On good morphisms of exact triangles

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Outline

Adams spectral sequence

Moss pairing

Good morphisms

Examples and non-examples

Lifting criterion

Classical Adams spectral sequence

Given finite spectra X and Y, the classical Adams spectral sequence has the form

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(H^*Y, \Sigma^t H^*X) \Rightarrow [\Sigma^{t-s}X, Y_p^\wedge]$$

where $\mathcal{A}$ denotes the mod p Steenrod algebra.

For E a nice ring spectrum (e.g. MU or BP), the E-based Adams spectral sequence is:

$$E_2^{s,t} = \mathrm{Ext}^s_{E_*E}(\Sigma^t E_*X, E_*Y) \Rightarrow [\Sigma^{t-s}X, L_E Y].$$

Triangulated version

- Brinkmann (1968): Adams spectral sequence in a triangulated category.
- Franke (1996): Application to $E(1)$-local ($=KU_{(p)}$-local) spectra.
- Further work related to Franke's construction (2007 and on): Roitzheim, Barnes, Patchkoria, and others. Applications to E-local spectra and E-module spectra for various E.
- Christensen (1998): Application to ghost lengths and stable module categories.

Projective and injective classes

Definition. A **projective class** in $\mathcal{T}$ is a pair $(\mathcal{P}, \mathcal{N})$, where $\mathcal{P} \subseteq \operatorname{ob} \mathcal{T}$ and $\mathcal{N} \subseteq \operatorname{mor} \mathcal{T}$, such that:

(i) $\mathcal{P}$ consists of exactly the objects P such that every composite $P \to X \to Y$ is zero for each $X \to Y$ in $\mathcal{N}$,

(ii) $\mathcal{N}$ consists of exactly the maps $X \to Y$ such that every composite $P \to X \to Y$ is zero for each P in $\mathcal{P}$,

(iii) for each X in $\mathcal{T}$, there is a triangle $P \to X \to Y$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{N}$.

An **injective class** in $\mathcal{T}$ is a projective class in $\mathcal{T}^{\mathrm{op}}$.

Examples

Example. In spectra, take:

$$\mathcal{P} = \text{retracts of wedges of spheres } \bigvee_i S^{n_i}$$

$$\mathcal{N} = \text{maps inducing zero on homotopy groups.}$$

$(\mathcal{P}, \mathcal{N})$ is the ghost projective class.

Example. For E any spectrum, take:

$$\mathcal{I} = \text{retracts of products } \prod_i \Sigma^{n_i} E$$

$$\mathcal{N} = \text{maps inducing zero on } E^*(-).$$

Then $(\mathcal{I}, \mathcal{N})$ is an injective class.

For $E = H\mathbb{F}_p$, this injective class leads to the classical (cohomological) Adams spectral sequence.

Examples

Example. For E a homotopy commutative ring spectrum, take:

$$\mathcal{I} = \text{retracts of } E \wedge W$$
$$\mathcal{N} = \text{maps } f \colon X \to Y \text{ with } E \wedge f \simeq 0 \colon E \wedge X \to E \wedge Y.$$

The injective class $(\mathcal{I}, \mathcal{N})$ leads to the E-based (homological) Adams spectral sequence.

Remark. We always assume that our projective and injective classes are stable, i.e., closed under suspension and desuspension.

Adams resolutions

Definition. An **Adams resolution** of an object Y in $\mathcal{T}$ with respect to an injective class $(\mathcal{I}, \mathcal{N})$ is a diagram

$$\begin{tikzcd}
Y = Y_0 \arrow[dr, "p_0"'] & & Y_1 \arrow[ll, "i_0"'] \arrow[dr, "p_1"'] & & Y_2 \arrow[ll, "i_1"'] \arrow[dr, "p_2"'] & & Y_3 \arrow[ll, "i_2"'] & & \cdots \arrow[ll] \\
& I_0 \arrow[ur, "\delta_0"'] & & I_1 \arrow[ur, "\delta_1"'] & & I_2 \arrow[ur, "\delta_2"'] & & \cdots &
\end{tikzcd}$$

where each I_s is injective, each map i_s is in $\mathcal{N}$, and the triangles are triangles.

Axiom (iii) says that you can form such a resolution.

Applying $\mathcal{T}(X, -)$ leads to an exact couple and therefore a spectral sequence, called the Adams spectral sequence.

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{I}}(\Sigma^t X, Y)$$

Some structural features

Classical	Triangulated version
d_r as an r^{th} order cohomology operation (Maunder 1964)	✓ (Christensen–F. 2017)
Pairing (Moss 1968)	In progress
Convergence theorem (Moss 1970)	Need the pairing

... But why?

- Fewer hypotheses.
- There's a lot we can do using only the triangulated structure.
- Get derived invariants.

Outline

Moss pairing

Theorem (Moss)**.** For spectra X, Y, and Z, there is a natural associative pairing of Adams spectral sequences

$$E_r^{s,t}(Y,Z) \otimes E_r^{s',t'}(X,Y) \to E_r^{s+s',t+t'}(X,Z)$$

satisfying the following properties:

1. It agrees with the Yoneda pairing of Ext classes on E_2 terms.
2. The differentials d_r satisfy the Leibniz rule.
3. The pairing on E_∞ terms is compatible with the composition product
$$[Y,Z] \otimes [X,Y] \xrightarrow{\circ} [X,Z].$$

Cofibers in an Adams tower

Starting point: an $\mathcal{I}$-Adams tower

$$X = X_0 \xleftarrow{x_1} X_1 \xleftarrow{x_2} X_2 \longleftarrow \cdots$$

For intervals $[n, m] \leq [n', m']$, there is a fill-in in the diagram

$$\begin{tikzcd}
X_{m'} \arrow[r] \arrow[d] & X_{n'} \arrow[r] \arrow[d] & X_{n'}/X_{m'} \arrow[r] \arrow[d, dashed] & \Sigma X_{m'} \arrow[d] \\
X_m \arrow[r] & X_n \arrow[r] & X_n/X_m \arrow[r] & \Sigma X_m.
\end{tikzcd}$$

Question. How convenient can those choices be made?

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Mapping cone

Definition (Neeman)**.** The **mapping cone** of a map of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

is the sequence

$$X' \oplus Y \xrightarrow{\begin{bmatrix} u' & g \\ 0 & -v \end{bmatrix}} Y' \oplus Z \xrightarrow{\begin{bmatrix} v' & h \\ 0 & -w \end{bmatrix}} Z' \oplus \Sigma X \xrightarrow{\begin{bmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{bmatrix}} \Sigma X' \oplus \Sigma Y.$$

The map of triangles (f, g, h) is **good** if its mapping cone is an exact triangle.

Middling good morphisms

Proposition (Neeman)**.** If the map of triangles (f, g, h) is good, then it extends to a 4×4 diagram

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] \arrow[d, "f'"'] & Y' \arrow[r, "v'"] \arrow[d, "g'"] & Z' \arrow[r, "w'"] \arrow[d, dashed, "h'"] & \Sigma X' \arrow[d, "\Sigma f'"] \\
X'' \arrow[r, dashed, "u''"] \arrow[d, "f''"'] & Y'' \arrow[r, dashed, "v''"] \arrow[d, "g''"] & Z'' \arrow[r, dashed, "w''"] \arrow[d, dashed, "h''"] \arrow[dr, phantom, "\boxed{-1}"] & \Sigma X'' \arrow[d, "\Sigma f''"] \\
\Sigma X \arrow[r, "\Sigma u"'] & \Sigma Y \arrow[r, "\Sigma v"'] & \Sigma Z \arrow[r, "\Sigma w"'] & \Sigma^2 X
\end{tikzcd}
$$

where the first three rows and columns are exact.

Definition (Neeman)**.** A map of triangles is **middling good** if it extends to a 4×4 diagram.

Verdier good morphisms

Definition. A map of triangles

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

is **Verdier good** if h can be constructed as in Verdier's proof of the 4×4 lemma.

Explicitly: ...

Verdier good morphisms, cont'd

... There exists an octahedron for the composite $X \xrightarrow{u} Y \xrightarrow{g} Y'$:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "\alpha_1"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "gu"] & Y' \arrow[r, "\widetilde{v}"] \arrow[d, "g'"] & A \arrow[r, "\widetilde{w}"] \arrow[d, dashed, "\beta_1"] & \Sigma X \\
& Y'' \arrow[r, equal] \arrow[d, "g''"] & Y'' \arrow[d, dashed, "\gamma_1"] & \\
& \Sigma Y \arrow[r, "\Sigma v"'] & \Sigma Z, &
\end{tikzcd}
$$

Verdier good morphisms, cont'd

... an octahedron for the composite $X \xrightarrow{f} X' \xrightarrow{u'} Y'$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & X' \arrow[r, "f'"] \arrow[d, "u'"] & X'' \arrow[r, "f''"] \arrow[d, dashed, "\alpha_2"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "u'f=gu"] & Y' \arrow[r, "\widetilde{v}"] \arrow[d, "v'"] & A \arrow[r, "\widetilde{w}"] \arrow[d, dashed, "\beta_2"] & \Sigma X \\
& Z' \arrow[r, equal] \arrow[d, "w'"] & Z' \arrow[d, dashed, "\gamma_2"] & \\
& \Sigma X' \arrow[r, "\Sigma f'"'] & \Sigma X'', &
\end{tikzcd}$$

and $h\colon Z \to Z'$ is given by $h = \beta_2 \circ \alpha_1$.

Enhanced 4×4 lemma

Lemma (Miller)**.** A map of triangles (f, g, h) is Verdier good if and only if it extends to a 4×4 diagram and there is an object A ($=$ cofiber of $gu\colon X \to Y'$) together with three diagrams:

$$\begin{tikzcd}
 & X \arrow[rr, "gu=u'f"] & & Y' \arrow[dl, "\widetilde{v}"'] \arrow[dr, "v'"] & \\
Z \arrow[ur, "w"] \arrow[rr, dashed, "\alpha_1"] & & A \arrow[ul, "\widetilde{w}"'] \arrow[rr, dashed, "\beta_2"] \arrow[dl, dashed, "\beta_1"'] & & Z' \arrow[dl, "(\Sigma f')w'"] \\
 & Y'' \arrow[ul, "(\Sigma v)g''"] & & X'', \arrow[ll, dashed, "u''"] \arrow[ul, dashed, "\alpha_2"'] &
\end{tikzcd}$$

Enhanced 4×4 lemma, cont'd

$$
\begin{tikzcd}
 & \Sigma X \arrow[dl, "\Sigma f"'] \arrow[dr, "\Sigma u"] & \\
\Sigma X' & & \Sigma Y \\
 & A \arrow[uu, "\widetilde{w}"'] \arrow[dl, dashed, "\beta_2"'] \arrow[dr, dashed, "\beta_1"] & \\
Z' \arrow[uu, "w'"] \arrow[dr, dashed, "h'"'] & & Y'' \arrow[uu, "g''"'] \arrow[dl, dashed, "v''"] \\
 & Z'' &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y' \arrow[dd, "\widetilde{v}"'] & \\
Y \arrow[ur, "g"] \arrow[dd, "v"'] & & X' \arrow[ul, "u'"'] \arrow[dd, "f'"] \\
 & A & \\
Z \arrow[ur, dashed, "\alpha_1"] & \boxed{-1} & X'' \arrow[ul, dashed, "\alpha_2"'] \\
 & \Sigma^{-1} Z'' \arrow[ul, dashed, "\Sigma^{-1} h''"] \arrow[ur, dashed, "\Sigma^{-1} w''"'] &
\end{tikzcd}
$$

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Fill-in of zero

Example (Neeman)**.** The map of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "0"'] & Y \arrow[r, "v"] \arrow[d, "0"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "0"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

is good $\Leftrightarrow h = v'\theta w$ for some $\theta\colon \Sigma X \to Y'$.

Fact. The map $(0, 0, h)$ above is Verdier good $\Leftrightarrow h = v'\theta w$ for some $\theta\colon \Sigma X \to Y'$.

Similarly for the maps $(0, g, 0)$ and $(f, 0, 0)$.

Remark. Goodness is invariant under rotation. What about Verdier goodness?

Middling good but not good

Example (Neeman)**.** The map of triangles

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "u"'] & Y \arrow[r, "v"] \arrow[d, "v"] & Z \arrow[r, "w"] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\
Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma X \arrow[r, "-\Sigma u"'] & \Sigma Y
\end{tikzcd}$$

is always middling good.

It is good $\Leftrightarrow w = w\theta w$ for some $\theta\colon \Sigma X \to Z$.

For instance, with $\mathcal{T}(\Sigma X, Z) = 0$ and $w \neq 0$, the map of triangles is *not* good.

Fact. The map (u, v, w) above is Verdier good $\Leftrightarrow w = w\theta w$ for some $\theta\colon \Sigma X \to Z$.

Split triangles

Example. If the top row or bottom row of

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

is a split triangle, then the map of triangles is Verdier good.

Remark. Goodness is invariant under chain homotopy. What about Verdier goodness?

Not middling good

Example. In the derived category $D(\mathbb{Z})$, the map of triangles

$$\begin{tikzcd}
\mathbb{Z}[0] \arrow[r, "n"] \arrow[d, "0"'] & \mathbb{Z}[0] \arrow[r, "q"] \arrow[d, "0"] & \mathbb{Z}/n[0] \arrow[r, "\epsilon"] \arrow[d, "\epsilon"] & \mathbb{Z}[1] \arrow[d, "0"] \\
\mathbb{Z}[0] \arrow[r, "q"'] & \mathbb{Z}/n[0] \arrow[r, "\epsilon"'] & \mathbb{Z}[1] \arrow[r, "-n"'] & \mathbb{Z}[1]
\end{tikzcd}$$

is *not* middling good.

Example. In the stable homotopy category, the map of triangles

$$\begin{tikzcd}
S^0 \arrow[r, "n"] \arrow[d, "0"'] & S^0 \arrow[r, "q"] \arrow[d, "0"] & M(n) \arrow[r, "\delta"] \arrow[d, "\delta"] & S^1 \arrow[d, "0"] \\
S^0 \arrow[r, "q"'] & M(n) \arrow[r, "\delta"'] & S^1 \arrow[r, "-n"'] & S^1
\end{tikzcd}$$

is *not* middling good.

Middling goodness and Toda brackets

Proposition. The map of triangles

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "0"'] & Y \arrow[r, "v"] \arrow[d, "0"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "0"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

is middling good $\Leftrightarrow$ the Toda bracket $\langle w', h, v\rangle \subseteq \mathcal{T}(\Sigma Y, \Sigma X')$ contains zero.

Middling goodness and Toda brackets, cont'd

Example. The map of triangles

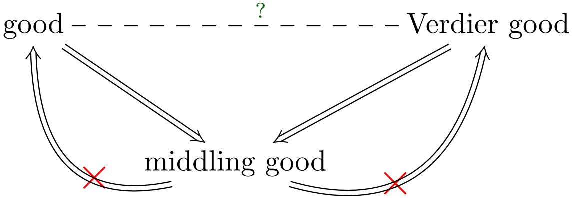
$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "0"'] & Y \arrow[r, "v"] \arrow[d, "0"] & Z \arrow[r, "w"] \arrow[d, "w"] & \Sigma X \arrow[d, "0"] \\
Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma X \arrow[r, "-\Sigma u"'] & \Sigma Y
\end{tikzcd}$$

is middling good $\Leftrightarrow$ $1_{\Sigma Y}$ lies in the indeterminacy subgroup:

$$1_{\Sigma Y} \in (\Sigma u)_* \mathcal{T}(\Sigma Y, \Sigma X) + (\Sigma v)^* \mathcal{T}(\Sigma Z, \Sigma Y).$$

Corollary. Middling goodness is *not* invariant under chain homotopy.

Summary



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Lifting criterion

Theorem (Christensen–F.)**.** In the diagram with exact rows

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dl, dashed, "k"', "?"] & Z \arrow[r, "w"] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X',
\end{tikzcd}
$$

there exists a lift $k \colon Y \to X'$ satisfying $ku = f$ and $u'k = g$
$\Leftrightarrow$ The map $0 \colon Z \to Z'$ is a good fill-in.

Remark. This situation appears in the Moss pairing.

Thank you!