum_{i\in I} \lambda_i \kappa_1(g, m)\right) \\ &= \sum_{i\in I} \lambda_i f(\kappa_1(g, m)) \\ &= \sum_{i\in I} \lambda_i \kappa_2(g, f(m)) \\ &= \sum_{i\in I} \lambda_i g_i \cdot_2 f(m), \end{aligned}$$

so $F(f) = f$ is indeed well-defined. The case for the right modules is analogous. This produces a natural correspondence from left G-modules to left $\mathbb{Z}[G]$-modules since F does not change the underlying groups or functions:

$$\begin{tikzcd} (M, \kappa_1) \arrow[r, "F"] \arrow[d, "f"'] & (M, \cdot_1) \arrow[d, "f"] \\ (N, \kappa_2) \arrow[r, "F"'] & (N, \cdot_2). \end{tikzcd}$$

> The proof of the natural correspondence from $\mathbb{Z}[G]$-modules to G-modules is similar to that from G-modules to $\mathbb{Z}[G]$-modules from before. In fact, by letting $H : \mathbb{Z}[G]\textbf{-Mod} \to G\textbf{-Mod}$ be the functor that associates a $\mathbb{Z}[G]$-module $(M, \cdot)$ with the G-module (M, κ) as described previously and leaves the morphisms unchanged, one can readily check that $HF : G\textbf{-Mod} \to G\textbf{-Mod}$ and $FH : \mathbb{Z}[G]\textbf{-Mod} \to \mathbb{Z}[G]\textbf{-Mod}$ are the identity functors. $\square$

If $\kappa : G \times M \to M$ is a group action, it is standard to write $g \cdot m$ for $\kappa(g, m)$. This notation will be adopted as it is consistent with the notation of the scalar product in a $\mathbb{Z}[G]$-module.

Definition I.4.8 ((Co)homology of a group)**.** Let G be a group, M be a left G-module, and n be a natural number. The *homology group of degree* n *of* G *with coefficients* M, denoted by $H_n(G, M)$, is defined by

$$H_n(G, M) = \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Whereas the *cohomology group of degree* n *of* G *with coefficients* M, denoted by $H^n(G, M)$, is defined by

$$H^n(G, M) = \operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Example I.4.9. Let $C_2 = \{1, g\}$ be the cyclic group of order 2. Let $\varepsilon : \mathbb{Z}[C_2] \to \mathbb{Z}$ be the augmentation homomorphism. Note that the kernel of ε is equal to the submodule $\{ne - ng \mid n \in \mathbb{Z}\} \subseteq \mathbb{Z}[C_2]$. Now, consider the homomorphisms from $\mathbb{Z}[C_2]$ to $\mathbb{Z}[C_2]$ defined by mapping $1 \mapsto 1 - g$ and $1 \mapsto 1 + g$ respectively, and extended by $\mathbb{Z}[C_2]$-linearity. Call these homomorphisms f_1 and f_2 respectively. Observe that

$$\operatorname{im}(f_1) = \ker(\varepsilon) = \ker(f_2) \quad \text{and} \quad \operatorname{im}(f_2) = \ker(f_1).$$

Hence,

$$\cdots \longrightarrow \mathbb{Z}[C_2] \xrightarrow{f_1} \mathbb{Z}[C_2] \xrightarrow{f_2} \mathbb{Z}[C_2] \xrightarrow{f_1} \mathbb{Z}[C_2] \xrightarrow{\varepsilon} \mathbb{Z}$$

is a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[C_2]$-module. Consider the $\mathbb{Z}[C_2]$-module $\mathbb{Z}/2$, which has trivial action by the elements of C_2 just as $\mathbb{Z}$ does. Removing the object $\mathbb{Z}$ in degree -1 and applying

$-\otimes_{\mathbb{Z}[C_2]} \mathbb{Z}/2$ to the previous resolution one has the chain complex

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}/2 \arrow[r, "f_2\otimes\mathrm{id}"] \arrow[d, "\cong"] & \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}/2 \arrow[r, "f_1\otimes\mathrm{id}"] \arrow[d, "\cong"] & \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z}/2 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, equal] \\
\cdots \arrow[r] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r] & 0,
\end{tikzcd}$$

where the mappings in each square (except for the rightmost square) occur in the following way:

$$\begin{tikzcd}
(n1+mg)\otimes x \arrow[r, mapsto] \arrow[d, mapsto] & ((n-m)1+(-n+m)g)\otimes g \arrow[d, mapsto] \\
(n+m)x \arrow[r, mapsto] & ((n-m)+(-n+m))x = 0
\end{tikzcd}$$

for $f_1 \otimes \mathrm{id}$, and

$$\begin{tikzcd}
(n1+mg)\otimes x \arrow[r, mapsto] \arrow[d, mapsto] & ((n+m)1+(n+m)g)\otimes g \arrow[d, mapsto] \\
(n+m)x \arrow[r, mapsto] & ((n+m)+(n+m))x = 0
\end{tikzcd}$$

for $f_2 \otimes \mathrm{id}$. Then,

$$H_n(C_2, \mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \text{for all } n.$$

On the other hand, by removing the object $\mathbb{Z}$ in degree -1 and applying $\mathrm{Hom}_{\mathbb{Z}[G]}(-, \mathbb{Z}/2)$ one has the chain complex

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}[C_2]}(\mathbb{Z}[C_2], \mathbb{Z}/2) \arrow[r, "f_1^*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbb{Z}[C_2]}(\mathbb{Z}[C_2], \mathbb{Z}/2) \arrow[r, "f_2^*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbb{Z}[C_2]}(\mathbb{Z}[C_2], \mathbb{Z}/2) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r] & \cdots
\end{tikzcd}$$

Then,

$$H^n(C_2, \mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \text{for all } n.$$

If the module $\mathbb{Z}/2$ were to be changed by the module $\mathbb{Z}$, by following a similar procedure as before, the homology groups $H_2(C_2, \mathbb{Z})$ would be computed from the chain complex

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[r, "f_1 \otimes \mathrm{id}"] \arrow[d, "\cong"] & \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[r, "f_2 \otimes \mathrm{id}"] \arrow[d, "\cong"] & \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[r, "f_1 \otimes \mathrm{id}"] \arrow[d, "\cong"] & \mathbb{Z}[C_2] \otimes_{\mathbb{Z}[C_2]} \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, equal] \\
\cdots \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

Therefore,

$$H_n(C_2, \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition I.4.10. *Let G be a group.*

(a) *For each $n \in \mathbb{N}$, $H_n(G, -)$ and $H^n(G, -)$ are covariant functors from G-**Mod** (or $\mathbb{Z}[G]$-**Mod**) to **Ab**.*

(b) *If P is a projective left G-module, then $H_n(G, P) = 0$ for all $n \geq 1$.*

(c) *If I is an injective left G-module, then $H_n(G, I) = 0$ for all $n \geq 1$.*

(d) *If*

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow M'' \longrightarrow 0$$

is a short exact sequence of left G-modules, then there are associated long exact sequences

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
H_{n+1}(G, M') \arrow[r] & H_{n+1}(G, M) \arrow[r] & H_{n+1}(G, M'') \arrow[dll, "\partial"] \\
H_n(G, M') \arrow[r] & H_n(G, M) \arrow[r] & H_n(G, M'') \arrow[dll] \\
\cdots & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
H^n(G,M') \arrow[r] & H^n(G,M) \arrow[r] & H^n(G,M'') \arrow[dll, "\partial"] \\
H^{n+1}(G,M') \arrow[r] & H^{n+1}(G,M) \arrow[r] & H^{n+1}(G,M'') \arrow[dll] \\
\cdots . & &
\end{tikzcd}$$

Proof. Follows directly from the properties of $\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},-)$ and $\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},-)$ □

Definition I.4.11 (Group of Coinvariants)**.** Let G be a group and M be a left G-module. The *group of coinvariants of* M, denoted by M_G, is the quotient group

$$M_G = M/\langle g \cdot m - m \mid g \in G, m \in M\rangle.$$

Definition I.4.12 (Group of Invariants)**.** Let G be a group and M be a left G-module. The *subgroup of invariants of* M, denoted by M^G, is the group

$$M^G = \{m \mid g \cdot m = m \text{ for all } g \in G\}$$

Theorem I.4.13. *Let G be a group and M be a left G-module. Then*

$$H_0(G,M) \cong M_G \quad \text{and} \quad H^0(G,M) \cong M^G.$$

Proof. Consider the short exact sequence of right $\mathbb{Z}[G]$-modules

$$0 \longrightarrow IG \xhookrightarrow{\text{inc}} \mathbb{Z}[G] \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z} \longrightarrow 0.$$

By applying the functor $-\otimes_{\mathbb{Z}[G]} M$ to such a sequence one gets the following exact sequence:

$$IG \otimes_{\mathbb{Z}[G]} M \xrightarrow{\text{inc}\otimes\text{id}} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M \overset{\varepsilon\otimes\text{id}}{\twoheadrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}[G]} M \longrightarrow 0.$$

Notice that

$$\mathbb{Z} \otimes_{\mathbb{Z}[G]} M \cong (\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M)/\ker(\varepsilon \otimes \mathrm{id}) \cong M/\mathrm{im}(\mathrm{inc} \otimes \mathrm{id}),$$

and

$$\begin{aligned}
\mathrm{im}(\mathrm{inc} \otimes \mathrm{id}) &= \Big\langle (g-1) \otimes_{\mathbb{Z}[G]} m \mid g \in G - \{1\}, m \in M \Big\rangle && \text{(Proposition I.4.4)} \\
&= \Big\langle 1 \otimes_{\mathbb{Z}[G]} (g-1) \cdot m \mid g \in G - \{1\}, m \in M \Big\rangle \\
&\cong \langle (g-1) \cdot m \mid g \in G - \{1\}, m \in M \rangle && (\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M \cong M) \\
&= \langle (g-1) \cdot m \mid g \in G, m \in M \rangle \\
&= \langle g \cdot m - m \mid g \in G, m \in M \rangle.
\end{aligned}$$

Therefore, $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M \cong M_G$. Since $H_0(G, M) = \mathrm{Tor}^{\mathbb{Z}[G]}(\mathbb{Z}, M) \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$, it follows that $H_0(G, M) \cong M_G$.

On the other hand, recall that $H^0(G, M) = \mathrm{Ext}^0_{\mathbb{Z}[G]}(\mathbb{Z}, M) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$. A $\mathbb{Z}[G]$-module homomorphism $f : \mathbb{Z} \to M$ must satisfy

$$g \cdot f(1) = f(g \cdot 1) = f(1),$$

so $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M) \cong M^G$ since $\mathbb{Z}[G]$-module homomorphisms from $\mathbb{Z}$ to any other module are prescribed by the value of $f(1)$ due to G acting trivially on $\mathbb{Z}$. $\square$

Lemma I.4.14. *Let G be a group and IG be its augmentation ideal. Then the additive group $IG/(IG)^2$ is isomorphic to the multiplicative group $G/[G, G]$, where $[G, G]$ denotes the commutator subgroup of G.*

Proof. The correspondence is given by

$$g[G, G] \leftrightarrow g - 1 + (IG)^2$$

and extended by linearity. See [LP05, IV 3.1] for details of the proof. $\square$

Theorem I.4.15. *Let G be a group. The homology and cohomology groups of degree 1 of G with* *coefficients in* $\mathbb{Z}$ *are*

$$H_1(G, \mathbb{Z}) \cong G/[G, G]$$

and

$$H^1(G, \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(G/[G, G], \mathbb{Z}).$$

Proof. Consider the short exact sequence

$$0 \longrightarrow IG \xhookrightarrow{\text{inc}} \mathbb{Z}[G] \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z} \longrightarrow 0.$$

By Proposition I.4.10 (d), the following is a long exact sequence:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, out=-170, in=20] \\
H_1(G, IG) \arrow[r] & H_1(G, \mathbb{Z}[G]) \arrow[r] & H_1(G, \mathbb{Z}) \arrow[dll, out=-170, in=20] \\
H_0(G, IG) \arrow[r] & H_0(G, \mathbb{Z}[G]) \arrow[r] & H_0(G, \mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathbb{Z}[G]$ is free as a $\mathbb{Z}[G]$ module, it is projective, so $H_1(G, \mathbb{Z}[G]) = 0$. It is clear that $H_0(G, \mathbb{Z}[G]) \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \cong \mathbb{Z}$, and by Theorem I.4.13,

$$H_0(G, \mathbb{Z}) \cong \mathbb{Z}_G = \mathbb{Z},$$

since G acts trivially on $\mathbb{Z}$. Therefore, the previous long exact sequence can be rewritten as

$$\begin{tikzcd}
 & & \cdots \arrow[dll, out=-170, in=20] \\
H_1(G, IG) \arrow[r] & 0 \arrow[r] & H_1(G, \mathbb{Z}) \arrow[dll, out=-170, in=20] \\
H_0(G, IG) \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

Observe that the map $H_0(G, \mathbb{Z}[G]) \to H_0(G, \mathbb{Z})$ must be surjective because of the exactness of the sequence. Even more, since any mapping from $\mathbb{Z}$ to $\mathbb{Z}$ is either injective or trivial, the map $H_0(G, \mathbb{Z}[G]) \to H_0(G, \mathbb{Z})$ must be an isomorphism.

Hence, it follows that

$$\begin{aligned} H_1(G,\mathbb{Z}) &\cong H_0(G, IG) \\ &\cong IG_G && \text{(Theorem I.4.13)} \\ &= IG/\langle g_0(g-1)-(g-1) \mid g, g_0 \in G\rangle \\ &= IG/\langle (g_0-1)(g-1) \mid g, g_0 \in G\rangle \\ &= IG/(IG)^2. \end{aligned}$$

Using Lemma I.4.14, one concludes that $H_1(G,\mathbb{Z}) \cong G/[G,G]$.

Before proceeding with the second part of the proof of Theorem I.4.15, it is useful to establish a connection between the concepts of group homology and group cohomology.

Lemma I.4.16. *Let G be a group and M, N be right G-modules such that N has trivial action. Then*

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M,N) \cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}[G]} \mathbb{Z}, N),$$

where, on the right-hand side, N is seen as a $\mathbb{Z}$-module.

Proof. Seeing both M and N as $\mathbb{Z}[G]$-modules, the lemma follows directly from the universal property of extension of scalars along the augmentation map $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$. This universal property establishes that for a ring homomorphism $f : R \to S$ and R-modules M and N,

$$\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M, f^*N),$$

where f^* denotes the restriction of scalars along $f : R \to S$. □

Theorem I.4.17 (Universal Coefficient Theorem for Cohomology). *Let G be a group, and N be a G-module with trivial action. For all $k \in \mathbb{Z}$, the sequences*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(G,\mathbb{Z}), N) \hookrightarrow H^k(G,\mathbb{Z}) \twoheadrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_k(G,\mathbb{Z}), N) \longrightarrow 0$$

are split short exact sequences (the splitting is not natural).

Proof. Let $P_\bullet$ be a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module. Let $C_\bullet$ be the chain complex $\mathbb{Z} \otimes_{\mathbb{Z}[G]} P_\bullet$. Notice that

$$H_k(G, \mathbb{Z}) \cong H_k(C_\bullet), \quad \text{and} \quad H^k(G, N) \cong H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(P_\bullet, N)) \cong H^k(\mathrm{Hom}_{\mathbb{Z}}(C_\bullet, N))$$

by Lemma I.4.16. Let

$$Z_k = \ker\Big(C_k \xrightarrow{\partial} C_{k-1}\Big), \qquad B_k = \mathrm{im}(C_{k+1} \xrightarrow{\partial} C_k), \quad \text{and} \quad H_k = H_k(G, \mathbb{Z}) = Z_k / B_k.$$

Since C_k is a free $\mathbb{Z}$-module, both Z_k and B_k are free $\mathbb{Z}$-modules. Consider the exact sequence

$$0 \longrightarrow Z_k \xhookrightarrow{\text{inc}} C_k \xrightarrow{\partial} B_{k-1} \longrightarrow 0.$$

Because B_{k-1} is free, the sequence splits, so

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(B_{k-1}, N) \xhookrightarrow{\partial^*} \mathrm{Hom}_{\mathbb{Z}}(C_k, N) \xrightarrow{\text{inc}^*} \mathrm{Hom}_{\mathbb{Z}}(Z_k, N) \longrightarrow 0$$

is a split exact sequence. In fact, these sequences assemble into a short exact sequence of cochain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(B_{k-2}, N) \arrow[r, hook] \arrow[d, "0"] & \mathrm{Hom}_{\mathbb{Z}}(C_{k-1}, N) \arrow[r, two heads] \arrow[d, "\partial^*"] & \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}, N) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(B_{k-1}, N) \arrow[r, hook] \arrow[d, "0"] & \mathrm{Hom}_{\mathbb{Z}}(C_k, N) \arrow[r, two heads] \arrow[d, "\partial^*"] & \mathrm{Hom}_{\mathbb{Z}}(Z_k, N) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(B_k, N) \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(C_{k+1}, N) \arrow[r, two heads] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(B_{k+1}, N) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots. &
\end{tikzcd}$$

The previous exact sequence induces a long exact sequence in cohomology:

$$
\begin{tikzcd}
 & & \cdots \arrow[dll] \\
\operatorname{Hom}_{\mathbb{Z}}(B_{k-2}, N) \arrow[r] & H^{k-1}(G, N) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(Z_{k-1}, N) \arrow[dll] \\
\operatorname{Hom}_{\mathbb{Z}}(B_{k-1}, N) \arrow[r] & H^{k}(G, N) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(Z_{k}, N) \arrow[dll] \\
\operatorname{Hom}_{\mathbb{Z}}(B_{k}, N) \arrow[r] & H^{k+1}(G, N) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(B_{k+1}, N) \arrow[dll] \\
\cdots .
\end{tikzcd}
$$

Furthermore, the exact sequence

$$
0 \longrightarrow B_k \xhookrightarrow{\text{inc}} Z_k \xrightarrow{q} H_k \longrightarrow 0
$$

induces a long exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(H_k, N) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(Z_k, N) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(B_k, N) \arrow[dll] \\
 & \operatorname{Ext}^1_{\mathbb{Z}}(H_k, N) \arrow[r] & \operatorname{Ext}^1_{\mathbb{Z}}(Z_k, N) \cong 0 \arrow[r] & \cdots
\end{tikzcd}
$$

for every $k \in \mathbb{Z}$. Observe that $\operatorname{Ext}^1_{\mathbb{Z}}(Z_k, N) \cong 0$ since Z_k is a free $\mathbb{Z}$-module. From the previous sequence follows that

$$
\ker(\operatorname{Hom}_{\mathbb{Z}}(Z_k, N) \to \operatorname{Hom}_{\mathbb{Z}}(B_k, N)) \cong \operatorname{Hom}_{\mathbb{Z}}(H_k, N)
$$

and

$$
\operatorname{coker}(\operatorname{Hom}_{\mathbb{Z}}(Z_k, N) \to \operatorname{Hom}_{\mathbb{Z}}(B_k, N)) \cong \operatorname{Ext}^1_{\mathbb{Z}}(H_k(G, \mathbb{Z}), N).
$$

Substituting in the induced long exact sequence in cohomology, one concludes that

$$
0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_{k-1}(G, \mathbb{Z}), N) \lhook\joinrel\longrightarrow H^k(G, \mathbb{Z}) \longrightarrow\mathrel{\mkern-14mu}\rightarrow \operatorname{Hom}_{\mathbb{Z}}(H_k(G, \mathbb{Z}), N) \longrightarrow 0
$$

is a short exact sequence for every $k \in \mathbb{Z}$. This sequence splits since the short exact sequence of cochain complexes it originates from splits. □

With this theorem established, the remaining part of the proof of Theorem I.4.15 becomes significantly simpler:

Continuation of the proof of Theorem I.4.15: Using Theorem I.4.17 with $N = \mathbb{Z}$ and $k = 1$, and the first part of the proof of Theorem I.4.15, it follows that the sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong 0 \longrightarrow H^1(G, H) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(G/[G,G], \mathbb{Z}) \longrightarrow 0$$

is exact. In particular, $H^1(G, H) \cong \mathrm{Hom}_{\mathbb{Z}}(G/[G,G], \mathbb{Z})$. □

Corollary I.4.18. *Let G be a group. Then $H^2(G, N) = 0$ for all G-modules N with trivial action if and only if $G/[G,G]$ is free as a $\mathbb{Z}$-module and $H_2(G, \mathbb{Z}) = 0$.*

Proof. The group $\mathrm{Ext}^1_{\mathbb{Z}}(H_1(G, \mathbb{Z}), N)$ is trivial for every G-module N with trivial action if and only if $H_1(G, \mathbb{Z}) \cong G/[G,G]$ is free. On the other hand, the group $\mathrm{Hom}_{\mathbb{Z}}(H_2(G, \mathbb{Z}), N) = 0$ for every G-module N with trivial action if and only if $H_2(G, \mathbb{Z}) = 0$.

Applying Theorem I.4.17, $H_2(G, \mathbb{Z}) = 0$ for every N-module G with trivial action if and only if $\mathrm{Ext}^1_{\mathbb{Z}}(H_1(G, \mathbb{Z}), N) = 0$ and $\mathrm{Hom}_{\mathbb{Z}}(H_2(G, \mathbb{Z}), N) = 0$ for every G-module N with trivial action. The latter is equivalent to $G/[G,G]$ being free and $H_2(G, \mathbb{Z}) = 0$. □

Theorem I.4.19. *Let G be a group, and A be an abelian group. The abelian group* $\mathrm{Ext}(G, A)$ *is naturally equivalent to $H^2(G, A)$, where A is seen as a trivial G-module.*

Proof. See [Ros94, 4.1.16] □

Corollary I.4.20. *Let G be a perfect group. A central extension (U, ν) of G is universal if and only if $H_1(U, \mathbb{Z}) = 0$ and $H_2(U, \mathbb{Z}) = 0$.*

Proof. Follows directly from Theorems I.3.11 and I.4.19. □

Theorem I.4.21. *Let G be a perfect group. The second homology group of G with coefficients in $\mathbb{Z}$ is*

$$H_2(G, \mathbb{Z}) \cong (R \cap [F, F])/[R, F],$$

where

$$1 \longrightarrow R \hookrightarrow F \twoheadrightarrow G \longrightarrow 1$$

is a presentation of G with F free.

Proof. By Theorem I.3.11, G has a universal central extension (U, ν). Let A be the kernel of ν, so that

$$1 \longrightarrow A \xhookrightarrow{\text{inc}} U \xtwoheadrightarrow{\nu} G \longrightarrow 1$$

is a short exact sequence. By the proof of Theorem I.3.11, $[F, F]/[R, F]$ is the universal central extension of G, so $U \cong [F, F]/[R, F]$. Therefore,

$$A = \ker\Big(U \xrightarrow{\nu} G\Big) = (R \cap [F, F])/[R, F]$$

because $G \cong F/R$. Now, consider the induced long exact sequence in homology

$$\cdots \longrightarrow H_2(U, \mathbb{Z}) \cong 0 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow H_1(A, \mathbb{Z}) \longrightarrow H_1(U, \mathbb{Z}) \cong 0 \longrightarrow \cdots,$$

where $H_2(U, \mathbb{Z}) \cong 0$ and $H_1(U, \mathbb{Z}) \cong 0$ because of Corollary I.4.20. Hence,

$$H_2(G, \mathbb{Z}) \cong H_1(A, \mathbb{Z}) \cong A/[A, A] \cong A$$

due to A being abelian. $\square$

Corollary I.4.22 (Recognition Theorem)**.** *Let G be a perfect group. Then,*

$$1 \longrightarrow H_2(G; \mathbb{Z}) \longrightarrow [F, F]/[R, F] \longrightarrow G \longrightarrow 1.$$

is the universal central extension of G, where

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

is a presentation of G.

Proof. Follows directly from the proof of Theorems I.3.11 and I.4.21. □

Chapter II

Classical Algebraic K-Theory

This chapter introduces the groups K_0, K_1, and K_2 are fundamental algebraic invariants in the theory of rings, aiming to capture different aspects of the structures of rings. The group K_0 classifies the isomorphism classes of projective modules, while K_1 is related to the general linear group $GL(R)$ and captures information about invertible matrices. On the other hand, the group K_2 is more complex and deals with studying the extent to which specific matrix identities characterize elementary matrices.

Although these groups provide a powerful way to analyze rings, their calculations can be complicated. However, the theory allows for the extension of exact sequences between these groups, which facilitates their study in particular contexts.

II.1 The Grothendieck Group of a Ring: K_0

The Grothendieck group associated with a ring R is a group constructed from the isomorphism classes of finitely generated projective R-modules. Throughout, several results are presented that show the relationship between the Grothendieck group and a way to measure how projective modules fail to be free modules.

Let R be a ring. Let $\mathfrak{P}(R)$ be the set of isomorphism classes of finitely generated right projective R-modules and $\mathfrak{F}(R)$ be the set of isomorphism classes of finitely generated right free R-modules.

Proposition II.1.1. *The set $\mathfrak{P}(R)$ is a commutative monoid with addition defined by the direct* *sum* $\oplus$. *Furthermore, if* R *is commutative, then* $\mathfrak{P}(R)$ *is a commutative semiring with multiplication defined by the tensor product* $\otimes_R$.

Proof. Follows directly from the properties of the direct sum and tensor product. $\square$

Definition II.1.2 (The Grothendieck Group of a Ring: $K_0(R)$)**.** Let R be a ring. The *Grothendieck Group* of R is defined as the group completion $(\mathfrak{P}(R), \oplus)^{\mathrm{gp}}$, and is denoted by $K_0(R)$.

Remark II.1.3. For the case where R is commutative, $K_0(R)$ is a commutative ring.

Proposition II.1.4. *The mapping* $K_0 : \mathbf{Ring} \to \mathbf{Ab}$ *is a functor, and so is its restriction* $K_0|_{\mathbf{CRing}} : \mathbf{CRing} \to \mathbf{CRing}$.

Proof. It is clear that K_0 maps a ring to an abelian group. Now, let $\varphi : R \to S$ be a ring homomorphism. Recall that by extension of scalars, any right R-module M can be mapped to a right S-module by taking the tensor product with S:

$$M \mapsto M \otimes_R S, \quad \text{where} \quad (m \otimes s) \cdot s' = m \otimes (ss').$$

Let P be a finitely generated right projective R-module, so that there exists an R-module Q and a non-negative integer n such that $P \oplus Q \cong R^n$. Observe that

$$(P \otimes_R S) \oplus (Q \otimes_R S) \cong (P \oplus Q) \otimes_R S \cong R^n \otimes_R S \cong S^n,$$

so that $P \otimes_R S$ is a finitely generated right projective S-module. Therefore, the function $\varphi_* : \mathfrak{P}(R) \to \mathfrak{P}(S)$ induced by $- \otimes_R S$ is a monoid homomorphism due to the properties of the direct sum and tensor product. Define $K_0(\varphi) := (\varphi_*)^{\mathrm{gp}}$. It is clear that for the ring homomorphism $\mathrm{id}_R : R \to R$, the equality $K_0(\mathrm{id}_R) = \mathrm{id}_{\mathfrak{P}(R)}$ holds since $M \otimes_R R \cong M$ for every right R-module M. Also, given another ring homomorphism $\psi : S \to T$, it follows that for every isomorphism class $[P]$ of finitely generated right projective R-modules,

$$\begin{aligned}(K_0(\psi) \circ K_0(\varphi))([P]) &= K_0(\psi)(K_0(\varphi)([P])) \\ &= K_0(\psi)([P \otimes_R S]) \\ &= [P \otimes_R S \otimes_S T] \\ &= [P \otimes_R (S \otimes_S T)] \\ &= [P \otimes_R T] \\ &= K_0(\psi \circ \varphi).\end{aligned}$$

Hence, $K_0 : \mathbf{Ring} \to \mathbf{Ab}$ is a functor.

For the case where K_0 is restricted to **CRing**, suppose that R and S are commutative rings and consider the monoid homomorphism φ_* as before. This homomorphism is in fact a semiring homomorphism since

$$\begin{aligned}(P \otimes_R Q) \otimes_R S &\cong P \otimes_R (Q \otimes_R S) \\ &\cong P \otimes_R (S \otimes_R Q) \\ &\cong ((P \otimes_R S) \otimes_S S) \otimes_R Q \\ &\cong (P \otimes_R S) \otimes_S (S \otimes_R Q) \\ &\cong (P \otimes_R S) \otimes_S (Q \otimes_R S).\end{aligned}$$

The rest of the proof is the same as in the previous case. □

Proposition II.1.5. *The monoid $(\mathfrak{F}(R), \oplus)$ is a cofinal submonoid of $(\mathfrak{P}(R), \oplus)$.*

Proof. Follows directly from the definition of projective modules. □

Proposition II.1.6. *Let P_1 and P_2 be finitely generated right projective R-modules. Then $[P_1] = [P_2]$ in $K_0(R)$ if and only if P_1 and P_2 are stably isomorphic.*

Proof. Follows directly from Propositions II.1.5 and I.2.15 (c). □

Corollary II.1.7. *A finitely generated right projective R-module P is stably free if and only if $[P] \in \mathbb{Z} \cdot [R]$ in $K_0(R)$.*

Proof. Suppose that P is stably free. Let m, n be integers such that $P \oplus R^m \cong R^n$, then $[P] = [R^n] - [R^m] = (n - m) \cdot [R]$ in $K_0(R)$.

Now, suppose that $[P] \in \mathbb{Z} \cdot [R]$, so there exists an integer s such that $[P] = s \cdot [R]$. Let s_0 be an integer such that $s + s_0 \geq 0$, then $[P \oplus R^{s_0}] = [P] + [R^{s_0}] \cong (s + s_0) \cdot [R] = [R^{s+s_0}]$. By Proposition II.1.6 there exists a non-negative integer t such that $(P \oplus R^{s_0}) \oplus R^t \cong R^{s+s_0} \oplus R^t$. Therefore $P \oplus R^{s_0+t} \cong R^{s+s_0+t}$, i.e., P is stably free. □

II.1.1 The Invariant Basis Number Property

In Corollary II.1.7, the subgroup $\mathbb{Z} \cdot [R]$ of $K_0(R)$ was considered to relate the ideas of finitely generated projectives module and stably free modules. Nonetheless, this subgroup may not be isomorphic to the additive group of the integer numbers since general modules do not have a well-defined idea of dimension as vector spaces do.

Definition II.1.8 (Invariant Basis Number (IBN))**.** A ring R is said to have the *Invariant Basis Number* property, abbreviated as *IBN* property, if as R-modules, $R^n \cong R^m$ implies $n = m$ for any two natural numbers n and m.

It should be evident that if R is a division ring, then it satisfies the IBN property due to the theory of vector spaces. This result helps to demonstrate that additional rings also satisfy the IBN property.

Proposition II.1.9. *Let S be a ring that satisfies the IBN property. If there exists a ring homomorphism $f : R \to S$, then R also satisfies the invariant basis property.*

Proof. Suppose that $R^n \cong R^m$ for some natural numbers n and m. Applying $- \otimes_R S$ to both sides, it follows that

$$S^n \cong R^n \otimes_R S \cong R^m \otimes_R S \cong S^m.$$

Since S satisfies the IBN property, it follows that $n = m$. □

Example II.1.10. Applying Proposition II.1.9, it follows that the following rings satisfy the IBN property:

(a) Non-zero commutative rings. Every such ring R has a maximal ideal $\mathfrak{m}$ due to Krull's Theorem, using this together with the fact that $R/\mathfrak{m}$ is a field and the quotient homomorphism $q : R \to R/\mathfrak{m}$ the IBN property follows.

(b) Local rings. Similar to the case of non-zero commutative rings, but using the fact that local rings have a unique maximal two-sided ideal.

(c) Group rings of the form $\mathbb{Z}[G]$. Consider the augmentation map $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$. Since $\mathbb{Z}$ is a non-zero commutative ring, it has the IBN property. Hence, $\mathbb{Z}[G]$ also has the IBN property.

Proposition II.1.11. *A ring R satisfies the IBN property if and only if $\mathbb{Z} \cdot [R] \cong \mathbb{Z}$ in $K_0(R)$.*

Proof. Suppose that $\mathbb{Z} \cdot [R] \cong \mathbb{Z}$ in $K_0(R)$. Then $R^n \cong R^m$ for some natural numbers n, m implies $(n - m) \cdot [R] = 0$ in $K_0(R)$, so $n = m$.

Now suppose that $\mathbb{Z} \cdot [R] \ncong \mathbb{Z}$ in $K_0(R)$. Since $\mathbb{Z} \cdot [R]$ is generated by one element, it is isomorphic to $\mathbb{Z}/n$ for some positive integer n, so $[R^n] = n \cdot [R] = 0$. By Proposition II.1.6, there exists a natural number t such that $R^n \oplus R^t \cong R^t$, so R does not satisfy the IBN property. □

Corollary II.1.12. *Let R be a ring such that it satisfies the IBN property, and all finitely generated projective R-modules are stably free. Then $K_0(R) \cong \mathbb{Z}$ with generator $[R]$.*

Proof. Follows directly from II.1.7 and II.1.11. □

Example II.1.13. Applying Corollary II.1.12, the following rings have a Grothendieck group isomorphic to $\mathbb{Z}$:

(a) Principal Ideal Domains (PIDs). It is well known that submodules of free modules over a PID are free (see [CL21, 5.4.1]); therefore all projective modules over a PID are free, in particular, they are stably free.

(b) Local rings. Similar to the previous case but using the fact that finitely generated projective modules over a local ring are free (see [Lam06, I.1.8]).

The idea from Corollary II.1.12 can be (not quite trivially) further developed to compute the Grothendieck group of polynomial rings over a field with a finite number of variables. This is known as the *Quillen-Suslin Theorem*, and stating it in terms of the functor K_0, it says that

$$K_0(k[x_1, \dots, x_n]) \cong \mathbb{Z},$$

where k is a field. For details see [Lan02, XXI §3] or [Lam06].

II.1.2 More on Projective Modules

Given a finitely generated projective R-module P, using the definition of being projective, it follows that there exists another R-module Q, and a natural number n such that $P \oplus Q \cong R^n$. Consider the map $p = (\mathrm{proj}_P, 0) : R^n \to P \oplus Q \cong R^n$ that acts as the projection on P, and the zero homomorphism on Q. It is clear that p is idempotent, i.e., $p^2 = p$. This idea yields a way to construct finitely generated projective modules: recall that any homomorphism from R^n to R^n can be represented by a $n \times n$ matrix, so any idempotent homomorphism from R^n to R^n can be represented by an idempotent $n \times n$ matrix, call this matrix A. The image of R^n under such a homomorphism results in the finitely generated R-module AR^n, which is indeed projective because the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (A-1)R^n \arrow[r, hook] & R^n \arrow[r, two heads, "A"] & AR^n \arrow[r] \arrow[l, bend left, "\mathrm{inc}_{AR^n}"] & 0 \end{tikzcd}$$

splits due to A being idempotent. Therefore, all projective R-modules can be associated with an idempotent matrix with entries in R. Nonetheless, different idempotent matrices can produce, up to isomorphism, the same finitely generated module; for example, when R is a field, any two idempotent matrices of the same rank produce the same finitely generated projective module by following the previous procedure.

Definition II.1.14 ($M(R)$ and $GL(R)$)**.** Given a ring R and a non-negative number n, let $M_n(R)$ denote the ring of $n \times n$ matrices with entries in R, and let $GL_n(R)$ denote the group (under the usual multiplication) of invertible $n \times n$ matrices with entries in R. Consider the non-unital ring homomorphism

$$M_n(R) \hookrightarrow M_{n+1}(R) \quad \text{given by} \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

and the group homomorphism

$$GL_n(R) \hookrightarrow GL_{n+1}(R) \quad \text{given by} \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe that each of these embeddings generate directed systems $(M_n(R))_{n\in\mathbb{Z}^+}$ and $(GL_n(R))_{n\in\mathbb{Z}^+}$ in **Grp**. Define

$$M(R) = \varinjlim M_n(R) \quad \text{and} \quad GL(R) = \varinjlim GL_n(R).$$

The groups $M(R)$ and $GL(R)$ are called the *infinite matrix group of* R and the *infinite general linear group of* R respectively.

Definition II.1.15 (Idem(R))**.** Let Idem(R) be the set of idempotent matrices in $M(R)$.

Observe that $GL(R)$ acts on Idem(R) by conjugation.

Theorem II.1.16. *Let R be a ring. Let $p, q \in \mathrm{Idem}(R)$, and P, Q be the finitely generated projective R-modules produced by p and q respectively. Then $P \cong Q$ if and only if p and q are conjugate under the action of $GL(R)$. In other words, $\mathfrak{P}(R)$ may be identified with the set of orbits of $GL(R)$ on* Idem(R)*. The (commutative) monoid operation is given by*

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

where, by abuse of notation, p and q are being identified to a matrix in $M_{n_p}(R)$ and $M_{n_q}(R)$ respectively for some non-negative integers n_p and n_q.

> *Proof.* See [Ros94, 1.2.1] for the first part. To verify that the operation $\oplus$ defines a monoid operation, it suffices to see that for $p, q, r \in \mathrm{Idem}(R)$, the following are satisfied:

- Associativity

$$p \oplus (q \oplus r) = \begin{pmatrix} p & 0 \\ 0 & \begin{pmatrix} q & 0 \\ 0 & r \end{pmatrix} \end{pmatrix} = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} & 0 \\ 0 & r \end{pmatrix} = (p \oplus q) \oplus r.$$

- Existence of identity

$$p \oplus 0 = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = (p) = p.$$

- Commutativity

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} = q \oplus p. \qquad \square$$

Proposition II.1.17. *Let $R = R_1 \times \cdots \times R_n$ be a product of rings. Then $K_0(R) \cong K_0(R_1) \oplus \cdots \oplus K_0(R_n)$.*

Proof. Identify the groups $M(R)$ and $M(R_1) \times \cdots \times M(R_n)$ via

$$\begin{pmatrix} ((r_1)_{11}, \ldots, (r_n)_{11}) & \cdots \\ \vdots & \ddots \end{pmatrix} \mapsto \left(\begin{pmatrix} (r_1)_{11} & \cdots \\ \vdots & \ddots \end{pmatrix}, \ldots, \begin{pmatrix} (r_n)_{11} & \cdots \\ \vdots & \ddots \end{pmatrix} \right).$$

This implies that $\mathrm{Idem}(R) = \mathrm{Idem}(R_1) \times \cdots \times \mathrm{Idem}(R_n)$ and $GL(R) = GL(R_1) \times \cdots \times GL(R_n)$. Hence, the set of orbits of $GL(R)$ on $\mathrm{Idem}(R)$ is isomorphic to the set of orbits of $GL(R_1) \times \cdots \times GL(R_n)$ on $\mathrm{Idem}(R_1) \times \cdots \times \mathrm{Idem}(R_n)$. By Theorem II.1.16 it follows that $\mathfrak{P}(R) \cong \mathfrak{P}(R_1) \times \cdots \times \mathfrak{P}(R_n)$ as commutative monoids, and finally, by Proposition I.2.17, $K_0(R) \cong K_0(R_1) \oplus \cdots \oplus K_0(R_n)$ $\square$

Definition II.1.18 (Milnor Squares)**.** Consider the following fibered product of rings:

$$\begin{tikzcd}
R \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & R_1 \arrow[d, "j_1"] \\
R_2 \arrow[r, "j_2"'] & R',
\end{tikzcd}$$

where at least one of the homomorphisms j_1 or j_2 is surjective. The previous commutative diagram is called a *Milnor square.*

Milnor squares can be used to construct projective modules over the fibered product R; in fact, all such projective modules can be constructed. The procedure is the following:

> Without loss of generality, suppose that j_2 is surjective. Let P_1 and P_2 be projective modules over R_1 and R_2 respectively, such that there exists an isomorphism
>
> $$h : P_1 \otimes_{R_1} R' \xrightarrow{\cong} P_2 \otimes_{R_2} R'$$
>
> of R'-modules. Let $M(P_1, P_2, h)$ be the fibered product obtained from the following diagram
>
> $$\begin{tikzcd}
M(P_1, P_2, h) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_1 \arrow[d, "h\circ(\mathrm{id}_{P_1}\otimes_{R_1} j_1)"] \\
P_2 \arrow[r, "\mathrm{id}_{P_2}\otimes_{R_2} j_2"'] & P_2 \otimes_{R_2} R'.
\end{tikzcd}$$
>
> Observe that $j_2 \otimes_{R_2} \mathrm{id}_{R'}$ is still surjective. Give $M(P_1, P_2, h)$ a right $R_1 \otimes_{R'} R_2$-module structure by setting
>
> $$r \cdot (m_1, m_2) \mapsto \big(i_1(r) \cdot m_1, i_2(r) \cdot m_2\big).$$
>
> To simplify notation, let $M = M(P_1, P_2, h)$ when P_1, P_2, and h are clear from the context.

Definition II.1.19 (Milnor Patching)**.** The process through which the module $M(P_1, P_2, h)$ is generated is called *Milnor patching.* Since $R \cong R_1 \times_{R'} R_2 \subseteq R_1 \times R_2$, the R-module $M(P_1, P_2, h)$ can be explicitly described by

$$M(P_1, P_2, h) = \{(m_1, m_2) \in P_1 \times P_2 \mid h(j_1(r_1 \otimes 1)) = j_2(r_2 \otimes 1)\}.$$

Notice that given an R_1-module P_1, and an R_2-module P_2, then every homomorphism $h : P_1 \otimes_{R_1} R' \to P_2 \otimes_{R_2} R'$ can be associated with an infinite matrix, in particular, if P_1 and P_2 are finitely generated, such a matrix belongs to $M(R')$. Furthermore, if the homomorphism f is also an isomorphism, the matrix must necessarily be in $GL(R')$.

Lemma II.1.20. *Consider the Milnor square*

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & R_1 \arrow[d, "j_1"] \\ R_2 \arrow[r, "j_2"'] & R'. \end{tikzcd}$$

Let F_1 be a free R_1-module and F_2 be a free R_2-module. Consider the R-module $M(F_1, F_2, h)$ obtained by Milnor patching. Let g_1 be a matrix in $GL(R_1)$ and g_2 be a matrix in $GL(R_2)$. Denote by $\overline{g_1}$ and $\overline{g_2}$ the images under $GL(j_1)$ and $GL(j_2)$ of g_1 and g_2 respectively. Then,

$$M(F_1, F_2, \overline{g_1} h \overline{g_2}) \cong M(F_1, F_2, h).$$

In particular, if h is the image of a matrix in $GL(R_2)$ under $GL(j_2)$, then $M(F_1, F_2, h)$ is free.

Proof. For the last part of the statement, see [Mil71, 2.4]. For the first part, observe that

$$M(F_1, F_2, h) \cong \{(m_1, m_2) \in F_1 \times F_2 \mid h(m_1 \otimes 1) = m_2 \otimes 1\}$$

and

$$\begin{aligned} M(F_1, F_2, \overline{g_1} h \overline{g_2}) &\cong \{(m_1, m_2) \in F_1 \times F_2 \mid \overline{g_1} h \overline{g_2}(m_1 \otimes 1) = m_2 \otimes 1\} \\ &\cong \left\{(m_1, m_2) \in F_1 \times F_2 \mid h \overline{g_2}(m_1 \otimes 1) = \overline{g_1}^{-1}(m_2 \otimes 1)\right\} \\ &\cong \left\{(m_1, m_2) \in F_1 \times F_2 \mid h(g_2 m_1 \otimes 1) = (g_1^{-1} m_2 \otimes 1)\right\}. \end{aligned}$$

Therefore, there exists an isomorphism

$$\begin{aligned} \varphi\colon \quad M(F_1, F_2, \overline{g_1} h \overline{g_2}) &\to M(F_1, F_2, h) \\ (m_1, m_2) &\longmapsto (g_1^{-1} m_1, g_2 m_2) \end{aligned}$$

since both g_1^{-1} and g_2 are invertible matrices. □

Theorem II.1.21. *Let $M = M(P_1, P_2, h)$ be the R-module obtained by Milnor patching the projective R_1-module P_1 and the projective R_2-module along h.*

(a) *The module M is projective over R. Furthermore, if P_1 and P_2 are finitely generated over R_1 and R_2 respectively, then M is finitely generated over R.*

(b) *Every projective R-module P is isomorphic to $M(P_1, P_2, h)$ for suitable choices of projective modules P_1 and P_2, and isomorphism $h : P_1 \otimes_{R_1} R' \to P_2 \otimes_{R_2} R'$.*

(c) *The modules P_1 and P_2 are naturally isomorphic to $M \otimes_R R_1$ and $M \otimes_R R_2$ respectively.*

Proof. See [Mil71, 2.1, 2.2 and 2.3]. □

Lemma II.1.22. *Let $f : R \twoheadrightarrow S$ be a surjective ring homomorphism. Let $A \in GL(S)$, then there exists a lift of $A \oplus A^{-1}$ to $GL(R)$.*

Proof. Observe that

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where each of the matrices on the right-hand side admits a lift to $GL(R)$, indeed, any lift of A to $M(R)$ works since triangular matrices with identities on the diagonal are invertible. □

Lemma II.1.23. *Consider the Milnor square*

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & R_1 \arrow[d, "j_1"] \\ R_2 \arrow[r, "j_2"'] & R'. \end{tikzcd}$$

Let P_1, Q_1 be finitely generated projective R_1-modules, and P_2, Q_2 finitely generated projective R_2-modules such that $P_1 \oplus Q_1$ is a free R_1-module, $P_2 \oplus Q_2$ is a free R_2-module, and there exist isomorphisms

$$g : P_1 \otimes_{R_1} R' \xrightarrow{\cong} P_2 \otimes_{R_2} R',$$
$$h : Q_1 \otimes_{R_1} R' \xrightarrow{\cong} Q_2 \otimes_{R_2} R',$$
$$f : P_1 \otimes_{R_1} R' \xrightarrow{\cong} Q_2 \otimes_{R_2} R'.$$

Then, $M(P_1, P_2, g) \oplus M(Q_1, Q_2, h) \cong M(Q_1, P_2, gf^{-1}h) \oplus M(P_1, Q_2, f)$.

Proof. Observe that $M(P_1, P_2, g) \oplus M(Q_1, Q_2, h) \cong M(P_1 \oplus Q_1, P_2 \oplus Q_2, g \oplus h)$, and that $M(Q_1, P_2, gf^{-1}h) \oplus M(P_1, Q_2, f) \cong M(Q_1 \oplus P_1, P_2 \oplus Q_2, gf^{-1}h \oplus f)$ due to the commutativity of limits.

By Lemma II.1.22, the matrix $h^{-1}f \oplus f^{-1}h$ admits a lift to $GL(R_2)$. Noticing that

$$\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} gf^{-1}h & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} h^{-1}f & 0 \\ 0 & f^{-1}h \end{pmatrix},$$

and applying Lemma II.1.20, one can readily conclude that $M(P_1 \oplus Q_1, P_2 \oplus Q_2, g \oplus h) \cong M(Q_1 \oplus P_1, P_2 \oplus Q_2, gf^{-1}h \oplus f)$. □

From the previous lemma, in the particular case that $P_1 = Q_1 = R_1^n$, $P_2 = Q_2 = R_2^n$ and $f = \mathrm{id}_{(R')^n}$, it follows that $M(R_1^n, R_2^n, g) \oplus M(R_1^n, R_2^n, h) \cong M(R_1^n, R_2^n, gh) \oplus M(R_1^n, R_2^n, \mathrm{id}_{(R')^n}) \cong M(R_1^n, R_2^n, gh) \oplus R^n$.

Proposition II.1.24 (The Mayer-Vietoris Sequence)**.** *Consider the Milnor square*

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & R_1 \arrow[d, "j_1"] \\ R_2 \arrow[r, "j_2"'] & R'. \end{tikzcd}$$

Then

$$GL(R') \xrightarrow{\delta} K_0(R) \xrightarrow{\begin{pmatrix} K_0(i_1) \\ K_0(i_2) \end{pmatrix}} K_0(R_1) \oplus K_0(R_2) \xrightarrow{\begin{pmatrix} K_0(j_1) & -K_0(j_2) \end{pmatrix}} K_0(R')$$

is an exact sequence, where δ is defined by

$$\delta(A) = [M(R_1^n, R_2^n, A)] - [R^n],$$

for $A \in GL_n(R') \subset GL(R')$. The matrix A is seen as a proper matrix on the left-hand side, and as an isomorphism on the right-hand side.

Proof. Exactness at $K_0(R_1) \oplus K_0(R_2) \cong K_0(R_1 \oplus R_2)$ follows from II.1.21. Now, to check that δ is well defined observe that $M(R_1^n, R_2^n, A) \oplus M(R_1, R_2, \mathrm{id}_{R'}) \cong M(R_1^{n+1}, R_2^{n+1}, A \oplus 1)$, so $[M(R_1^n, R_2^n, A)] \oplus [R] \cong [M(R_1^{n+1}, R_2^{n+1}, A \oplus 1)]$. Then,

$$\begin{aligned}
\delta(A) &= [M(R_1^n, R_2^n, A)] - [R^n] \\
&= \Big([M(R_1^{n+1}, R_2^{n+1}, A \oplus 1)] - [R]\Big) - [R^n] \\
&= [M(R_1^{n+1}, R_2^{n+1}, A \oplus 1)] - [R^n] \\
&= \delta(A \oplus 1).
\end{aligned}$$

To see that δ defines a homomorphism, consider $A, B \in GL(R')$, which can be thought of as $n \times n$ matrices for sufficiently large n. Applying Lemma II.1.23 it follows that

$$[M(R_1^n, R_2^n, A)] + [M(R_1^n, R_2^n, B)] \cong [M(R_1^n, R_2^n, AB)] + [R^n],$$

so

$$\begin{aligned}\delta(AB) &= [M(R_1^n, R_2^n, AB)] - [R^n] \\ &= ([M(R_1^n, R_2^n, A)] + [M(R_1^n, R_2^n, B)] - [R^n]) - [R^n] \\ &= ([M(R_1^n, R_2^n, A)] - [R^n]) + ([M(R_1^n, R_2^n, B)] - [R^n]) \\ &= \delta(A) + \delta(B).\end{aligned}$$

Finally, to prove exactness at $K_0(R)$ observe that

$$\begin{aligned}\begin{pmatrix} K_0(i_1) \\ K_0(i_2) \end{pmatrix} \delta(A) &= \begin{pmatrix} K_0(i_1) \\ K_0(i_2) \end{pmatrix} \Big(M(R_1^n, R_2^n, A) - [R^n] \Big) \\ &= \begin{pmatrix} [R_1^n] - [R_1^n] \\ [R_2^n] - [R_2^n] \end{pmatrix} && \text{(by Theorem II.1.21 (c))} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},\end{aligned}$$

and if $\begin{pmatrix} K_0(i_1) \\ K_0(i_2) \end{pmatrix} \Big([P]\Big) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for some finitely generated right projective R-module P, then $P \otimes_R R_1$ and $P \otimes_R R_2$ are stably isomorphic to 0, so for sufficiently large N, one has that

$$(P \otimes_R R_1) \oplus R_1^N \cong R_1^N \quad \text{and} \quad (P \otimes_R R_2) \oplus R_2^N \cong R_2^N.$$

Let $A \in GL_N(R')$ denote the matrix such that, when seen as an isomorphism of R'-modules, $P \oplus R^N \cong M(R_1^N, R_2^N, A)$. This matrix exists because of Theorem II.1.21. Then,

$$\delta(A) = [P \oplus R^N] - [R^N] = [P],$$

verifying the exactness of the whole sequence. □

Corollary II.1.25. *The image of* $\delta : GL(R') \to K_0(R)$ *is in direct correspondence with the double coset space* $\mathrm{im}(GL(j_1)) \backslash GL(R') / \mathrm{im}(GL(j_2))$.

Proof. The proof is similar to that one of Lemma II.1.20. $\square$

II.1.3 The Relative K_0-Group and Excision

Let R be a ring and I be be two-sided ideal of R. Consider the following fibered product:

$$\begin{tikzcd} R \times_{R/I} R \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, two heads, "q"] \\ R \arrow[r, two heads, "q"'] & R/I, \end{tikzcd}$$

where explicitly,

$$R \times_{R/I} R = \{(r_1, r_2) \in R \times R \mid r_1 - r_2 \in I\}.$$

Let $p_1 : R \times_{R/I} R \to R$ be the induced projection onto the first coordinate. Observe that

$$\ker(p_1) = \Big\{(r_1, r_2) \in R \times_{R/I} R \mid r_1 = 0\Big\} = \{(0, r_2) \in R \times R : r_2 \in I\} \cong I.$$

Following this idea, the homomorphism of R-modules

$$\begin{aligned} f\colon \qquad R \times I &\to R \times_{R/I} R \\ (r, i) &\mapsto (r, r + i) \\ (r, r - s) &\mapsfrom (r, s) \end{aligned}$$

is an isomorphism, so $R \times I \cong R \times_{R/I} R$.

Definition II.1.26 (Relative K_0-group)**.** Let R be a ring and I be a two-sided ideal of R. The *relative K_0-group* of R with respect to I, denoted by $K_0(R, I)$, is defined as

$$K_0(R, I) = \ker\Big(K_0(R \times_{R/I} R) \xrightarrow{K_0(p_1)} K_0(R)\Big).$$

Theorem II.1.27. *Let R be a ring and I be a two-sided ideal of R. Then there exists a natural exact sequence*

$$K_0(R, I) \xrightarrow{K_0(p)} K_0(R) \xrightarrow{K_0(q)} K_0(R/I), \tag{1}$$

where $p = p_2|_{\ker(p_1)}$, *and* $q : R \to R/I$ *is the quotient homomorphism.*

Proof. Consider the Milnor square

$$\begin{tikzcd} R \times I \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, two heads, "q"] \\ R \arrow[r, two heads, "q"'] & R/I. \end{tikzcd}$$

By taking only the last three terms of the Mayer-Vietoris sequence obtained from it, one readily has that

$$K_0(R \times I) \xrightarrow{\begin{pmatrix} K_0(p_1) \\ K_0(p_2) \end{pmatrix}} K_0(R) \oplus K_0(R) \xrightarrow{\begin{pmatrix} K_0(q) & -K_0(q) \end{pmatrix}} K_0(R/I)$$

is exact. Recall that $K_0(R, I) = \ker\Big(K_0(R \times_{R/I} R) \xrightarrow{K_0(p_1)} K_0(R)\Big)$, so the previous exact sequence can be changed to

$$K_0(R, I) \xrightarrow{K_0(p)} K_0(R) \xrightarrow{K_0(q)} K_0(R/I),$$

while preserving exactness.

Actually, it was not necessary to only consider the last three terms of the Mayer-Vietoris sequence. Observe that $\operatorname{im}(\delta) = \ker\left(\begin{pmatrix} K_0(p_1) \\ K_0(p_2) \end{pmatrix}\right) = \ker(K_0(p_1)) \cap \ker(K_0(p_2)) = \ker(K_0(p))$. Hence, the exact sequence (1) can be extended by one term to the left:

$$GL(R/I) \xrightarrow{\delta} K_0(R, I) \xrightarrow{K_0(p)} K_0(R) \xrightarrow{K_0(q)} K_0(R/I),$$

where $\delta : GL(R/I) \to K_0(R, I)$ only needs to have its codomain modified. $\square$

Definition II.1.28 (I_+)**.** Let I be a ring without a multiplicative identity (rng). The ring obtained by adjoining a multiplicative identity to I consists of the abelian group $I \oplus \mathbb{Z}$ together with a multiplicative operation $\times$ defined by

$$(x, n) \times (y, m) = (xy + ny + mx, nm), \qquad \text{for all } x, y \in I \text{ and } n, m \in \mathbb{Z}.$$

The ring defined previously is denoted by I_+.

Proposition II.1.29. *Given a rng I, the construction I_+ from Definition II.1.28 defines a ring. Even more, this construction defines a functor from* **Rng** *to* **Ring**.

Definition II.1.30 (Grothendieck Group of a Rng)**.** Let I be a rng. Define

$$K_0(I) = \ker\left(K_0(I_+) \xrightarrow{K_0(\mathrm{proj}_{\mathbb{Z}})} K_0(\mathbb{Z}) \cong \mathbb{Z}\right).$$

Proposition II.1.31. *Let R be a ring. The definition of $K_0(R)$ as in Definition II.1.30 agrees with Definition II.1.2.*

Proof. Appying K_0 (as in II.1.2) to $R_+ = R \oplus \mathbb{Z}$, it follows that $K_0(R_+) = K_0(R \oplus \mathbb{Z})$. Then, because the ring structure on the summand $\mathbb{Z}$ of $\mathbb{R} \oplus \mathbb{Z}$ is the same as the ring structure on $\mathbb{Z}$ of the product $R \times \mathbb{Z}$, $K_0(R) \cong \ker\left(K_0(R \oplus \mathbb{Z}) \xrightarrow{K_0(\mathrm{proj}_{\mathbb{Z}})} K_0(\mathbb{Z})\right)$, which is the definition given in II.1.30. □

Theorem II.1.32 (Excision)**.** *Let R be a ring and I a two-sided ideal of R. Then $K_0(R, I) \cong K_0(I)$*

Proof. Let $\gamma : I_+ \to R \times_{R/I} R$ be the ring homomorphism defined by

$$\gamma(x, n) = (n \cdot 1, n \cdot 1 + x)$$

for all $x \in I$ and $n \in \mathbb{Z}$. Observe that the following diagram commutes:

$$\begin{tikzcd} I_+ \arrow[r, "\gamma"] \arrow[d, "\mathrm{proj}_{\mathbb{Z}}"'] & R \times_{R/I} R \arrow[d, "p_1 = \mathrm{proj}_1"] \\ \mathbb{Z} \arrow[r, "\iota"'] & R. \end{tikzcd}$$

By functoriality of K_0, it follows that the diagram

$$\begin{tikzcd} K_0(I_+) \arrow[r, "K_0(\gamma)"] \arrow[d, "K_0(\mathrm{proj}_{\mathbb{Z}})"'] & K_0(R \times_{R/I} R) \arrow[d, "K_0(p_1)"] \\ \mathbb{Z} \arrow[r, "K_0(\iota)"'] & K_0(R) \end{tikzcd}$$

commutes, and $K_0(\gamma)(K_0(I)) = K_0(\ker(K_0(\mathrm{proj}_{\mathbb{Z}}))) \subseteq \ker(K_0(p_1)) = K_0(R, I)$. It will be proven that $K_0(\gamma)$ induces an isomorphism between $K_0(I)$ and $K_0(R, I)$.

To prove surjectivity, let $[A] - [B] \in K_0(R, I)$, where, by abuse of notation, A and B are the finitely generated projective $R \times_{R/I} R$-modules generated from the idempotent matrices in $M(R \times_{R/I} R)$ with the same names. Observe that these matrices can be written as

$$A = (A_1, A_2) \quad \text{and} \quad B = (B_1, B_2),$$

where $[A_1] = [B_1]$ in $K_0(R)$ because $K_0(p_1)([A] - [B]) = 0$.

The idea of the proof is to replace the matrices A and B in a way that the value $[A]-[B]$ remains unchanged in each step. Notice that by Theorem II.1.16, $A_1 = gB_1g^{-1}$ for some $g \in GL(R)$, and by the same argument B and be replaced by $(g, g)B(g^{-1}, g^{-1}) = (gB_1g^{-1}, gB_2g^{-1}) = (A_1, gB_2g^{-1})$ without altering the value of $[B]$, so now

$$A = (A_1, A_2) \quad \text{and} \quad B = (A_1, gB_2g^{-1}).$$

Since adding the same matrix to A and B does not change the value of $[A] - [B]$, set

$$A = (A_1, A_2) \oplus (1 - A_1, 1 - A_1) \quad \text{and} \quad B = (A_1, gB_2g^{-1}) \oplus (1 - A_1, 1 - A_1).$$

Observe that for sufficiently large n, $A_1R^n \cong (A_1 \oplus 1 - A_1)R^n$ via $x \mapsto (x, 1 - x)$, so there exists a matrix $h \in GL_n(R)$ such that $h(A_1 \oplus 1 - A_1)h^{-1} = 1_n \oplus 0_n$. Conjugating A and B by $(h, h) \in GL(R \times_{R/I} R)$ it follows that

$$A = (1_n \oplus 0_n, h(A_2 \oplus (1 - A_1))h^{-1})) \quad \text{and} \quad B = (1_n \oplus 0_n, h((gB_2g^{-1}) \oplus (1 - A_1))h^{-1}).$$

Let $A_3 = h(A_2 \oplus (1 - A_1))h^{-1}$ and $B_3 = h((gB_2g^{-1}) \oplus (1 - A_1))h^{-1}$. Since $1_n \oplus 0_n - A_3$ and $1_n \oplus 0_n - B_3$ belong to $GL(I)$ because $A, B \in R \times_{R/I} R$, then A and B are the images under $K_0(\gamma)$ of $(A_3 - 1_n \oplus 0_n, 1_n \oplus 0_n)$ and $(B_3 - 1_n \oplus 0_n, 1_n \oplus 0_n)$, where, in the second entry of each matrix $1_n \oplus 0_n \in GL(\mathbb{Z})$. Thus, $K_0(\gamma)$ is surjective.

Injectivity is proved in a similar way. For further details, see [Ros94, 1.5.9]. □

Corollary II.1.33. *The exact sequence* (1) *can be rewritten as*

$$K_0(I) \longrightarrow K_0(R) \longrightarrow K_0(R/I).$$

Proof. Because of Theorem II.1.32, $K_0(R,I)$ can be substituted by $K_0(I)$ in (1). □

Corollary II.2.18 from the following chapter provides a condition under which the previous exact sequence becomes a short exact sequence.

II.2 The Whitehead Group of a Ring: K_1

The Whitehead group and its relative versions are groups related to the general linear group $GL(R)$ of a ring. It is shown that the group $K_1(R)$ measures the failure of an invertible matrix with coefficients in R to be reduced to the identity using only elementary operations. From there, concrete examples of Whitehead groups are computed.

The discussion is extended to the relative Whitehead group $K_1(R,I)$ and it is shown that this group can be related to a Mayer-Vietoris exact sequence, where the groups K_0 and K_1 appear.

Definition II.2.1 (The Whitehead Group of a Ring: $K_1(R)$)**.** Let R be a ring. The *Whitehead group* of R is defined as the quotient $GL(R)/[GL(R), GL(R)]$, and is denoted by $K_1(R)$.

Observe that with this definition, $K_1(R) \cong H_1(GL(R), \mathbb{Z})$ by Theorem I.4.15.

Proposition II.2.2. *The mapping* $K_1 : \mathbf{Ring} \to \mathbf{Ab}$ *is a functor.*

Proof. Observe that a ring homomorphism $f : R \to S$ induces a group homomorphism $GL(f) : GL(R) \to GL(S)$ by applying f entrywise. Let $K_1(f)$ be the group homomorphism obtained by the universal property of abelianization of a group applied to $q_S \circ GL(f)$, where $q_S : GL(S) \to K_1(S)$ denotes the quotient homomorphism.

$$R \xrightarrow{f} S$$

$$\begin{tikzcd}
GL(R) \arrow[r, "GL(f)"] \arrow[d, "q_R"'] & GL(S) \arrow[d, "q_S"] \\
K_1(R) \arrow[r, dashed, "K_1(f)"'] & K_1(S).
\end{tikzcd}$$

It is clear that $K_1(\mathrm{id}_R) = \mathrm{id}_{K_1(R)}$. Now, consider another ring homomorphism $g : S \to T$. It is straightforward that $GL(g) \circ GL(f) = GL(g \circ f)$. If $q_T : GL(T) \to K_1(T)$ denotes the quotient homomorphism, then $q_T \circ GL(g) \circ GL(f) = q_T \circ GL(g \circ f)$.

$$\begin{tikzcd}
GL(R) \arrow[r, "GL(f)"] \arrow[d, "q_R"] \arrow[rr, bend left, "GL(g\circ f)"] & GL(S) \arrow[r, "GL(g)"] \arrow[d, "q_S"] & GL(T) \arrow[d, "q_T"] \\
K_1(R) \arrow[r, "K_1(f)"'] \arrow[rr, bend right, "K_1(g\circ f)"'] & K_1(S) \arrow[r, "K_1(g)"'] & K_1(T).
\end{tikzcd}$$

By the uniqueness part of the universal property of the abelianization of a group, it follows that $K_1(g \circ f) = K_1(g) \circ K_1(f)$. □

Proposition II.2.3. *Let $R = R_1 \times \cdots \times R_n$ be a product of rings. Then $K_1(R) \cong K_1(R_1) \oplus \cdots \oplus K_1(R_n)$.*

Proof. Recall that $GL(R) \cong GL(R_1)\times\cdots\times GL(R_n)$. From this, it follows that $[GL(R), GL(R)] \cong [GL(R_1), GL(R_1)] \times \cdots \times [GL(R_n), GL(R_n)]$. Therefore, $K_1(R) \cong K_1(R_1) \oplus \cdots \oplus K_1(R_n)$. □

Definition II.2.4 (Elementary Matrix)**.** Let R be a ring. A matrix in $GL_n(R)$ is called *elementary* if it coincides with the identity matrix of $GL_n(R)$, except, possibly, in an off-diagonal entry. Denote by $e_{ij}^{(n)}(r)$ the elementary matrix which differs from the identity matrix in the entry i, j, with $i \neq j$, and has $r \in R$ instead of 0 in such entry.

Definition II.2.5 ($E(R)$)**.** Let R be a ring. Denote by $E_n(R)$ the subgroup of $GL_n(R)$ generated by all elementary matrices. Observe that the inclusion $GL_n(R) \hookrightarrow GL_{n+1}(R)$ restricts to an inclusion

$E_n(R) \hookrightarrow E_{n+1}(R)$, generating a directed system $(E_n(R))_{n\in\mathbb{Z}^+}$. Define

$$E(R) = \varinjlim E_n(R).$$

Matrices in $E(R)$ will still be called *elementary*, and $E(R)$ is called the *infinite elementary matrix group of R*.

Remark II.2.6. The superscript in $e_{ij}^{(n)}(r)$ now becomes irrelevant when thinking of such a matrix in $E(R)$, so it will be dropped from the notation.

Remark II.2.7. The group $E(R)$ is also the subgroup of $GL(R)$ generated by elementary matrices.

Proposition II.2.8. *The following relations of elementary matrices hold:*

$$\begin{cases} e_{ij}(r)e_{ij}(s) = e_{ij}(r+s), & \\ [e_{ij}(r), e_{k\ell}(s)] = 1 & \text{when } i \neq \ell \text{ and } j \neq k, \\ [e_{ij}(r), e_{j\ell}(s)] = e_{i\ell}(rs) & \text{when } i \neq \ell, \\ [e_{ij}(r), e_{ki}(s)] = e_{kj}(-sr) & \text{when } j \neq k, \end{cases}$$

where $[e_{ij}(r), e_{k\ell}(s)] = e_{ij}(r)e_{k\ell}(s)e_{ij}(r)^{-1}e_{k\ell}(s)^{-1}$.

Proof. Follows from direct computation in $GL_N(R)$ for sufficiently large N. □

Proposition II.2.9. *Any upper-triangular or lower-triangular matrix with entries in R and 1's on the diagonal is elementary. Even more, if $A \in GL_n(R)$, then $A \oplus A^{-1} \in E_{2n}(R)$.*

Proof. See [Ros94, 2.1.2 and 2.1.3]. □

Lemma II.2.10. *Let R be a ring.*

(a) *For $n \geq 3$, the group $E_n(R)$ is perfect, i.e., $E_n(R) = [E_n(R), E_n(R)]$.*

(b) *The group $E(R)$ is perfect.*

Proof.

(a) Notice that the inclusion $[E_n(R), E_n(R)] \subseteq E_n(R)$ is clear. For the other inclusion, observe that for an arbitrary elementary matrix $e_{ij}(R)$, one has that $e_{ij}(R) = [e_{ik}(r), e_{kj}(r)]$ for some $k \in \{1, \ldots, n\} - \{i, j\}$.

(b) Follows from the fact that $E(R)$ is the colimit of the directed system $(E_n(R))_{n \in \mathbb{Z}^+ - \{1,2\}}$. □

Proposition II.2.11 (Whitehead's Lemma)**.** *Let R be a ring. The subgroup $E(R)$ of $GL(R)$ equals the commutator subgroup $[GL(R), GL(R)]$.*

Proof. Observe that by Lemma II.2.10 (b), it follows that $E(R) \subseteq [GL(R), GL(R)]$. Conversely, let $A, B \in GL(R)$. There exists sufficiently large N such that $A, B \in GL_N(R)$. Observe that

$$[A, B] = \begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix},$$

where, by Proposition II.2.9, all the matrices in the right-hand side belong to $E(R)$. Therefore $E(R) = [GL(R), GL(R)]$. □

Using Whitehead's Lemma, the definition of $K_1(R)$ can be restated as

$$K_1(R) = GL(R)/E(R).$$

From this, it can be said that the Whitehead Group measures the failure of an invertible matrix with entries in R to be reduced to the identity matrix using only 'elementary operations', consisting of multiplying such a matrix by elementary matrices.

II.2.1 Some Computations of the Whitehead Group

Recall that when R is a commutative ring, there is a well-defined determinant in $GL_n(R)$ for every positive integer n. Denote this morphism by $\det_n : GL_n(R) \to R^\times$. By the definition of colimit, there exists a unique induced homomorphism

$$\det : GL(R) \to R^\times,$$

that corresponds to first thinking of a matrix $A \in GL(R)$ as belonging to $GL_N(R)$ for sufficiently large N and then applying $\det_N$.

Definition II.2.12 ($SL(R)$)**.** Let R be a ring. Define

$$SL(R) = \ker\Big(GL(R) \xrightarrow{\det} R^\times\Big).$$

The group $SL(R)$ is called the *infinite special linear group of R.*

Observe that $E(R) \subseteq SL(R)$, so $\det : GL(R) \to R^\times$ induces an homomorphism

$$\det|_{K_1(R)} : K_1(R) \to R^\times.$$

Even more, such induced homomorphism admits a section s defined in the following way:

$$\begin{array}{rcccccccc} s\colon & R^\times & \cong & GL_1(R) & \longrightarrow & GL(R) & \longrightarrow & K_1(R) \\ & r & \longmapsto & (r) & \longmapsto & \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} & \longmapsto & \left[\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\right] \end{array}$$

Let $SK_1(R) = \ker\Big(K_1(R) \xrightarrow{\det|_{K_1(R)}} R^\times\Big) = SL(R)/E(R)$, so that the following exact sequence of abelian groups splits:

$$0 \longrightarrow SK_1(R) \hookrightarrow K_1(R) \twoheadrightarrow R^\times \longrightarrow 0$$

This implies that $K_1(R) \cong SK_1(R) \oplus R^\times$, so the computation of $K_1(R)$ is reduced to the computation of $SK_1(R)$. In particular, if $SK_1(R) = 0$, then $K_1(R) \cong R^\times$.

Proposition II.2.13. *If R is an Euclidean domain, then $SK_1(R) = 0$.*

> *Proof.* It suffices to prove that $SL(R) \subseteq E(R)$. Let $d : R \to \mathbb{Z}^+ \cup \{-\infty\}$ be the Euclidean norm of R, and let $A = (a_{ij}) \in SL_n(R)$.
>
> Let $\varepsilon_1 = \min\Big\{d\Big(a_{1j}^{(1)}\Big) : 1 \le j \le n\Big\}$, where $A^{(1)} = (a_{ij}^{(1)}) = (a_{ij})$. Observe that ε_1 is well-defined because of the well-ordering principle and the fact that invertible matrices cannot have a row

of zeroes. Suppose that ε_k and $A^{(k)}$ have been defined and $\varepsilon_k \neq 1$. Let $a_{1m}^{(k)}$ be such that $d(a_{1m}^{(k)}) = \varepsilon_k$. By subtracting adequate multiples of the m-th column of $A^{(k)}$ to the rest of the columns, it is possible to have $d\left(a_{1j}^{(k)}{}'\right) < \varepsilon_k$ for all $j \neq m$, where $a_{1j}^{(k)}{}'$ denotes $a_{1j}^{(k)}$ minus the corresponding appropriate multiple of $a_{1m}^{(k)}$.

Let $A^{(k+1)}$ be the matrix obtained after the column operations applied to $A^{(k)}$, and define $\varepsilon_{k+1} = \min\left\{d\left(a_{1j}^{(k+1)}\right) : 1 \leq j \leq n\right\} < \varepsilon_k$. Observe that this process must end, and it can only do so once $\varepsilon_N = 1$ for sufficiently large N since $\det(A) = 1$. Let $d\left(a_{1M}^{(N)}\right) = 1$. If $M \neq 1$, then substracting $\left(a_{11}^{(N)} - 1\right)\left(a_{1M}^{(N)}\right)^{-1}$ times the M-th column to the first column, so that the $1,1$ entry of the resulting matrix is 1, one can assume that $M = 1$ since the beginning.

After performing the appropriate row and column operations on $A^{(N)}$, all the entries in the first row and column except for a_{11}^N can be changed to 0. Now, repeat this process to the $(1,1)$ minor of $A^{(N)}$ until A is converted to the identity matrix. Observe that this only required a finite number of row and column operations, i.e., multiplication by elementary matrices. Therefore, $A \in E_n(R)$, which implies that $SL(R) = E(R)$. $\square$

Example II.2.14. Applying Proposition II.2.13, the following Whitehead groups can be computed:

(a) Let F be a field, then $K_1(F) = F^{\times}$.

(b) Let F be a field, then $K_1(F[x]) = F^{\times}$.

(c) $K_1(\mathbb{Z}) = \{1, -1\} \cong \mathbb{Z}/2$.

(d) $K_1(\mathbb{Z}[i]) = \{1, i, -1, -i\} \cong \mathbb{Z}/4$.

II.2.2 The Mayer-Vietoris Sequence

Recall that Proposition II.1.24 gives the exact sequence

$$GL(R') \xrightarrow{\delta} K_0(R) \xrightarrow{\begin{pmatrix} K_0(i_1) \\ K_0(i_2) \end{pmatrix}} K_0(R_1) \oplus K_0(R_2) \xrightarrow{\begin{pmatrix} K_0(j_1) & -K_0(j_2) \end{pmatrix}} K_0(R'),$$

under the hypothesis of a Milnor square. The purpose of this subsection is to extend the given sequence to the left.

Proposition II.2.15 (The Mayer-Vietoris Sequence)**.** *Consider the Milnor square*

$$\begin{tikzcd} R \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & R_1 \arrow[d, "j_1"] \\ R_2 \arrow[r, "j_2"'] & R'. \end{tikzcd}$$

Then

$$K_1(R) \xrightarrow{\Delta_1} K_1(R_1) \oplus K_1(R_2) \xrightarrow{\nabla_1} K_1(R') \xrightarrow{\delta} K_0(R) \xrightarrow{\Delta_0} K_0(R_1) \oplus K_0(R_2) \xrightarrow{\nabla_0} K_0(R')$$

is an exact sequence, where

$$\Delta_n = \begin{pmatrix} K_n(i_1) \\ K_n(i_2) \end{pmatrix} \quad \text{and} \quad \nabla_n = \begin{pmatrix} K_n(j_1) & -K_n(j_2) \end{pmatrix}$$

for $n \in \{1, 2\}$, and, by abuse of notation, δ denotes the quotient map of $GL(R') \to K_0(R)$ as defined in the proof of Theorem II.1.27.

Proof. Exactness at $K_0(R_1) \oplus K_0(R_2)$ is given by Proposition II.1.24. Since $K_1(R')$ is defined as the quotient $GL(R')/[GL(R'), GL(R')]$, to prove exactness at $K_0(R)$ it suffices to see that δ is well-defined by seeing $\delta([GL(R'), GL(R')]) = \{0\} \subseteq K_0(R)$. This follows directly from the proof of Proposition II.2.11, Lemma II.1.22, and Lemma II.1.20. Now, for exactness at $K_1(R')$ notice that by Corollary II.1.25, the kernel of δ is the subgroup generated by the images of $K_1(R_1)$ and $K_1(R_2)$ because the K_1-groups are abelian. Finally, exactness at $K_1(R_1) \oplus K_1(R_2)$ follows from the observation that

$$\begin{tikzcd} GL(R) \arrow[r, "GL(i_1)"] \arrow[d, "GL(i_2)"'] \arrow[dr, phantom, "\lrcorner", very near start] & GL(R_1) \arrow[d, "GL(j_1)"] \\ GL(R_2) \arrow[r, "GL(j_2)"'] & GL(R') \end{tikzcd}$$

is still a Milnor square, and elementary matrices get mapped to elementary matrices in each step. $\square$

II.2.3 The Relative K_1-Group

The definition of relative K_1-groups is similar to relative K_0-groups. Recall the fibered product diagram:

$$\begin{tikzcd} R \times_{R/I} R \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, two heads, "q"] \\ R \arrow[r, two heads, "q"] & R/I. \end{tikzcd}$$

Definition II.2.16 (Relative K_1-group)**.** Let R be a ring and I be a two-sided ideal of R. The *relative K_1-group* of R with respect to I, denoted by $K_1(R, I)$, is defined as

$$K_1(R, I) = \ker\Big(K_1(R \times_{R/I} R) \xrightarrow{K_1(p_1)} K_1(R)\Big).$$

Theorem II.2.17. *Let R be a ring and I be a two-sided ideal of R. Then there exists a natural exact sequence*

$$K_1(R, I) \xrightarrow{K_1(p)} K_1(R) \xrightarrow{K_1(q)} K_1(R/I) \xrightarrow{\delta} K_0(R, I) \xrightarrow{K_0(p)} K_0(R) \xrightarrow{K_0(q)} K_0(R/I) \tag{2}$$

where $p = p_2|_{\ker(p_1)}$, $q : R \to R/I$ is the quotient homomorphism, and δ is the map defined in Proposition II.2.15.

Proof. The proof follows directly from Proposition II.2.15 and the proof of Theorem II.1.27. □

Corollary II.2.18. *Let R be a ring and I be a two-sided ideal of R. If the quotient ring homomorphism $q : R \to R/I$ splits, then*

$$0 \longrightarrow K_0(R, I) \xrightarrow{K_0(p)} K_0(R) \xrightarrow{K_0(q)} K_0(R/I) \longrightarrow 0$$

splits.

Proof. Because of functoriality of K_0 and K_1, both $K_0(q)$ and $K_1(q)$ split. Applying that $K_1(q)$ splits to (2), the quotient over $[GL(R/I), GL(R/I)]$ of any matrix in $GL(R/I)$ is the image of an element in $K_1(R)$, i.e., the image of δ is the zero group. Because (2) is exact, this implies that $K_0(p)$ is injective. □

The relative K_1-group can be studied from a different perspective:

Definition II.2.19 ($GL(I)$)**.** Let R be a ring and I be a two-sided ideal of R. Define

$$GL(I) = \ker\Big(GL(R) \xrightarrow{GL(q)} GL(R/I)\Big),$$

where q denotes the quotient homomorphism.

Remark II.2.20. The notation of $GL(I)$ suggests that this group does not depend on R. Indeed, the kernel of $GL(q)$ can be explicitly described as matrices of the form $1 + A$, where 1 is the identity matrix in $GL(R)$ and A is a matrix in $M(R)$ whose entries all belong to I.

Definition II.2.21 ($E(R, I)$)**.** Let R be a ring and I be a two-sided ideal of R. Define $E(R, I)$ to be the smallest normal subgroup of $E(R)$ containing the elementary matrices $e_{ij}(a)$ with $a \in I$.

Remark II.2.22. If $E_n(R, I)$ were to denote the normal subgroup generated by the matrices $e_{ij}^{(n)}(a)$ with $a \in I$, then it is straightforward to see that $E(R, I) = \varinjlim E_n(R, I)$.

Proposition II.2.23 (Relative Whitehead's Lemma)**.** *Let R be a ring and I be a two-sided ideal of R. Then $E(R, I)$ is a normal subgroup of $GL(I)$. Furthermore, the quotient $GL(I)/E(R, I)$ is isomorphic to $K_1(R, I)$.*

Proof. See [Ros94, 2.5.3]. □

II.3 The Milnor Group of a Ring: K_2

The Milnor group of a ring is a construction that measures the degree to which certain matrix identities characterize the group $E(R)$, using the Steinberg group for this purpose. Calculations involving this group can be complex, and the generality achieved in the previous sections of extending the Mayer-Vietoris sequence to any fibered product is lost. Despite this, it is still possible to obtain information about this group through a more restricted version, allowing an extension of the exact sequences (1) and (2).

II.3.1 The Steinberg Group

Definition II.3.1 (Steinberg Group)**.** Let R be a ring. For an integer $n \geq 3$, the *Steinberg group of order n over R*, denoted by $\mathrm{St}_n(R)$, is defined as the free group on generators $x_{ij}^{(n)}(r)$, with $i, j \in \{1, \ldots, n\}$, $i \neq j$, and $r \in R$ modulo the normal subgroup generated by the following relations:

$$\begin{cases} x_{ij}^{(n)}(r)x_{ij}^{(n)}(s) = x_{ij}^{(n)}(r+s), & \\ [x_{ij}^{(n)}(r), x_{k\ell}^{(n)}(s)] = 1 & \text{when } i \neq \ell \text{ and } j \neq k, \\ [x_{ij}^{(n)}(r), x_{j\ell}^{(n)}(s)] = x_{i\ell}^{(n)}(rs) & \text{when } i \neq \ell, \\ [x_{ij}^{(n)}(r), x_{ki}^{(n)}(s)] = x_{kj}^{(n)}(-sr) & \text{when } j \neq k. \end{cases}$$

By considering the natural homomorphism $\mathrm{St}_n(R) \to \mathrm{St}_{n+1}(R)$ given by $x_{ij}^{(n)}(r) \mapsto x_{ij}^{(n+1)}(r)$, $(\mathrm{St}_n(R))_{n\geq 3}$ becomes a directed system. The *Steinberg group over R*, denoted by $\mathrm{St}(R)$, is defined as the direct limit

$$\mathrm{St}(R) = \varinjlim \mathrm{St}_n(R).$$

Remark II.3.2. The natural homomorphism $\mathrm{St}_n(R) \to \mathrm{St}_{n+1}(R)$ need not be injective.

Remark II.3.3. In a similar way as in the case of elementary matrices, the superscript in $x_{ij}^{(n)}(r)$ now becomes irrelevant when thinking of such an element in $\mathrm{St}(R)$, so it will often be dropped from the notation.

Remark II.3.4. The relation $[x_{ij}^{(n)}(r), x_{ki}^{(n)}(s)] = x_{kj}^{(n)}(-sr)$ when $j \neq k$ is redundant in the definition of the Steinberg group of order n over R. See [Ros94, 4.2] for details.

Notice the similarity from the relations satisfied by elementary matrices defined in Proposition II.2.8 to the previously presented relations satisfied by the elements of the Steinberg group. These relations will be referred to as the obvious relations among elementary matrices over a ring.

Proposition II.3.5. *Let R be a ring. The Steinberg group* $\mathrm{St}(R)$ *is perfect, and there exists a canonical homomorphism* $\varphi : \mathrm{St}(R) \twoheadrightarrow E(R) \subseteq GL(R)$.

Proof. For the first part, let $x_{ij}(r) \in \mathrm{St}(R)$. There exists a sufficiently large natural number n such that $x_{ij}(r)$ can be thought of as an element in $\mathrm{St}_n(R)$. Because of the relations that $\mathrm{St}_n(R)$ satisfies,

$$[x_{ik}^{(n)}(r), x_{kj}^{(n)}(1)] = x_{ij}^{(n)}(r)$$

holds for k distinct from i and j. Therefore, $\mathrm{St}_n(R)$ is perfect since it is generated by commutators. The claim follows from the fact that $\mathrm{St}(R)$ is the directed limit of perfect groups. □

Remark II.3.6. The group homomorphism φ discussed in Proposition II.3.5 depends on the ring R. If it is necessary to explicitly specify the ring on which the homomorphism depends, it will be denoted by φ_R.

II.3.2 The Milnor Group of a Ring

The idea of the Milnor group of a ring R is to measure the non-obvious relations among elementary matrices over R.

Definition II.3.7 (The Milnor Group of a Ring: K_2)**.** Let R be a ring. The Milnor Group of R is defined as the kernel $\ker(\varphi : \mathrm{St}(R) \twoheadrightarrow E(R))$, and is denoted by $K_2(R)$.

Observe that by this definition, there is an exact sequence of groups

$$0 \longrightarrow K_2(R) \xhookrightarrow{\text{inc}} \mathrm{St}(R) \xrightarrow{\varphi} GL(R) \xrightarrow{q} K_1(R) \longrightarrow 0.$$

Proposition II.3.8. *Let R be a ring. Then $K_2(R) \cong Z(\mathrm{St}(R))$.*

Proof. First, observe that the center of $E(R)$ is trivial. Let $A \in Z(E(R))$, so there exists sufficiently large n such that $A \in E_n(R)$. Then, in $E_{2n}(R)$,

$$\begin{pmatrix} A & A \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus, $A = 1$, which proves the claim. Therefore, if $x \in Z(\mathrm{St}(R))$, then $\varphi(x) = 1 \in E(R)$, so $Z(\mathrm{St}(R)) \subseteq K_2(R)$.

Conversely, let $x \in K_2(R) = \ker(\mathrm{St}(R) \xrightarrow{\varphi} E(R))$. Write

$$x = x_{i_1 j_1}(r_1) \cdots x_{i_n j_n}(r_n), \quad \text{where} \quad e_{i_1 j_1}(r_1) \cdots e_{i_n j_n}(r_n) = 1.$$

Let $N = \max\{i_1, \dots i_n, j_1, \dots, j_n\} + 1$. Let A_N be the subgroup of $\mathrm{St}(R)$ generated by $\{x_{iN}(r) \in \mathrm{St}(R) \mid i \neq N, r \in R\}$. For $1 \leq \ell \leq n$, $1 \leq k < N$, and $r \in R$,

$$x_{i_\ell j_\ell}(r_0) x_{kN}(r) x_{i_\ell j_\ell}(r_0)^{-1} = \begin{cases} x_{kN}(r) & \text{when } k \neq j_\ell, \\ x_{i_\ell N}(r_0 r) x_{kN}(r) & \text{when } k = j_\ell. \end{cases}$$

Therefore, x normalizes A_N. Notice that A_N is abelian by the first relation of the definition of $\mathrm{St}(R)$, and the image under φ of $x_{iN}(r)$ with $i \neq N$ corresponds to a matrix in $E(R)$ that only differs from the identity matrix in the i, N entry by r. Thus, $\varphi|_{A_N}$ is injective. Let $y \in A_N$, then

$$\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1} = \varphi(y)\varphi(y)^{-1} = \varphi(yy^{-1}).$$

Because x normalizes A_N and $\varphi|_{A_N}$ is injective, it follows that $xyx^{-1} = y$. Hence, x commutes with all the elements in A_N.

Let B_N be the subgroup of $\mathrm{St}(R)$ generated by $\{x_{Nj}(r) \in \mathrm{St}(R) \mid j \neq N, r \in R\}$. By an analogous argument as before, x commutes with all the elements in B_N. Furthermore, A_N and B_N generate $\mathrm{St}(R)$ because

$$\begin{aligned} x_{iN}(r) &\in A_N, \\ x_{Nj}(r) &\in B_N, \\ x_{ij}(r) = [x_{iN}(r), x_{Nj}(1)] &\in [A_N, B_N] \qquad\qquad \text{if } i \neq N \text{ and } j \neq N, \end{aligned}$$

so x is central in $\mathrm{St}(R)$. □

Corollary II.3.9. *Let R be a ring. Then the group $K_2(R)$ is abelian.*

Proof. Follows directly from Proposition II.3.8. □

Corollary II.3.10. *Let R be a ring. Then* $\mathrm{St}(R)$ *is a central extension of* $E(R)$ *by* $K_2(R)$.

Proof. Follows directly from Definition I.3.1. □

Proposition II.3.11. *Let R be a ring. Then* $\mathrm{St}(R)$ *is the universal central extension of* $E(R)$.

Sketch of proof. By Proposition II.3.5 and Corollary II.3.10, it suffices to prove that all central extensions (E, φ) of $\mathrm{St}(R)$ are trivial. This can be done by defining a section $s : \mathrm{St}(R) \to E$ of φ. The construction is as follows: define elements $s_{ij}(r) \in E$ with $i \neq j$ and $r \in \mathbb{R}$ by

$$s_{ij}(r) = [\varphi^{-1}(x_{ik}(1)), \varphi^{-1}(x_{kj}(r))],$$

for some k distinct from i and j. Notice that there are several choices made here, but $s_{ij}(r)$ is well-defined nonetheless. Let $s(x_{ij}(r)) = s_{ij}(r)$. Following [Ros94, 4.2.6, 4.2.7, 4.2.8, 4.2.9], one can see that s is well-defined and defines a section for φ. □

Proposition II.3.12. *Let R be a ring. Then $K_2(R) \cong H_2(E(R), \mathbb{Z})$.*

Proof. Follows directly from Corollary I.4.22 and Proposition II.3.11. □

Proposition II.3.13. *The mapping $K_2 : \mathbf{Ring} \to \mathbf{Ab}$ is a functor.*

Proof. It is clear that $\mathrm{St}(-)$ and $GL(-)$ are both functorial. The claim follows from the exact sequence of groups

$$0 \longrightarrow K_2(R) \xhookrightarrow{\text{inc}} \mathrm{St}(R) \xrightarrow{\varphi} GL(R) \xrightarrow{q} K_1(R) \longrightarrow 0$$

discussed previously. □

Proposition II.3.14. *Let $R = R_1 \times \cdots \times R_n$ be a product of rings. Then $K_2(R_1 \times \cdots \times R_n) \cong K_2(R_1) \oplus \cdots \oplus K_2(R_n)$.*

Proof. See [LP05, V.3.5] □

Computing examples of the Milnor group of a ring can become more complicated and require a fair amount of work, even in specific cases. For reference, the reader may consult [Mil71, §10], where it is shown that $K_2(\mathbb{Z}) = \mathbb{Z}/2$.

II.3.3 The Relative K_2-Group

Defining the relative K_2-groups in a similar way to the relative K_0 and K_1-groups proves not to be adequate to extend the exact sequence (2) for a functor K_3, see [Swa71]. Instead, the definition depends on the Steinberg group $\mathrm{St}(R \times_{R/I} R)$, which is used in place of the group $K_2(R \times_{R/I} R)$ to produce the relative K_2-groups.

Recall the fibered product diagram:

$$\begin{tikzcd} R \times_{R/I} R \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, two heads, "q"] \\ R \arrow[r, two heads, "q"'] & R/I. \end{tikzcd}$$

Definition II.3.15 (Relative Steinberg Group)**.** Let R be a ring and I be a two-sided ideal of R. Let

$$\mathrm{St}'(R,I) = \ker\Big(\mathrm{St}(R \times_{R/I} R) \xrightarrow{\mathrm{St}(p_1)} \mathrm{St}(R)\Big).$$

The *relative Steinberg group*, denoted by $\mathrm{St}(R,I)$ is defined as the quotient of $\mathrm{St}'(R,I)$ modulo the normal subgroup generated by expressions of the form $[x_{ij}(0,a), x_{k\ell}(b,0)]$ with $a,b \in I$.

Observe that the homomorphism $\mathrm{St}(p_2)|_{\mathrm{St}'(R,I)} : \mathrm{St}'(R,I) \to \mathrm{St}(R)$ induces the homomorphism (which by abuse of notation will be denoted by $\mathrm{St}(p)$) $\mathrm{St}(p) : \mathrm{St}(R,I) \to \mathrm{St}(R)$ since

$\mathrm{St}(p)([x_{ij}(0,a), x_{k\ell}(b,0)]) = x_{ij}(a)x_{k\ell}(0)x_{ij}(a)^{-1}x_{k\ell}(0)^{-1} = 1.$

$$\begin{tikzcd}
\mathrm{St}(R) & & \\
\mathrm{St}(R \times_{R/I} R) \arrow[u, "\mathrm{St}(p_1)"'] & & \\
\mathrm{St}'(R,I) \arrow[u, hook] \arrow[uu, bend left=60, "\mathrm{St}(p_2)"] \arrow[r, two heads, "\varphi_{R\times_{R/I}R}"] \arrow[d] & E(0\times R, 0\times I) \arrow[r, "\cong"] & E(R,I) \\
\mathrm{St}(R,I). \arrow[uuu, dashed, bend left=80, "\exists!\,\mathrm{St}(p)"] \arrow[urr, dashed, two heads, "\exists!\,\psi"'] & &
\end{tikzcd}$$

Additionally, since $R \times_{R/I} R \cong R \times I$, it follows that $\varphi_{R\times_{R/I}R}(\mathrm{St}'(R,I)) \cong E(0\times R, 0\times I) \cong E(R,I)$ because of how $\mathrm{St}'(R,I)$ was defined, and $\varphi_{R\times_{R/I}R}([x_{ij}(0,a), x_{k\ell}(b,0)]) = 1$ for all possible choices of i, j, k and ℓ, there exists a homomorphism induced by $\varphi_{R\times_{R/I}R}$ from $\mathrm{St}(R,I)$ to $GL(R)$ with image $E(R,I)$.

Lemma II.3.16. *Let R be a ring and I be a two-sided ideal of R. Then the following sequence is exact:*

$$\mathrm{St}(R,I) \xrightarrow{\mathrm{St}(p)} \mathrm{St}(R) \xrightarrow{\mathrm{St}(q)} \mathrm{St}(R/I) \longrightarrow 1.$$

Proof. Note that $\mathrm{St}(R,I)$ is generated by the classes of elements of the form $sx_{ij}(0,a)s^{-1}$, where $s \in \mathrm{St}(R \times_{R/I} R)$ and $a \in I$. Thus, $\mathrm{St}(p)([sx_{ij}(0,a)s^{-1}])$ is of the form $tx_{ij}(a)t^{-1}$ for some $t \in \mathrm{St}(R)$. This is the form of an arbitrary element in $\ker(\mathrm{St}(q))$, implying that $\mathrm{St}(p_2)(\mathrm{St}'(R,I)) = \ker(\mathrm{St}(q))$. On the other hand, $\mathrm{St}(q)$ is surjective due to q being surjective and the definition of $\mathrm{St}(-)$. $\square$

Definition II.3.17 (Relative K_2-group)**.** Let R be a ring and I be a two-sided ideal of R. The *relative K_2-group* of R with respect to I, denoted by $K_2(R,I)$, is defined as

$$K_2(R,I) = \ker\Big(\mathrm{St}(R,I) \xrightarrow{\psi} E(R,I)\Big).$$

Theorem II.3.18. *Let R be a ring and I be a two-sided ideal of R. Then there exists a natural* *exact sequence*

$$\begin{tikzcd}
K_2(R,I) \arrow[r, "K_2(p)"] & K_2(R) \arrow[r, "K_2(q)"] & K_2(R/I) \arrow[dll, "\delta_2"] \\
K_1(R,I) \arrow[r, "K_1(p)"] & K_1(R) \arrow[r, "K_1(q)"] & K_1(R/I) \arrow[dll, "\delta_1"] \\
K_0(R,I) \arrow[r, "K_0(p)"] & K_0(R) \arrow[r, "K_0(q)"] & K_0(R/I)
\end{tikzcd} \tag{3}$$

where $p = p_2|_{\ker(p_1)}$, $q : R \to R/I$ *is the quotient homomorphism,* δ_1 *is the map* δ *defined in Theorem II.2.17, and* δ_2 *is the map induced by the following diagram:*

$$\begin{tikzcd}
 & & K_2(R/I) \arrow[d, hook] \arrow[d, bend right, blue] \\
E(R,I) \arrow[d, hook] & \mathrm{St}(R) \arrow[r, two heads, "\mathrm{St}(q)"] \arrow[d, "\varphi_R"] \arrow[d, bend right, blue] & \mathrm{St}(R/I) \arrow[l, bend left, blue] \\
GL(I) \arrow[r, hook] \arrow[d, two heads, "\pi"] \arrow[d, bend right, blue] & GL(R) \arrow[l, bend left, blue] & \\
K_1(R,I). & &
\end{tikzcd}$$

where $\pi : GL(I) \twoheadrightarrow K_1(R,I) \cong GL(I)/E(R,I)$ *denotes the corresponding quotient map.*

Proof. By Theorem II.2.17, it suffices to show that δ_2 is well-defined, and exactness at $K_2(R)$, $K_2(R/I)$ and $K_1(R,I)$. Observe that the following diagram has exact rows and columns:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
 & K_2(R,I) \arrow[r, "K_2(p)"] \arrow[d, hook] & K_2(R) \arrow[r, "K_2(q)"] \arrow[d, hook] & K_2(R/I) \arrow[d, hook] & \\
 & \mathrm{St}(R,I) \arrow[r, "\mathrm{St}(p)"] \arrow[d, "\psi"] & \mathrm{St}(R) \arrow[r, two heads, "\mathrm{St}(q)"] \arrow[d, "\varphi_R"] & \mathrm{St}(R/I) \arrow[r] \arrow[d, "\varphi_{R/I}"] & 1 \\
1 \arrow[r] & GL(I) \arrow[r, hook] \arrow[d, two heads] & GL(R) \arrow[r] \arrow[d, two heads] & GL(R/I) \arrow[d, two heads] & \\
 & K_1(R,I) \arrow[r, "K_1(p)"] \arrow[d] & K_1(R) \arrow[r, "K_1(q)"] \arrow[d] & K_1(R/I) \arrow[d] & \\
 & 1 & 1 & 1. &
\end{tikzcd}$$

The claim follows by applying the Snake Lemma to the middle rows. □

Notice how the exact sequence (3) expanded on the sequence (2), which in turn expanded the sequence (1). The same idea of extending the Mayer-Vietoris sequence of Proposition II.2.15 cannot be done in full generality, but the constrained case still proves to be useful:

Proposition II.3.19 (The Mayer-Vietoris Sequence). *Let $f : R \twoheadrightarrow S$ be a surjective ring homomorphism. Let I be a two-sided ideal of both R and S. Then*

$$\begin{tikzcd}
K_2(R) \arrow[r, "\Delta_2"] & K_2(S) \oplus K_2(R/I) \arrow[r, "\nabla_2"] & K_2(S/I) \arrow[dll, "\delta_2"'] \\
K_1(R) \arrow[r, "\Delta_1"] & K_1(S) \oplus K_1(R/I) \arrow[r, "\nabla_1"] & K_1(S/I) \arrow[dll, "\delta_1"'] \\
K_0(R) \arrow[r, "\Delta_0"] & K_0(S) \oplus K_0(R/I) \arrow[r, "\nabla_0"] & K_0(S/I)
\end{tikzcd}$$

is an exact sequence, where

$$\Delta_n = \begin{pmatrix} K_n(f) \\ K_n(q_R) \end{pmatrix} \quad \text{and} \quad \nabla_n = \begin{pmatrix} K_n(q_S) & -K_n(\overline{f}) \end{pmatrix}$$

for $n \in \{1, 2\}$, δ_1 is the map δ defined in Proposition II.2.15 and δ_2 is the map

$$K_2(S/I) \xrightarrow{\delta_2} K_1(S, I) \xrightarrow{\cong} K_1(R, I) \xrightarrow{K_1(p)} K_1(R).$$

Proof. First, notice that $K_1(R, I) \cong K_1(S, I)$ by Proposition II.2.23. By definition of the relative Steinberg groups, it is clear that $\mathrm{St}(R, I)$ maps onto $\mathrm{St}(S, I)$, and since $E(R, I) \cong E(S, I)$, it follows that $K_2(R, I)$ maps onto $K_2(S, I)$. Hence, the following diagram is commutative with exact rows:

$$\begin{tikzcd}
K_2(R, I) \arrow[r] \arrow[d, two heads] & K_2(R) \arrow[r] \arrow[d] & K_2(R/I) \arrow[r, "\delta_2"] \arrow[d] & K_1(R, I) \arrow[r] \arrow[d, "\cong"] & K_1(R) \arrow[r] \arrow[d] & K_1(R/I) \arrow[d] \\
K_2(S, I) \arrow[r] & K_2(S) \arrow[r] & K_2(S/I) \arrow[r, "\delta_2"] & K_1(S, I) \arrow[r] & K_1(S) \arrow[r] & K_1(S/I).
\end{tikzcd}$$

Chasing the diagram yields an exact sequence of the form

$$K_2(R) \longrightarrow K_2(R/I) \oplus K_2(S) \longrightarrow K_2(S/I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \oplus K_1(S).$$

Observing that R is isomorphic to the fibered product $R/I \times_{S/I} S$ gives the remaining part of the sequence. $\square$

Chapter III

Higher K-Theory

This chapter discusses some key constructions in algebraic topology that provide a deep understanding of the algebraic and topological structures associated with spaces and rings.

It begins with the construction of the classifying space of a topological group and explores some of its properties. Next, the plus construction is introduced, which is a topological technique used to modify spaces in order to eliminate perfect normal subgroups from their fundamental groups. Finally, the higher K-groups of Quillen are mentioned; an extension of classical algebraic K-theory that allows for the encoding of algebraic invariants of rings in terms of homotopy groups of spaces specifically built for this purpose.

III.1 The Classifying Space of a Group

The purpose of the classifying space is to construct a topological space from a group in such a way that its homology groups with local coefficients, viewed as topological invariants, are isomorphic to the homology groups of the original group, which are algebraic invariants.

In Appendix A.2, universal G-bundles are defined as principal G-bundles satisfying a certain bijective correspondence. In this section, a different characterization for universal G-bundles will be given so that they can be computed more easily, as well as an explicit construction for them.

Definition III.1.1 (Topological Group). A topological space G endowed with a group structure is a *topological group* if the group operation

$$\begin{aligned} \cdot : G \times G &\to G \\ (g, h) &\mapsto gh, \end{aligned}$$

and the inversion map

$$\begin{aligned} (-)^{-1} : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are continuous.

Example III.1.2. The following are examples of topological groups:

(a) A group G with the discrete topology.

(b) The real numbers $\mathbb{R}$ with the usual addition and the Euclidean topology.

(c) The general linear group $GL(\mathbb{R})$, with the topology induced by the Euclidean metric.

Definition III.1.3 (Classifying space)**.** Given a topological group G, its *classifying space*, denoted by BG, is the base space of the universal G-bundle $\xi_G = (EG, BG, p_G)$.

Remark III.1.4. Note that classifying spaces are only defined up to homotopy equivalence.

Definition III.1.5 (G-CW-complex)**.** Let G be a topological group. A space X is called a *G-CW-complex* if it is the colimit of G-spaces X^n, with inclusions $i_n : X^n \to X^{n+1}$, such that X^0 is the disjoint union of orbits G/H_α with H_α a closed subgroup of G, and X^{n+1} is obtained from X^n by attaching G-equivariant $(n+1)$-cells $G/H_\alpha \times D^{n+1}$ via G-maps, as described in the following fibered coproduct diagram:

$$\begin{tikzcd} \coprod_\alpha G/H_\alpha \times S^n \arrow[r] \arrow[d, hook] & X^n \arrow[d] \\ \coprod_\alpha G/H_\alpha \times D^{n+1} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X^{n+1}, \end{tikzcd}$$

where G acts trivially on S^n and D^{n+1}.

Proposition III.1.6. *Let X be a G-CW-complex. Then the orbit space X/G is a CW-complex.*

Proof. By taking the quotient space of the 0-skeleton X^0, it follows that X^0/G is a discrete space. Observe that the quotient of the $(n+1)$-skeleton is given by the fibered coproduct

$$\begin{tikzcd}
\coprod_\alpha S^n \arrow[r] \arrow[d, hook] & X^n/G \arrow[d] \\
\coprod_\alpha D^{n+1} \arrow[r] & X^{n+1}/G, \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

so $\varinjlim(X^n/G)$ is a CW-complex. Finally, notice that the functor from the category of topological G-spaces (**GTop**) to **Top** that maps a G-space to its orbit space is left adjoint to the functor F from **Top** to **GTop** that considers a topological space as a G-space with trivial action, i.e.,

$$\mathbf{Top}(X/G, Y) \cong \mathbf{GTop}(X, FY),$$

due to the universal property of the quotient. Therefore

$$\varinjlim(X^n/G) = (\varinjlim X^n)/G = X/G.$$

Thus, the orbit space is a CW-complex. □

Remark III.1.7. In a G-CW-complex with free G-action, i.e. $x \cdot g = x$ implies that g is the identity of G, all disk orbits are of the form $D^n \times G$.

Theorem III.1.8. *A principal G-bundle $\xi_G = (EG, BG, p_G)$ is universal if and only if EG is contractible.*

The proof of the theorem will only be done for the case when the spaces of principal G-bundles are both CW-complexes, and the total space has the structure of a G-CW-complex. An alternative proof for the general statement can be found in [tD08, 14.4.12].

Proof. Suppose that EG is contractible. Let $\xi = (E, B, p)$ be a principal G-bundle as described in Appendix A.2. A map $f : B \to BG$ will be constructed such that $\xi \cong f^*(\xi_G)$.

First, it will be proven by induction over the skeleta of E that there exists a G-equivariant map $h : E \to EG$ that maps each orbit eG homeomorphically onto its image $h(e)G$. Recall that $E^{-1} = \emptyset$, so a map $h^{-1} : E^{-1} \to EG$ with such characteristics exists trivially. Suppose given a map $h^n : E^n \to EG$ satisfying the induction hypothesis.

Let $D^{n+1} \times G$ be a disk orbit in the G-CW-complex structure of E^{n+1}. Consider the disk $D^{n+1} \times \{1\} \subseteq D^{n+1} \times G$. Then the map h^n extends to $D^{n+1} \times \{1\}$ if and only if the composition

$$S^n \times \{1\} \to E^n \to EG$$

is null homotopic, which always holds due to EG being contractible. Extending h^n to all the $(n+1)$-cells of the orbit $D^{n+1} \times G$, and then to the rest of $(n+1)$-cell orbits of E, it follows that h^n can be extended to a G-equivariant $h^{n+1} : E^{n+1} \to EG$ with the required property. Having completed the inductive step, one now has a G-equivariant map $h : E \to EG$ that is a homeomorphism on the orbits. Hence, it induces a map on the orbit space $f : E/G \cong B \to BG$ making the following diagram commute:

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "p"'] & EG \arrow[d, "p_G"] \\ B \arrow[r, "f"'] & BG. \end{tikzcd}$$

Since h induces a homeomorphism on each orbit by construction, the maps h and f determine an isomorphism $\xi \cong f^*(\xi_G)$.

Now, suppose that two principal G-bundles (E_1, X, p_1) and (E_2, X, p_2) are isomorphic. By the previous part, there exist maps f_1 and f_2 from X to BG such that $f_1^*(\xi_G) = (E_1, X, p_1)$ and $f_2^*(\xi_G) = (E_2, X, p_2)$, and a homomorphism of total spaces $\varphi : f_1^*(\xi_G) \to f_2^*(\xi_2)$. It suffices to prove that f_1 and f_2 are homotopic.

By the Cellular Approximation Theorem, f_1 and f_2 can be assumed to be cellular. This and the fact that EG is assumed to be a G-CW-complex induces a G-CW-complex in the total spaces $f_1^*(\xi_G)$ and $f_2^*(\xi_2)$. Define a principal G-bundle $(E, X \times [0,1], p)$ by setting

$$E = f_1^*(\xi_G) \times [0, 1/2] \cup_\varphi f_2^*(\xi_2) \times [1/2, 1]$$

and

$$p(e, x) = \begin{cases} (p_1(e), x) & \text{if } 0 \leq x \leq 1/2, \\ (p_2(e), x) & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

By the same argument as before, there exists a G-equivariant map $H : E \to EG$ that induces a homomorphism on each orbit, and extends the maps $f_1^*(\xi_G) \times \{0\} \to EG$ and $f_2^*(\xi_G) \times \{1\} \to EG$ obtained from each fibered product. This induces a map $F : E/G \cong X \times [0,1] \to BG$ that is a homotopy between f_1 and f_2.

Hence, if EG is contractible it verifies the definition of a universal G-bundle.

The reciprocal follows by constructing a principal G-bundle with total space contractible for any topological group G (done in Theorem III.1.11) and Proposition A.2.10. □

At this moment, the existence of classifying spaces is not yet guaranteed. There are several constructions for them, such as Milnor's infinite join construction or the bar construction. This work discusses the former.

III.1.1 Milnor's Construction of the Classifying Space

Definition III.1.9 (Join of spaces)**.** For a finite collection of spaces $\{X_i\}$, the *join of* $\{X_i\}$, denoted by $X_1 \star X_2 \star \cdots \star X_n$, is defined as the space with elements of the form

$$(t_1 x_1, t_2 x_2, \ldots, t_n x_n)$$

with

(*i*) $t_1, \ldots, t_n$ non-negative real numbers satisfying $t_1 + \cdots + t_n = 1$;

(*ii*) $x_i \in X_i$ for all i such that $t_i \neq 0$. In case $t_i = 0$, the element x_i is omitted, and topology given by the coarsest topology such that the coordinate functions

$$t_i : X_1 \star X_2 \star \cdots \star X_n \to [0,1] \qquad \text{and} \qquad x_i : t_i^{-1}(0,1] \to X_i$$

are continuous. The join of (countably many) infinitely many spaces is defined in the same manner, with the restriction that all but finitely many t_i are equal to zero. In fact, the *infinite join of spaces* can be more formally defined as the direct limit

$$X_1 \star X_2 \star \cdots = \varinjlim (X_1 \star \cdots \star X_n),$$

by identifying $X_1 \star \cdots \star X_n$ in $X_1 \star \cdots \star X_{n+1}$ via

$$(t_1x_1, t_2x_2, \ldots, t_nx_n) \mapsto (t_1x_1, t_2x_2, \ldots, t_nx_n, 0).$$

Example III.1.10.

(a) The join $X \star \{*\}$ is homeomorphic to the cone $CX = (X \times I)/(X \times \{1\})$.

(b) Let $Y = \{y_1, y_2\}$ be a discrete space. Then the join $X \star Y$ is homeomorphic to the suspension $\Sigma X = (CX)/(X \times \{0\})$

(c) The join of two arbitrary spaces X and Y can be visualized as follows:

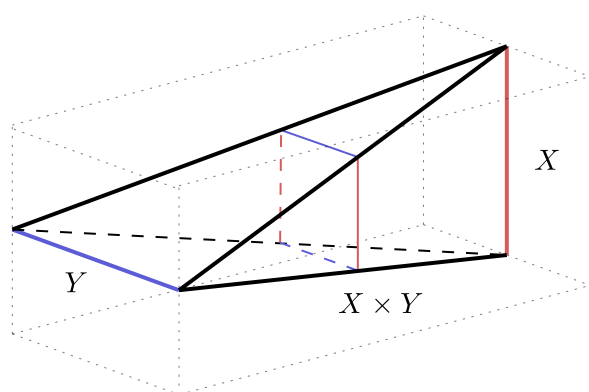

Figure III.1. The join $X \star Y$.

From this, now it can be seen that $X \star Y \cong (CX \times Y) \cup_{X \times Y} (X \times CY)$.

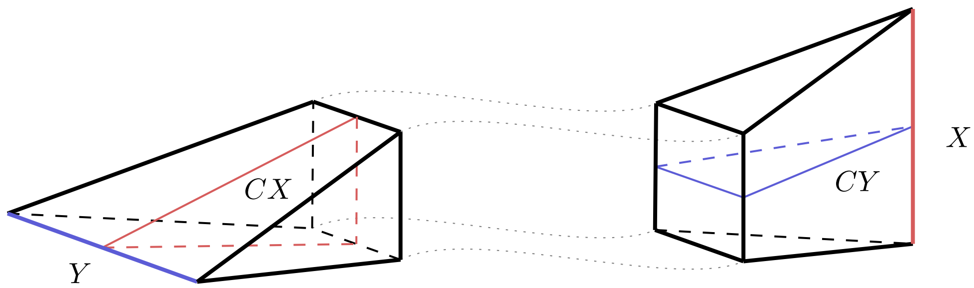

Figure III.2. The join $X \star Y$ pictured as the fibered coproduct $(CX \times Y) \cup_{X \times Y} (X \times CY)$.

(d) The join of spheres $S^m \star S^n$ is homeomorphic to S^{m+n+1}. To see this, observe that

$$\begin{aligned} S^m \star S^n &\cong (CS^m \times S^n) \cup_{S^m \times S^n} (S^m \times CS^n) \\ &\cong (D^{m+1} \times S^n) \cup_{S^m \times S^n} (S^m \times D^{n+1}) \\ &\cong (D^{m+1} \times \partial D^{n+1}) \cup_\partial (\partial D^{m+1} \times D^{n+1}) \\ &\cong \partial(D^{m+1} \times D^{n+1}) \\ &\cong S^{m+n+1}. \end{aligned}$$

(e) The join of CW-complexes is again a CW-complex. This follows from the fact that cones and products of CW-complexes are again CW-complexes (assuming the spaces are compactly generated), and the previous examples.

If G is a topological group, the join $G \star \cdots \star G$ can be endowed with a right G-action by defining

$$(t_1 g_1, t_2 g_2, \ldots, t_n g_n) \cdot g = (t_1 g_1 g, t_2 g_2 g, \ldots, t_n g_n g).$$

Notice that the join $G \star \cdots \star G$ with G a topological group is a G-CW-complex with free action, i.e., for all cells of the CW-complex, the orbits G/H_α are all equal to G. By considering the quotient space $(G \star \cdots \star G)/G$ given by identifying

$$(t_1 g_1, t_2 g_2, \ldots, t_n g_n) \sim (t_1 g_1, t_2 g_2, \ldots, t_n g_n) \cdot g$$

for all $g \in G$, one gets a principal G-bundle $\big(G \star \cdots \star G, (G \star \cdots \star G)/G, p\big)$ where both the total space and the base space are CW-complexes. The same occurs for an infinite join.

Theorem III.1.11. *Let G be a topological group with the structure of a CW-complex. Then the universal G-bundle $\xi_G = (EG, BG, p_G)$ exists, where*

$$EG = G \star G \star \cdots \qquad \text{and} \qquad BG = EG/G,$$

with both EG and BG being CW-complexes.

Proof. Set $EG = G \star G \star \cdots$ and $BG = EG/G$. By the previous discussion, both EG and BG are CW-complexes, so by Theorem III.1.8 it suffices to show that EG is contractible.

Let $\gamma : S^n \to EG$ be a map. Since S^n is compact, $\gamma(S^n)$ is compact, so there exists $N \geq 0$ such that $\gamma(S^n) \subseteq \underbrace{G \star \cdots \star G}_{N \text{ times}} \subseteq EG$ due to EG being a CW-complex. Denote $\underbrace{G \star \cdots \star G}_{N \text{ times}}$ by G_N, and take an element $g \in G$. Notice that the cone CG_N is embedded in $E_G \star G \subseteq EG$, so the map γ can be factored as $\gamma : S^n \to G_N \to G_N \star \{g\} \cong CG_N \hookrightarrow EG$, where $CG_N \hookrightarrow EG$ is null-homotopic. Thus γ is null-homotopic.

Therefore, $\pi_n(EG) = 0$ for all non-negative n, making EG weakly contractible. By the Whitehead Theorem, it follows that EG is contractible. $\square$

The previous theorem only gives information for the case when the topological group G is a CW-complex, which is enough for the purposes of this work. For a general statement, see [Mil56].

Example III.1.12. The following are examples of classifying spaces:

(a) If $G = \mathbb{Z}$ with usual addition endowed with the discrete topology, then the universal covering map of the circle, $p : \mathbb{R} \to S^1$, is a principal $\mathbb{Z}$-bundle with total space contractible, so $B\mathbb{Z} \simeq S^1$.

(b) If $G = C_2$, the cyclic group of order 2, endowed with the discrete topology, then the construction given for $E(C_2)$ corresponds to the infinite join of 0-dimensional spheres, this because $S^0 = \{-1, 1\} \cong C_2$. Since C_2 acts freely in S^n by the antipodal map, and the inclusions $S^n \subseteq S^{n+1}$ are equivariant with respect to this action, then there is an induced action of C_2 over $E(C_2) \simeq S^\infty$ by the antipodal map. Hence

$$B(C_2) \simeq S^\infty/\text{antipodal points} \cong \mathbb{RP}^\infty.$$

(c) If $G = S^1$ endowed with the Euclidean topology, then, similarly to the previous example, $ES^1 \simeq S^\infty$ and

$$BS^1 \simeq S^\infty / S^1 \cong \mathbb{CP}^\infty.$$

III.1.2 Properties of the Classifying Space

Proposition III.1.13. *Let G be a topological group. Then $\pi_n(BG) \cong \pi_{n-1}(G)$ for all $n \geq 1$.*

Proof. By Proposition A.3.3, the universal G-bundle ξ_G is a Serre fibration, and by Theorem A.3.6, there exists a long exact sequence of the form

$$\cdots \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0).$$

Since EG is contractible, the previous long exact sequence reduces to the fact that $\pi_n(BG) \cong \pi_{n-1}(G)$. $\square$

Proposition III.1.14. *The mapping $B : \mathbf{Grp} \to \mathrm{Ho}(\mathbf{CW})$ that associates a group, seen as a discrete topological group, with its classifying space is a functor.*

Proof. Let $f : G \to H$ be a group homomorphism. First, using the model for the total space given in Theorem III.1.11, notice that f induces a map $f_* : EG \to EH$ by mapping

$$(t_1 g_1, t_2 g_2, \dots) \mapsto (t_1 f(g_1), t_2 f(g_2), \dots).$$

Since f_* satisfies $f_*(x \cdot g) = f_*(x) \cdot f(g)$, it induces a map Bf from $EG/G \simeq BG$ to $EH/H \simeq BH$. The functoriality conditions follow from the construction of Bf and the universal property of a quotient space.

One may even consider this functor to be defined from $\mathbf{Grp}$ to $\mathrm{Ho}(\mathbf{CW})_*$ by picking as base point the element belonging to the equivalence class of the identity 1_G from the first entry of the infinite join $G \star G \star \cdots$. By abuse of notation, denote such a point by 1_G. $\square$

Theorem III.1.15. *Let G be a group and BG be the classifying space of G seen as a topological group with the discrete topology. Then for any G-module M, there is a natural isomorphism between group homology $H_\bullet(G, M)$ and homology with local coefficients $H_\bullet(BG; M)$.*

Proof. Using the CW-model for BG of Theorem III.1.11, it follows that EG is the universal covering space of BG due to the universal bundle being a principal G-bundle with EG contractible. Furthermore, $\pi_1(BG) \cong \pi_0(G) = G$. Thus, the cellular chain complex $C_\bullet(BG; M)$ is isomorphic to the chain complex $C_\bullet(EG) \otimes_G M$. Since EG is contractible, the chain $C_\bullet(EG)$ is acyclic except in dimension 0, where $C_0(EG) \cong \mathbb{Z}$. Therefore, $C_\bullet(EG)$ is a projective resolution of $\mathbb{Z}$ by free G-modules.

Observe that $H_\bullet(G, M)$ is computed from a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules tensored by M, which is exactly how $C_\bullet(BG; M)$ is defined. □

Example III.1.16. By applying Theorem III.1.15, the homology groups of the infinite real projective space $\mathbb{RP}^\infty$ can be computed as the homology groups of C_2 with integer coefficients, which was already done in Example I.4.9.

III.2 The Plus Construction

Given a path-connected space (X, x_0) with the homotopy type of a CW-complex, a commonly known construction due to Quillen, often referred to as the *plus construction* and denoted by $i : (X, x_0) \to (X^+, x_0)$, allows a space to be modified in such a way as to eliminate or simplify its fundamental group, without altering its homology or cohomology. It operates by killing a perfect subgroup of the fundamental group $\pi_1(X, x_0)$ typically with the goal of abelianizing the fundamental group $\pi_1(X^+, x_0)$ while preserving the space's homological properties.

The Plus Construction. Suppose given a pair $((X, x_0), N)$ consisting of a path-connected space (X, x_0) with the homotopy type of a CW-complex, and a perfect normal subgroup N of $\pi_1(X, x_0)$.

Choose a minimal set of generators $x_\alpha \in N$ running over the index set $\mathcal{A}$. For each of these classes, choose a representing loop γ_α and construct the space X_1 as the fibered coproduct of the following diagram:

$$\begin{tikzcd}
\coprod_{\alpha\in\mathcal{A}} S^1 \arrow[r, "\coprod_{\alpha\in\mathcal{A}}\gamma_\alpha"] \arrow[d] & X \arrow[d] \\
\coprod_{\alpha\in\mathcal{A}} D^2 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X_1.
\end{tikzcd}$$

By the Seifert-Van Kampen Theorem, the fundamental group $\pi_1(X_1, x_0)$ is isomorphic to the quotient of $\pi_1(X, x_0)$ by the normal subgroup generated by the classes x_α, i.e., it is isomorphic to $\pi_1(X, x_0)/N$.

Consider the universal covering space $\widetilde{X_1} \xrightarrow{p_1} X_1$. Construct the covering space $\widehat{X}$ of X as the fibered product of the following diagram:

$$\begin{tikzcd}
\widehat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X_1} \arrow[d, "p_1"] \\
X \arrow[r] & X_1.
\end{tikzcd}$$

Observe that $\widehat{X}$ is obtained from $\widetilde{X_1}$ by removing the inverse image of the 2-cells attached to X to produce X_1, and since $\widetilde{X_1}$ is connected, then $\widehat{X}$ is also connected. Notice that $\widehat{X} \to X$ is constructed as the fibered product of a principal $\pi_1(X_1, x_0)$-bundle, thus $\widehat{X} \to X$ is a principal $\pi_1(X, x_0)/N$ bundle. From this and the observation that $\widehat{X}$ is constructed from a simply connected space by removing specific 2-cells, it follows that $\pi_1(\widehat{X}, \widehat{x}_0) = N$.

For each 2-cell e^2_α of the relative CW-complex (X_1, X), the fundamental group $\pi_1(X, x_0)$ acts transitively on the 2-cells $p_1^{-1}(e^2_\alpha)$ and has isotropy group N. Hence, $H_2(\widetilde{X_1}, \widehat{X}; \mathbb{Z}) \cong C_2(\widetilde{X_1}, \widehat{X}; \mathbb{Z})$ is a free $(\pi_1(X, x_0)/N)$-module on generators $\left[\widetilde{e^2_\alpha}\right]$, running over all the 2-cells $\widetilde{e^2_\alpha}$ in $p_1^{-1}(e^2_\alpha)$ for all $\alpha \in \mathcal{A}$.

Consider the following diagram consisting of the long exact homotopy sequence and the long exact sequence in homology of the pair $(\widetilde{X_1}, \widehat{X})$, related by Hurewicz maps:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2(\widehat{X}, \hat{x}_0) \arrow[r] \arrow[d] & \pi_2(\widetilde{X_1}, \tilde{x}_0) \arrow[r] \arrow[d] & \pi_2(\widetilde{X_1}, \widehat{X}, \tilde{x}_0) \arrow[r] \arrow[d] & \pi_1(\widehat{X}, \hat{x}_0) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_2(\widehat{X}; \mathbb{Z}) \arrow[r] & H_2(\widetilde{X_1}; \mathbb{Z}) \arrow[r, "j"] & H_2(\widetilde{X_1}, \widehat{X}; \mathbb{Z}) \arrow[r] & H_1(\widehat{X}; \mathbb{Z}) \arrow[r] & \cdots .
\end{tikzcd}$$

Notice that

- $H_1(\widehat{X}; \mathbb{Z}) \cong \pi_1(\widehat{X}, x_0)^{\mathrm{ab}} \cong N^{\mathrm{ab}} \cong 0$ since N is perfect.
- For being a universal covering space, $\pi_1(\widetilde{X_1}, x_0) = 0$, so by the Hurewicz Theorem $\pi_2(\widetilde{X_1}) \cong H_2(\widetilde{X_1}; \mathbb{Z})$.
- By exactness of the bottom row and the fact that $H_1(\widehat{X}; \mathbb{Z}) = 0$, the map $H_2(\widetilde{X_1}; \mathbb{Z}) \to H_2(\widetilde{X_1}, \widehat{X}; \mathbb{Z})$ is surjective. Thus, the composite $\pi_2(\widetilde{X_1}, x_0) \to H_2(\widetilde{X_1}; \mathbb{Z}) \to H_2(\widetilde{X_1}, \widehat{X}; \mathbb{Z})$ is surjective. Therefore, for any 2-cell $\widetilde{e^2_\alpha} \in p_1^{-1}(e^2_\alpha)$ in $(\widetilde{X_1}, \widehat{X})$, there exists a map $\widetilde{\delta_\alpha} : S^2 \to \widetilde{X_1}$ such that the class $\left[\widetilde{\delta_\alpha}\right]$ in $\pi_2(\widetilde{X_1}, x_0)$ maps to $\left[\widetilde{e^2_\alpha}\right]$ in $H_2(\widetilde{X_1}, \widehat{X}, \mathbb{Z})$ along this composite map.

Let $\delta_\alpha = p_1 \circ \widetilde{\delta_\alpha}$. Construct the space X^+ as the fibered coproduct of the following diagram:

$$\begin{tikzcd}
\coprod\limits_{\alpha \in \mathcal{A}} S^2 \arrow[r, "\coprod_{\alpha \in \mathcal{A}} \delta_\alpha"] \arrow[d] & X_1 \arrow[d] \\
\coprod\limits_{\alpha \in \mathcal{A}} D^3 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X^+ .
\end{tikzcd}$$

The space X^+ is called the plus construction of X, where X is a subspace of X^+ via the composite inclusion map $i : X \to X_1 \to X^+$.

III.2.1 Properties of the Plus Construction

Proposition III.2.1. *Let (X, x_0) be a path-connected space with the homotopy type of a CW-complex, and N be a perfect normal subgroup of $\pi_1(X, x_0)$. Let $i : X \to X^+$ be the plus construction of the pair $\big((X, x_0), N\big)$. Then the following properties hold:*

(a) *The sequence*

$$0 \longrightarrow N \longrightarrow \pi_1(X, x_0) \xrightarrow{i_*} \pi_1(X^+, x_0) \longrightarrow 0$$

is exact.

(b) *For any $(\pi_1(X, x_0)/N)$-module viewed as a local coefficient system on X^+, the homology groups $H_n(X, i^*M)$ and $H_n(X^+, M)$ are isomorphic.*

Proof.

(a) The construction of $i : X \to X^+$ implies that the relative CW-complex (X^+, X_1) only has 3-dimensional cells, so $\pi_1(X^+, x_0) \cong \pi_1(X_1, x_0) \cong \pi_1(X, x_0)/N$. The induced map i_* is a surjection since X^+ is obtained from X by attaching cells of dimensions 2 and 3, which makes i_* a quotient map.

(b) let $\widetilde{X^+} \xrightarrow{p^+} X^+$ be the universal covering space of X^+, so that $\widetilde{X^+}$ is obtained from $\widetilde{X_1}$ by attaching 3-cells. This observation makes every square of the following commutative diagram a fibered product:

$$\begin{tikzcd}
\widehat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X_1} \arrow[r] \arrow[d, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X^+} \arrow[d, "p^+"] \\
X \arrow[r] & X_1 \arrow[r] & X^+.
\end{tikzcd}$$

Since the relative CW-complex $(\widetilde{X^+}, \widehat{X})$ only has cells in dimensions 2 and 3, the corresponding relative cellular chain complex has the form

$$\cdots \longrightarrow 0 \longrightarrow C_3(\widetilde{X^+}, \widehat{X}) \xrightarrow{\partial} C_2(\widetilde{X^+}, \widehat{X}) \longrightarrow 0 \longrightarrow \cdots .$$

Furthermore,

$$C_3(\widetilde{X^+}, \widehat{X}) \cong C_3(\widetilde{X^+}, \widetilde{X_1}) \cong H_3(\widetilde{X^+}, \widetilde{X_1}; \mathbb{Z})$$

and

$$C_2(\widetilde{X^+}, \widehat{X}) \cong C_2(\widetilde{X_1}, \widehat{X}) \cong H_2(\widetilde{X_1}, \widehat{X}; \mathbb{Z}).$$

Therefore, the boundary map $C_3(\widetilde{X^+}, \widehat{X}) \xrightarrow{\partial} C_2(\widetilde{X^+}, \widehat{X})$ can be identified with the composite map $j \circ d$, where d is the boundary map $H_3(\widetilde{X^+}, \widetilde{X}_1; \mathbb{Z}) \to H_2(\widetilde{X}_1; \mathbb{Z})$ obtained from the long exact sequence in homology of the pair $(\widetilde{X^+}, \widetilde{X}_1)$:

$$\cdots \longrightarrow H_3(\widetilde{X^+}; \mathbb{Z}) \longrightarrow H_3(\widetilde{X^+}, \widetilde{X}_1; \mathbb{Z}) \xrightarrow{d} H_2(\widetilde{X}_1; \mathbb{Z}) \longrightarrow H_2(\widetilde{X^+}; \mathbb{Z}) \longrightarrow \cdots,$$

and j is the map obtained from the long exact sequence in homology of the pair $(\widetilde{X}_1, \widehat{X})$:

$$\cdots \longrightarrow H_2(\widehat{X}; \mathbb{Z}) \longrightarrow H_2(\widetilde{X}_1; \mathbb{Z}) \xrightarrow{j} H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z}) \longrightarrow H_1(\widehat{X}; \mathbb{Z}) \longrightarrow \cdots.$$

Consider the following diagram consisting of the long exact homotopy sequence and the long exact sequence in homology of the pair $(\widetilde{X^+}, \widetilde{X}_1)$, related by Hurewicz maps:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_3(\widetilde{X^+}, x_0) \arrow[r] \arrow[d] & \pi_3(\widetilde{X^+}, \widetilde{X}_1, x_0) \arrow[r] \arrow[d] & \pi_2(\widetilde{X}_1, x_0) \arrow[r] \arrow[d] & \pi_2(\widetilde{X^+}, x_0) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_3(\widetilde{X^+}; \mathbb{Z}) \arrow[r] & H_3(\widetilde{X^+}, \widetilde{X}_1; \mathbb{Z}) \arrow[r, "d"] & H_2(\widetilde{X}_1; \mathbb{Z}) \arrow[r] & H_2(\widetilde{X^+}; \mathbb{Z}) \arrow[r] & \cdots.
\end{tikzcd}$$

Let $\widetilde{e^3_\alpha}$ be a 3-cell of the relative CW-complex $(\widetilde{X^+}, \widetilde{X}_1)$ with attaching map $\widetilde{\delta_\alpha}$. Note that $\widetilde{e^3_\alpha}$ determines an element $\left[\widetilde{\Delta_\alpha}\right]$ in $\pi_3(\widetilde{X^+}, \widetilde{X}_1)$ by considering the homotopy class of the map obtained by extending $\widetilde{\delta_\alpha}$ to the corresponding disk $\widetilde{D^3_\alpha}$ in $\widetilde{X^+}$ containing the cell $\widetilde{e^3_\alpha}$.

The image of $\left[\widetilde{\Delta_\alpha}\right]$ under the Hurewicz map is the class $\left[\widetilde{e^3_\alpha}\right]$ corresponding to the 3-cell $\widetilde{e^3_\alpha}$. On the other hand, the image of $\left[\widetilde{\Delta_\alpha}\right]$ in the long exact homotopy sequence is the class $\left[\widetilde{\delta_\alpha}\right]$ represented by the attaching map $\widetilde{\delta_\alpha}$. Since $d\Big(\left[\widetilde{e^3_\alpha}\right]\Big) = \left[\widetilde{e^2_\alpha}\right]$, where $\widetilde{e^2_\alpha}$ is the 2-cell in $\widetilde{X}_1$ used to attach $\widetilde{e^3_\alpha}$, the commutativity of the diagram implies that the Hurewicz map sends $\left[\widetilde{\delta_\alpha}\right]$ to $\left[\widetilde{e^2_\alpha}\right]$. Hence, $\left[\widetilde{\delta_\alpha}\right] \in \pi_2(\widetilde{X}_1)$ maps to $\left[\widetilde{e^2_\alpha}\right] \in H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$ under the composite $\pi_2(\widetilde{X}_1, x_0) \to H_2(\widetilde{X}_1; \mathbb{Z}) \xrightarrow{j} H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$.

One readily concludes that $j \circ d : H_3(\widetilde{X^+}, \widetilde{X}_1; \mathbb{Z}) \to H_2(\widetilde{X}_1, \widehat{X}; \mathbb{Z})$ sends $\left[\widetilde{e^3_\alpha}\right]$ to $\left[\widetilde{e^2_\alpha}\right]$. Observe that, similarly as in the construction of X^+, $H_3(\widetilde{X^+}, \widetilde{X}_1; \mathbb{Z})$ is a free $(\pi_1(X, x_0)/N)$-module on generators $\left[\widetilde{e^3_\alpha}\right]$, running over all the 3-cells $\widetilde{e^3_\alpha}$ in $(p^+)^{-1}(e^3_\alpha)$ for all $\alpha \in \mathcal{A}$.

Hence, both $C_3(\widetilde{X^+}, \widehat{X})$ and $C_2(\widetilde{X^+}, \widehat{X})$ are free $(\pi_1(X, x_0)/N)$-modules on generators indexed by the same set. Thus, ∂ is an isomorphism.

Let M be a left $(\pi_1(X, x_0)/N)$-module. The relative homology group with local coefficients $H_n(X^+, X; M)$ is defined as the n-th homology group of the complex $C_\bullet(\widetilde{X^+}, \widehat{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X,x_0)/N]} M$, which is acyclic since the chain complex $C_\bullet(\widetilde{X^+}, \widehat{X}; \mathbb{Z})$ is concentrated in degrees 2 and 3, and these modules are isomorphic. Thus, $H_n(X^+, X; M) = 0$ for all n. By the long exact sequence in homology with local coefficients of the pair (X^+, X), it follows that the homology groups with local coefficients $H_n(X, i^*M)$ and $H_n(X^+, M)$ are isomorphic. $\square$

Theorem III.2.2. *Let (X, x_0) be a path-connected space with the homotopy type of a CW-complex, and N be a perfect normal subgroup of $\pi_1(X, x_0)$. If (Y, y_0) is a space with the homotopy type of a CW-complex equipped with a continuous map $f : X \to Y$ satisfying $N = \ker\left(f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)\right)$, and conditions* (a) *and* (b) *from Proposition III.2.1, then (Y, y_0) is homotopy equivalent to (X^+, x_0), the plus construction of the pair $\left((X, x_0), N\right)$.*

Proof. Start by extending f to the cells added in the construction of X^+ from X. This extension can be done to the 2-cells if and only if, for each attaching map γ_α, the composite map $f \circ \gamma_\alpha$ is null-homotopic. This can be proven by seeing that the homotopy class of the loops γ_α belongs to N and the fact that $\ker(f_*) = N$. Hence, there exists an extension $f_1 : X_1 \to Y$ of f.

The extension to the 3-cells of the map f_1 can be done if and only if, for each attaching map δ_α, the composite map $f_1 \circ \delta_\alpha$ is null-homotopic. Let $\widetilde{Y} \to Y$ be the universal covering space of Y.

Then

$$\begin{aligned}
\pi_2(Y, y_0) &\cong \pi_2(\widetilde{Y}, \tilde{y}_0) \\
&\cong H_2(\widetilde{Y}; \mathbb{Z}) && \text{(Hurewicz Theorem)} \\
&\cong H_2(\widehat{X}; \mathbb{Z}) && \text{(hypothesis)} \\
&\cong H_2(\widetilde{X^+}; \mathbb{Z}) && \text{(Proposition III.2.1)} \\
&\cong \pi_2(\widetilde{X^+}, \tilde{x}_0) \\
&\cong \pi_2(X^+, x_0)
\end{aligned}$$

which implies that for $f_1 \circ \delta_\alpha$ to be null-homotopic if and only if δ_α is homotopic to a constant map in X^+. The latter is true because of the 3-cell e^3_α attached along the map δ_α to construct X^+. Therefore, there exists an extension $g : X^+ \to Y$ of f.

By the commutativity of the diagram of topological spaces on the left, there is an induced commutative diagram of fundamental groups:

$$\begin{tikzcd}
(X, x_0) \arrow[r, "f"] \arrow[d, "i"'] & (Y, y_0) \\
(X^+, x_0) \arrow[ur, "g"']
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, two heads, "f_*"] \arrow[d, two heads, "i_*"'] & \pi_1(Y, y_0) \\
\pi_1(X^+, x_0) \arrow[ur, "g_*"']
\end{tikzcd}$$

Since i_* is a quotient map, by the universal property of the quotient, the map g_* is an isomorphism. Observe that for any $(\pi_1(X, x_0)/N)$-module M, the homology groups $H_n(X; i^*M)$ and $H_n(X; f^*M)$ are isomorphic. Thus g induces an isomorphism of the homology groups $H_n(X^+; M)$ and $H_n(Y; M)$.

Notice that $H_n(X^+; \mathbb{Z}[\pi_1(X^+, x_0)]) \cong H_n(\widetilde{X^+}; \mathbb{Z})$ and $H_n(Y; \mathbb{Z}[\pi_1(Y, y_0)]) \cong H_n(\widetilde{Y}; \mathbb{Z})$ for all n. Consider the commutative diagram of universal covers on the left, which induces a commutative diagram in homology with local coefficients:

$$\begin{tikzcd} \widetilde{X^+} \arrow[r, "\tilde{g}"] \arrow[d] & \widetilde{Y} \arrow[d] \\ X^+ \arrow[r, "g"] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} H_n(\widetilde{X^+};\mathbb{Z}) \arrow[r, "\widetilde{g}_*"] \arrow[d, "\cong"'] & H_n(\widetilde{Y};\mathbb{Z}) \arrow[d, "\cong"] \\ H_n(X^+;\mathbb{Z}[\pi_1(X^+,x_0)]) \arrow[r, "g_*"] & H_n(Y;\mathbb{Z}[\pi_1(Y,y_0)]) \end{tikzcd}$$

Since g_* is an isomorphism, it follows that $\widetilde{g}_*$ is an isomorphism. Now, consider the induced commutative diagram of homotopy:

$$\begin{tikzcd} \pi_n(\widetilde{X^+},\tilde{x}_0) \arrow[r] \arrow[d] & \pi_n(\widetilde{Y},\tilde{y}_0) \arrow[d] \\ \pi_n(X^+,x_0) \arrow[r] & \pi_n(Y,y_0). \end{tikzcd}$$

For $n \geq 2$, the vertical maps are isomorphisms because of the long exact sequence of a fibration. On the other hand, because both X^+ and Y have the homotopy type of a CW-complex, the universal covers $\widetilde{X^+}$ and $\widetilde{Y}$ also have the homotopy type of a CW-complex. By the Homology Whitehead Theorem, $\widetilde{X^+}$ and $\widetilde{Y}$ are homotopy equivalent. Hence, by commutativity of the diagram, the Whitehead Theorem, it follows that X^+ and Y are homotopy equivalent. $\square$

Proposition III.2.3. *The plus construction is functorial in the following sense: Let $f : (X, x_0) \to (Y, y_0)$ be a map of path-connected spaces with both X and Y having the homotopy type of a CW-complex, and N_1 and N_2 be perfect normal subgroups of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ respectively with $f_*(N_1) \subseteq N_2$. Then f induces a map $f^+ : X^+ \to Y^+$, where X^+ is the plus construction of the pair $((X, x_0), N_1)$ and Y^+ is the plus construction of the pair $((Y, y_0), N_2)$. Furthermore, f is unique up to homotopy.*

Proof. See [Ros94, 5.2.4]. $\square$

III.3 Quillen's Higher K-theory

In his work on higher K-theory, Daniel Quillen introduced a topological construction that encodes the information of K-groups of a ring into the homotopy groups of a suitably constructed topological space.

Definition III.3.1 (K-Theory Space of a Ring)**.** Let R be a ring. Endow the groups $GL(R)$ and $K_0(R)$ with the discrete topology. The *Algebraic K-Theory Space of R*, denoted by $K(R)$, is the space

$$K(R) = \left(K_0(R) \times BGL(R)^+, (0_{K_0(R)}, 1_{GL(R)})\right),$$

where $BGL(R)^+$ is the plus construction of the pair $\left((BGL(R), 1_{GL(R)}), E(R)\right)$.

Remark III.3.2. Note that the K-theory space of a ring is only well-defined up to homotopy equivalence.

Proposition III.3.3. *Let R be a ring. Then*

$$\pi_0(K(R)) \cong K_0(R), \qquad \pi_1(K(R)) \cong K_1(R), \qquad \text{and} \qquad \pi_2(K(R)) \cong K_2(R).$$

Proof. Since $BGL(R)^+$ is connected, it follows that $\pi_0(K(R)) = K_0(R)$. Now, by Propositions III.2.1 and III.1.13,

$$\pi_1(BGL(R)^+) \cong \pi_1(BGL(R))/E(R) \cong \pi_0(GL(R))/E(R) = GL(R)/E(R) \cong K_1(R).$$

For the last expression, observe that $E(R)$, for being a normal subgroup of $GL(R)$, acts freely and properly discontinuously on $EGL(R)$, so $EGL(R) \to EGL(R)/E(R)$ is a principal $E(R)$-bundle, and therefore the base space $BGL(R)/E(R)$ is a model for $BE(R)$. Thus, $EGL(R)/E(R) \to EGL(R)/GL(R) \simeq BGL(R)$ is a normal covering map of $BGL(R)$, with covering group $GL(R)/E(R)$.

In the following commutative diagram, the cells attached to $BGL(R)$ to construct $BGL(R)^+$ get lifted to cells in $BE(R)^+$ that kill the perfect group $E(R)$ of $\pi_1(BE(R)) \cong E(R)$, i.e., the covering space $BE(R)^+ \to BGL(R)^+$ is the universal cover of $BGL(R)^+$:

$$\begin{array}{ccc} BE(R) & \longrightarrow & BE(R)^+ \\ \big\downarrow & & \big\downarrow \\ BGL(R) & \longrightarrow & BGL(R)^+. \end{array}$$

In particular, $\pi_n(BE(R)^+) \cong \pi_n(BGL(R)^+)$ for $n \geq 2$, and $\pi_1(BE(R)^+) = 0$. By the Hurewicz Theorem,

$$\pi_2(BE(R)^+) \cong H_2(BE(R)^+; \mathbb{Z}) \cong H_2(BE(R); \mathbb{Z}) \cong H_2(E(R); \mathbb{Z}).$$

Hence, by Proposition II.3.12, it follows that $\pi_2(K(R)) \cong \pi_2(BGL(R)^+) \cong K_2(R)$. □

This motivates the following definition:

Definition III.3.4 (Higher K-Theory Groups of a Ring)**.** Let R be a ring. For all $n \in \mathbb{N}$, define the *n-th K-theory group*, denoted by $K_n(R)$, as

$$K_n(R) = \pi_n\big(K(R), (0_{K_0(R)}, 1_{GL(R)})\big).$$

Proposition III.3.5. *The mapping $K_n(-) : \mathbf{Ring} \to \mathbf{Ab}$ is a functor.*

Proof. For any ring R, all the steps in the construction of $K_n(R)$ can be seen as functors, so the desired mapping is a functor too. □

Unfortunately, the study of further properties of the higher K-groups of a ring requires additional tools from algebraic topology beyond the scope of this work. Nevertheless, a few notable results can be mentioned briefly.

Let R be a ring and I be a two-sided ideal of R.

1. All of the K-groups commute with finite products.

2. The K-group $K_3(R)$ can be completely defined in only algebraic terms as $H_3(\mathrm{St}(R), \mathbb{Z})$.

3. The quotient map $q : R \to R/I$ induces a map $q_* : K(R) \to K(R/I)$. Let $F(q_*)$ be the homotopy fiber of q_*. By defining the relative K-groups $K_n(R, I)$ as $K_n(R, I) = \pi_n(F(q_*))$, one can verify that these relative groups agree with the relative K-groups defined previously. Even more, one obtains a long exact homotopy sequence isomorphic to

$$\cdots \longrightarrow K_{n+1}(R/I) \longrightarrow K_n(R, I) \longrightarrow K_n(R) \longrightarrow K_n(R/I) \longrightarrow K_{n-1}(R, I) \longrightarrow \cdots,$$

   which extends the exact sequence (3).

4. A more advanced computation shows that for every finite field $\mathbb{F}_q$,

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/(q^k - 1) & \text{if } n = 2k - 1 \text{ for } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this result was already known for $n = 1$ due to Example II.2.14, because it is known that the multiplicative group of a field: $\mathbb{F}^\times$, is cyclic. In this case, it is isomorphic to the cyclic group of q elements, which coincides with $\mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)$.

Appendix

This appendix presents several technical results from algebraic topology. The development of these results is somewhat intricate and tangential to the main focus of the thesis, and they are only used in the final chapter. Relevant references are provided for those interested in exploring the topic in more detail.

A.1 Theorems in Homotopy Theory

Theorem A.1.1 (Long Exact Homotopy Sequence)**.** *Let (X, A) be a pair of spaces with basepoint $x_0 \in A$. Denote by i the inclusion $(A, x_0) \hookrightarrow (X, x_0)$ and by j the inclusion $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. Then the homotopy sequence*

$$\cdots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \cdots \xrightarrow{i_*} \pi_0(X, x_0)$$

is exact.

Proof. See [Hat02, 4.3]. □

Theorem A.1.2 (Whitehead Theorem)**.** *If a map $f : X \to Y$ between connected CW-complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence.*

Proof. See [Hat02, 4.5]. □

Lemma A.1.3 (Extension Lemma)**.** *Given a CW-pair (X, A) and a map $f : A \to Y$ with Y path-connected, then f can be extended to a map $X \to Y$ if $\pi_n(Y) = 0$ for all n such that $X - A$ has cells of dimension n.*

Proof. See [Hat02, 4.7]. □

Theorem A.1.4 (Cellular Approximation Theorem). *Every map $f : X \to Y$ of CW-complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subseteq X$, the homotopy may be taken stationary on A.*

Proof. See [Hat02, 4.8]. □

Theorem A.1.5 (Hurewicz Theorem). *If X is a $(n-1)$-connected CW-complex for $n \geq 1$, then $\widetilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X)^{\mathrm{ab}} \cong H_1(X; \mathbb{Z})$.*

Proof. See [Hat02, 2.A1 and 4.32]. □

Theorem A.1.6 (Homology Whitehead Theorem). *A map $f : X \to Y$ between simply-connected CW-complexes is a homotopy equivalence if $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

Proof. See [Hat02, 4.33]. □

A.2 Principal and Universal G-bundles

Definition A.2.1 (Fiber Bundle). A *fiber bundle* with fiber F is a triple $\xi = (E, B, p)$, where $p : E \to B$ is a continuous map, such that B admits an open cover $\{U_\alpha\}$ together with a collection of homeomorphisms $\{\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F\}$ making the following diagram commute:

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "p"'] & U_\alpha \times F \arrow[dl, "\pi_1"] \\ U_\alpha, & \end{tikzcd}$$

with π_1 being the projection onto the first factor. The space B is called the *base space*, the space E is called the *total space*, and the map p is called the *projection* of the fiber bundle.

Remark A.2.2. In a fiber bundle $\xi = (E, B, p)$, if the base space and projection map of a fiber bundle are understood from the context, it is common to write ξ for the total space E.

Example A.2.3. The following are examples of fiber bundles:

(a) All covering spaces are fiber bundles with discrete fiber.

(b) The Möbius band.

(c) The tangent and normal bundles.

Definition A.2.4 (Principal G-Bundle)**.** Let G be a topological group, and let $\xi = (E, B, p)$ be a fiber bundle with fiber G. Let $\kappa : E \times G \to E$ be a right action of G on E. The pair (ξ, κ) is called a *principal G-bundle* if for $x \in E$ and $g \in G$, $p(\kappa(x, g)) = p(x)$. This is equivalent to the following commutative diagram of G-spaces:

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "p"'] & U_\alpha \times G \arrow[dl, "\pi_1"] \\ U_\alpha, & \end{tikzcd}$$

where G acts on the right of $U_\alpha \times G$ by $((b, g), h) \mapsto (b, gh)$, and trivially on U_α.

Definition A.2.5 (Pullback Bundle)**.** Let $\xi = (E, B, p)$ be a fiber bundle with fiber F. Given a map $f : B' \to B$, consider the fibered product

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B. \end{tikzcd}$$

The *pullback bundle*, denoted by $f^*(\xi)$, is the fiber bundle (E', B', p').

Proposition A.2.6. *Pullback bundles are indeed fiber bundles. Furthermore, if ξ is a principal G-bundle, then $f^*(\xi)$ is a principal G-bundle.*

Proof. See [Hus94, Chapter 4, 4.1] □

Observe that f^* has functorial properties, in particular, given a fiber bundle $\xi : (E, B, p)$, and maps $g : B_2 \to B_1$ and $f : B_1 \to B$, one has that $(f \circ g)^*(\xi) \cong g^*(f^*(\xi))$ via $(b_2, x) \mapsto (b_2, (g(b_2), x))$. This is illustrated by the following commutative diagram:

$$
\begin{tikzcd}
(f\circ g)^*(\xi) \arrow[rr, "\cong"] \arrow[dr] \arrow[rrrr, bend left] & & g^*(f^*(\xi)) \arrow[r] \arrow[dl] & f^*(\xi) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
& B_2 \arrow[rr, "g"'] & & B_1 \arrow[r, "f"'] & B,
\end{tikzcd}
$$

where the elements get mapped in the following way

$$
\begin{tikzcd}
(b_2,x) \arrow[rr, mapsto] \arrow[dr, mapsto] \arrow[rrrr, mapsto, bend left] & & (b_2,(g(b_2),x)) \arrow[r, mapsto] \arrow[dl, mapsto] & (g(b_2),x) \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto, "p"] \\
& b_2 \arrow[rr, mapsto, "g"'] & & g(b_2) \arrow[r, mapsto, "f"'] & (f\circ g)(b_2) = p(x).
\end{tikzcd}
$$

Theorem A.2.7. *Let $\xi = (E, B, p)$ be a principal G-bundle, and let f and g be homotopic maps from B' to B, with B' Hausdorff and paracompact. Then $f^*(\xi)$ and $f^*(g)$ are isomorphic as principal G-bundles over B'.*

Proof. A more general statement can be found in [tD08, 14.3.3] or [Hus94, Chapter 4, 9.9]. The precise statement can be found in [Coh, 4.1]. $\square$

Corollary A.2.8. *Let B be a contractible Hausdorff paracompact space. Then every principal G-bundle over B is isomorphic to the trivial bundle $B \times G$.*

Proof. Let $\xi = (E, B, p)$ be an arbitrary principal G-bundle. Since B is contractible, the constant map $\mathrm{cst}_{b_0} : B \to B$ is homotopic to the identity map, so $\mathrm{cst}^*_{b_0}(\xi) \cong \mathrm{Id}^*_B(\xi)$. Notice that $\mathrm{Id}^*_B(\xi) \cong \xi$, and $\mathrm{cst}^*_{b_0}(\xi) \cong (B \times G, B, \pi_1)$ since the total space of $\mathrm{cst}^*_{b_0}(\xi)$ is $B \times_B E = B \times p^{-1}(b_0) \cong B \times G$. $\square$

Definition A.2.9 (Universal G-bundle)**.** Let $[X, Y]$ denote to homotopy classes of maps from X to Y, and let $\mathrm{Princ}_G(B)$ denote the set of isomorphism classes of principal G-bundles over B. A *universal G-bundle* $\xi_G = (EG, BG, p_G)$ is a principal G-bundle such that BG is a Hausdorff paracompact space, and for all Hausdorff paracompact spaces B, the following correspondence is a bijection:

$$[B, BG] \longrightarrow \mathrm{Princ}_G(B)$$
$$[f] \longmapsto f^*(\xi_G).$$

Aditionally, a principal G-bundle $\xi_G = (EG, BG, p_G)$ is called a *universal G-bundle on CW-complexes* if BG is a Hausdorff paracompact space, and for all CW complexes X, the following correspondence is a bijection:

$$[X, BG] \longrightarrow \mathrm{Princ}_G(X)$$
$$[f] \longmapsto f^*(\xi_G).$$

Observe that the correspondence given in the definition of universal G bundles is well-defined because of Theorem A.2.7.

Proposition A.2.10. *Let B and B' be Hausdorff paracompact spaces. If $\xi_G = (E, B, p)$ and $\xi'_G = (E', B', p')$ are universal G-bundles, then B and B' are homotopy equivalent. Moreover, E and E' are also homotopy equivalent.*

Proof. From the definition of universal G bundles, there exist bijections

$$[B', B] \longleftrightarrow \mathrm{Princ}_G(B) \qquad \text{and} \qquad [B, B'] \longleftrightarrow \mathrm{Princ}_G(B').$$

Let $f : B' \to B$ and $f' : B \to B'$ be maps such that $f^*(\xi_G) = \xi'_G$ and $(f')^*(\xi'_G) = \xi_G$ respectively. Then

$$(f \circ f')^*(\xi_G) = (f')^*(f^*(\xi_G)) = (f')^*(\xi'_G) = \xi_G = (\mathrm{id}_B)^*(\xi_G),$$

so $f \circ f' \simeq \mathrm{id}_B$. Similarly, $f' \circ f \simeq \mathrm{Id}_{B'}$. Therefore, B and B' are homotopy equivalent.

To see that E and E' are also homotopy equivalent, it suffices to use the Covering Homotopy Theorem from [Coh, 4.2], to induce a homotopy equivalence from E to E' given a homotopy from B to B' □

From the previous proposition, it follows that a universal G-bundle is unique up to homotopy of base and total spaces. Therefore, it is appropiate to refer to the universal G-bundle of a topological group (rather than a universal G-bundle) as long as only the homotopy type of the spaces is considered.

A.3 Fibrations

Definition A.3.1 (Homotopy Lifting Property)**.** A continuous map $p : E \to B$ is said to have the *homotopy lifting property* with respect to a space X if, given a maps $f : X \to E$ and $H : X \times I \to B$, there exists a map $\tilde{H} : X \times I \to E$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "(\mathrm{id},\mathrm{cst}_0)"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists\, \tilde{H}"] & B. \end{tikzcd}$$

Definition A.3.2 (Serre Fibration)**.** A continuous map $p : E \to B$ is called a *Serre fibration* if it satisfies the homotopy lifting property with respect to all disks D^n. A *pointed Serre fibration* is a Serre fibration $p : E \to B$ equipped with basepoints $b_0 \in B$ and $e_0 \in E$ such that $p(e_0) = b_0$.

Proposition A.3.3. *The projection map of a fiber bundle $\xi = (E, B, p)$ is a Serre fibration.*

Proof. See [Hat02, 4.48]. □

Proposition A.3.4. *Let $p : E \to B$ be a Serre fibration. Then for any points b_0, b_1 from the same path component of B, the fibers $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are weakly homotopy equivalent.*

Proof. See [FF16, §9.6]. □

Lemma A.3.5. *Let $p : E \to B$ be a Serre fibration. Let $e \in E$ be a point. Denote $b = p(e)$ and $F = p^{-1}(b)$. Then the map*

$$p_* : \pi_n(E, F, e) \to \pi_n(B, b)$$

is an isomorphism for all n.

Proof. See [FF16, §9.8]. □

Theorem A.3.6 (Homotopy Sequence of a Fibration)**.** *Let $p : (E, e_0) \to (B, b_0)$ be a pointed Serre fibration, and let $F = p^{-1}(b_0)$ be the fiber over the basepoint b_0. If B is path-connected, then there is a natural long exact sequence*

$$\cdots \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0),$$

where this is a long exact sequence of groups for $n \geq 1$, and a long exact sequence of pointed sets for all n.

Proof. Follows directly from Theorem A.1.1 and Lemma A.3.5. □

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