Cohomologie de Quillen des algèbres à puissances divisées sur une opérade

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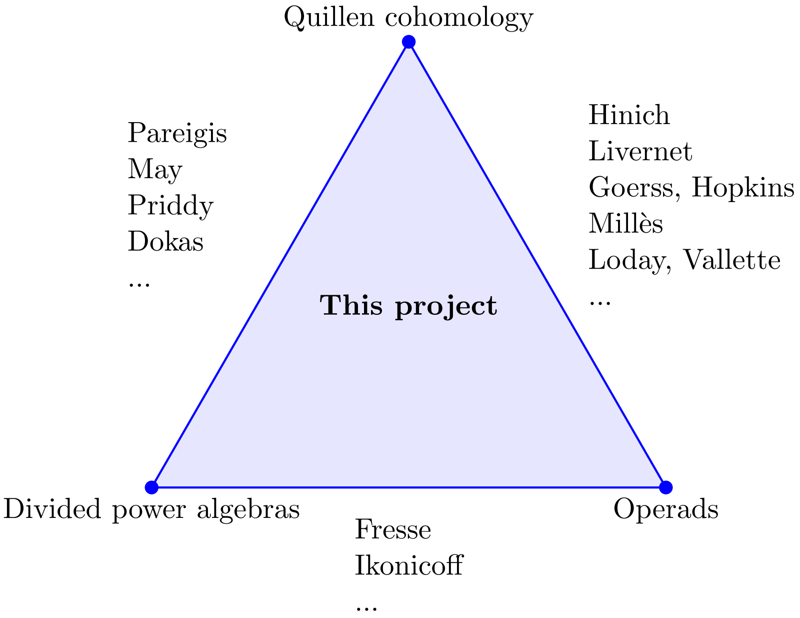
Outline

Overview


Quillen cohomology
Pareigis
May
Priddy
Dokas
...
Hinich
Livernet
Goerss, Hopkins
Millès
Loday, Vallette
...
This project
Divided power algebras
Operads
Fresse
Ikonicoff
...

André–Quillen (co)homology

- Cohomology theory for commutative rings.
- Developed by André and Quillen in the 1960s.
- Non-additive derived functors constructed using simplicial methods.
- Used to solve problems in commutative algebra and algebraic geometry.
- Makes sense for any algebraic structure.

Applications in topology

A sampler of applications in topology.

- Unstable Adams spectral sequence (Miller, Goerss).
- Obstruction theory for ring spectra (Goerss–Hopkins–Miller, Lurie)
- Realization and classification problems (Blanc, Blanc–Dwyer–Goerss, F., Biedermann–Raptis–Stelzer).
- Higher homotopy operations (Baues–Blanc, Blanc–Johnson–Turner).
- Knot theory: Quillen homology of racks and quandles (Szymik, Berest).

Outline

Cohomology in algebra

Cohomology theories for algebraic structures:

- Group cohomology
- Lie algebra cohomology
- Hochschild cohomology of associative algebras
- André–Quillen cohomology of commutative rings
- etc.

Unified approach: Barr–Beck triple cohomology, based on simplicial resolutions (1969).

Put into the framework of model categories by Quillen (1967).

Idea: cohomology $\approx$ derived functors of derivations.

Beck modules

Setup: An "algebraic" category $\mathcal{C}$, e.g., groups, abelian groups, rings, commutative rings, R-modules, Lie algebras, etc.

Need a good notion of *coefficient module* M over X:

$$H^*(X; M).$$

Definition (Beck 1967)**.** For an object X in a category $\mathcal{C}$, a **Beck module** over X is an abelian group object in the slice category $\mathcal{C}/X$.

Denote the category of Beck modules over X

$$\mathrm{Mod}(X) \coloneqq (\mathcal{C}/X)_{\mathrm{ab}}.$$

Pullback and pushforward

Definition. The pullback functor $f^*\colon \mathcal{C}/Y \to \mathcal{C}/X$ induces a functor

$$f^*\colon \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$$

also called the **pullback**.

Its left adjoint

$$f_!\colon \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$$

is called the **pushforward** along f.

pullback = "restriction of scalars"

pushforward = "extension of scalars"

Groups

$\mathcal{C} = \mathrm{Gp}$, the category of groups.

For a group G:

$$\begin{aligned}\mathrm{Mod}(G) &\cong G\text{-modules in the usual sense}\\ &\cong \mathbb{Z}G\text{-Mod}.\end{aligned}$$

A Beck module over G is a split extension of G with abelian kernel:

$$1 \longrightarrow K \longrightarrow G \ltimes K \underset{e}{\overset{p}{\rightleftarrows}} G \longrightarrow 1.$$

The G-action on K is given by $e(g)k = (g, g \cdot k)$. In other words:

$$(g,k)(g',k') = (gg', k + g \cdot k').$$

Groups, cont'd

For a map of groups $f\colon G \to H$, the pushforward functor is

$$\begin{aligned} f_! \colon \mathrm{Mod}(G) &\to \mathrm{Mod}(H) \\ f_!(M) &= \mathbb{Z}H \otimes_{\mathbb{Z}G} M. \end{aligned}$$

Rings

$\mathcal{C} = \mathrm{Alg}_{\Bbbk}$, the category of (unital) algebras over a commutative ring $\Bbbk$.

For a $\Bbbk$-algebra A:

$$\mathrm{Mod}(A) \cong {}_A\mathrm{Bimod}_A.$$

A Beck module over A is a split extension of A with square zero kernel:

$$0 \longrightarrow M \longrightarrow A \oplus M \underset{s}{\overset{p}{\rightleftarrows}} A \longrightarrow 0.$$

The two actions on M are given by

$$(a, m)(a', m') = (aa', a \cdot m' + m \cdot a')$$

and they coincide for scalars in $\Bbbk$.

Rings, cont'd

For a map of $\Bbbk$-algebras $f\colon A \to B$, the pullback functor

$$f^*\colon \mathrm{Mod}(B) \to \mathrm{Mod}(A)$$

is the usual restriction of scalars:

$$a \cdot m \cdot a' \coloneqq f(a) \cdot m \cdot f(a').$$

The pushforward functor is

$$\begin{aligned} f_!\colon \mathrm{Mod}(A) &\to \mathrm{Mod}(B) \\ f_!(M) &= B \otimes_A M \otimes_A B. \end{aligned}$$

Commutative rings

$\mathcal{C} = \mathrm{Com}_{\Bbbk}$, the category of commutative $\Bbbk$-algebras.

For a commutative $\Bbbk$-algebra A:

$$\mathrm{Mod}(A) \cong \mathrm{Mod}_A \quad \text{in the usual sense.}$$

Same correspondence as for algebras, except that $A \oplus M$ must be commutative. This forces the two actions to coincide:

$$a \cdot m = m \cdot a.$$

For a map of commutative k-algebras $f \colon A \to B$, the pushforward functor is extension of scalars:

$$f_! \colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$
$$f_!(M) = B \otimes_A M.$$

Remark. The notion of Beck module depends on the ambient category! A Beck module over A in $\mathrm{Com}_{\Bbbk}$ is **not** the same as in $\mathrm{Alg}_{\Bbbk}$.

Lie algebras

$\Bbbk$ = a field.

$\mathcal{C} = \mathrm{Lie}_{\Bbbk}$, the category of Lie algebras over $\Bbbk$.

For a Lie algebra L:

$$\begin{aligned} \mathrm{Mod}(L) &\cong L\text{-modules in the usual sense} \\ &\cong U(L)\text{-Mod} \end{aligned}$$

where $U(L)$ is the universal enveloping algebra of L.

A Beck module over L is a split extension of L with abelian kernel:

$$0 \longrightarrow M \longrightarrow L \oplus M \underset{s}{\overset{p}{\rightleftarrows}} L \longrightarrow 0$$

with $[m, m'] = 0$ for all $m, m' \in M$.

Lie algebras (cont'd)

The action of L on M is given by

$$[(\ell, 0), (0, m)] = (0, \ell \cdot m),$$

which satisfies

$$[\ell, \ell'] \cdot m = \ell \cdot (\ell' \cdot m) - \ell' \cdot (\ell \cdot m).$$

Universal enveloping algebra

Usually, the category of Beck modules over X is equivalent to (left) modules over a ring $\mathbb{U}_{\mathcal{C}}(X)$, called the **universal enveloping algebra** of X:

$$\mathrm{Mod}(X) \cong \mathbb{U}_{\mathcal{C}}(X)\text{-Mod}.$$

Examples of universal enveloping algebras:

1. For a group G, the group ring $\mathbb{U}_{\mathrm{Gp}}(G) = \mathbb{Z}G$.
2. For a ring A, the enveloping ring $\mathbb{U}_{\mathrm{Alg}_{\mathbb{Z}}}(A) = A \otimes A^{\mathrm{op}}$.
3. For a commutative ring R, the ring itself $\mathbb{U}_{\mathrm{Com}_{\mathbb{Z}}}(R) = R$.
4. For a Lie algebra L over $\Bbbk$, the (classical) universal enveloping algebra $\mathbb{U}_{\mathrm{Lie}_{\Bbbk}}(L) = U(L)$.

Derivations and differentials

Definition. A **derivation** of X with coefficients in a Beck module $p\colon E \to X$ is a section of p:

$$\begin{array}{ccc} X & \xrightarrow{s} & E \\ & \searrow^{=} & \downarrow{\scriptstyle p} \\ & & X. \end{array}$$

The abelian group of derivations is

$$\begin{aligned} \mathrm{Der}(X, E) &\coloneqq \mathrm{Hom}_{\mathcal{C}/X}(X \xrightarrow{\mathrm{id}} X, E \xrightarrow{p} X) \\ &\cong \mathrm{Hom}_{(\mathcal{C}/X)_{\mathrm{ab}}}(Ab_X(X \xrightarrow{\mathrm{id}} X), E \xrightarrow{p} X), \end{aligned}$$

where $Ab_X \colon \mathcal{C}/X \to (\mathcal{C}/X)_{\mathrm{ab}}$ is the **abelianization** functor, i.e., the left adjoint to the forgetful functor $(\mathcal{C}/X)_{\mathrm{ab}} \to \mathcal{C}/X$.

The module of **Kähler differentials** is $\Omega_{\mathcal{C}}(X) = Ab_X X$, which represents derivations.

Example: Commutative rings

Take $\mathcal{C} = \mathrm{Com}_{\Bbbk}$. Given a commutative $\Bbbk$-algebra A and an A-module M, a $\Bbbk$-linear map

$$(\mathrm{id}, f)\colon A \to A \oplus M$$

is a (unital) ring homomorphism if and only if

$$\begin{cases} f(1_A) = 0 \\ f(ab) = a \cdot f(b) + f(a) \cdot b & \rightsquigarrow \text{Leibniz rule.} \end{cases}$$

$$\mathrm{Der}(A, M) = \text{derivations in the usual sense.}$$

The A-module $Ab_A A$ is:

$$Ab_A A = I_A / I_A^2 = \Omega_{A/k},$$

where $I_A = \ker(\mu\colon A \otimes_{\Bbbk} A \to A)$. Classical module of Kähler differentials, which represents k-derivations:

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \cong \mathrm{Der}_k(A, M).$$

Geometric interpretation

Analogy:

$$\begin{array}{c}\text{Kähler differentials in algebraic geometry}\\ \Updownarrow\\ \text{differential 1-forms in differential geometry}\end{array}$$

Example. 1. $A = \Bbbk[x, y]$

$$\begin{aligned}\Omega_{A/\Bbbk} &\cong \{p\,dx + q\,dy \mid p, q \in \Bbbk[x, y]\}\\ &\cong A\,\langle dx, dy\rangle \quad \text{free } A\text{-module.}\end{aligned}$$

2. $A = \Bbbk[x, y]/(f)$

$$\Omega_{A/\Bbbk} \cong A\,\langle dx, dy\rangle \,/\, \left\langle \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \right\rangle.$$

Quillen (co)homology

Quillen homology of $X \approx$ (simplicially) derived functors of Kähler differentials.

Quillen cohomology of X with coefficients in a module $M \approx$ (simplicially) derived functors of derivations with coefficients in M.

The "package":

1. Beck modules
2. Universal enveloping algebra
3. Derivations
4. Kähler differentials

Outline

Divided power algebras

Introduced by Cartan in the 1950s. Appear in positive characteristic.

Divided power operations ≈ operations γ_n that look like $\gamma_n(x) = \frac{x^n}{n!}$. More precisely...

Divided power algebras (cont'd)

Definition. Let A be a commutative $\Bbbk$-algebra. A **system of divided powers** on an ideal $I \subseteq A$ is a collection of maps $\gamma_i \colon I \to A$ for $i \geq 0$ satisfying:

$$\begin{aligned}
\gamma_0(a) &= 1 \\
\gamma_1(a) &= a \\
\gamma_i(a) &\in I,\ i \geq 1 \\
\gamma_i(a+b) &= \sum_{k=0}^{k=i} \gamma_k(a)\gamma_{i-k}(b),\ a,b \in I,\ i \geq 0 \\
\gamma_i(ab) &= a^i \gamma_i(b),\ a \in A,\ i \geq 0 \\
\gamma_i(a)\gamma_j(a) &= \frac{(i+j)!}{i!j!}\gamma_{i+j}(a),\ a \in I,\ i,j \geq 0 \\
\gamma_i(\gamma_j(a)) &= \frac{(ij)!}{i!(j!)^i}\gamma_{ij}(a),\ a \in I,\ i \geq 0,\ j \geq 1.
\end{aligned}$$

Usually A has an augmentation $\epsilon \colon A \to \Bbbk$ and $I = \ker(\epsilon)$.

Examples

Example. The free divided power algebra over $\mathbb{Z}$ on one generator x is

$$\mathbb{Z}[x, \frac{x^2}{2}, \dots, \frac{x^n}{n!}, \dots] \subset \mathbb{Q}[x].$$

Example. Over a field $\Bbbk$ of characteristic 0, divided powers must in fact be given by

$$\gamma_n(x) = \frac{x^n}{n!}.$$

$\rightsquigarrow$ No additional structure in that case.

Divided powers are a positive characteristic phenomenon.

In positive characteristic

Proposition (Soublin 1987)**.** Over a field $\Bbbk$ of characteristic $p > 0$, a system of divided power operations is the same data as just the operation $\pi = \gamma_p \colon I \to I$ satisfying:

$$
\begin{aligned}
a^p &= 0,\ a \in I \\
\pi(a+b) &= \pi(a) + \pi(b) + \sum_{k=1}^{k=p-1} \frac{(-1)^k}{k} a^k b^{p-k},\ a, b \in I \\
\pi(ab) &= 0,\ a, b \in I \\
\pi(\lambda a) &= \lambda^p \pi(a),\ a \in I, \lambda \in \Bbbk.
\end{aligned}
$$

Note: The operation π is non-additive!

Theorem (Dokas 2009)**.** Worked out the "package" for divided power algebras over a field $\Bbbk$ of characteristic $p > 0$.

Outline

Restricted Lie algebras

Over a field $\Bbbk$ of characteristic $p > 0$, Lie algebras usually come with extra structure, that of a *restricted* Lie algebra.

Textbook account in Jacobson (1962).

Important in topology. See work of Milnor–Moore, May, Rector, Bousfield, Curtis, Priddy, etc. in the 1960s.

Restricted Lie algebras (cont'd)

Definition. A **restricted Lie algebra** L over $\Bbbk$ is a Lie algebra together with a map $(-)^{[p]} \colon L \to L$ called the p-map satisfying:

$$(\alpha x)^{[p]} = \alpha^p \, x^{[p]}, \; \alpha \in \Bbbk$$

$$[x, y^{[p]}] = [\cdots [x, \underbrace{y], y], \cdots, y}_{p}]$$

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$$

where $is_i(x,y)$ is the coefficient of λ^{i-1} in $\mathrm{ad}_{\lambda x + y}^{p-1}(x)$.

Here $\mathrm{ad}_x \colon L \to L$ denotes the adjoint representation

$$\mathrm{ad}_x(y) \coloneqq [y, x], \; x, y \in L.$$

Example. For $p = 2$, the last equation becomes:

$$(x+y)^{[2]} = x^{[2]} + [x,y] + y^{[2]}.$$

Examples

Example. For a $\mathbb{k}$-algebra A, the underlying restricted Lie algebra of A has the commutator bracket

$$[x, y] = xy - yx$$

and the p^{th} power map

$$x^{[p]} = x^p.$$

Example. The free restricted Lie algebra on a $\mathbb{k}$-vector space V is the subspace

$$L^r(V) \subseteq T(V)$$

obtained by taking V and closing it under commutators and p^{th} powers.

Theorem (Dokas 2004)**.** Worked out the "package" for restricted Lie algebras over a field $\mathbb{k}$ of characteristic $p > 0$.

Outline

Introduction

Quillen cohomology

Divided power algebras (classical)

Restricted Lie algebras

Operads

Divided power algebras (operadic)

Operads in sets

Idea: Encode a type of algebraic structure as

$$\mathcal{P}(n) = \{\text{the allowed } n\text{-ary operations}\}.$$

Definition. A (symmetric) **operad** in sets $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ is a sequence of sets $\mathcal{P}(n)$ with a right action of the symmetric group Σ_n together with composition maps

$$\circ\colon \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k)$$

and a unit $\eta \in \mathcal{P}(1)$ such that composition is associative, unital, and equivariant.

Algebras over operads

Example. For any set X, the **endomorphism operad** of X is:

$$\begin{cases} \mathrm{End}_X(n) \coloneqq \mathrm{Hom}(X^n, X) \\ \circ = \text{ usual composition} \\ \eta = \mathrm{id}_X \in \mathrm{Hom}(X, X) \\ \Sigma_n \text{ acts by permuting the factors of } X^n. \end{cases}$$

Definition. A **$\mathcal{P}$-algebra** is a set X equipped with action maps

$$\alpha_n \colon \mathcal{P}(n) \to \mathrm{Hom}(X^n, X)$$

compatible with composition, unit, and equivariance. In other words, an operad map

$$\alpha \colon \mathcal{P} \to \mathrm{End}_X .$$

Each n-ary operation symbol $\mu \in \mathcal{P}(n)$ yields an actual n-ary operation on X.

Examples

Example. The (unital) associative operad:

$$\mathrm{As}^{\mathrm{un}}(n) = \Sigma_n,\ n \geq 0.$$

$\mathrm{As}^{\mathrm{un}}$-algebra = monoid.

Example. The (non-unital) associative operad:

$$\mathrm{As}(n) = \begin{cases} \Sigma_n, & n \geq 1 \\ \emptyset, & n = 0. \end{cases}$$

As-algebra = semigroup.

Examples (cont'd)

Example. The (unital) commutative operad:

$$\mathrm{Com}^{\mathrm{un}}(n) = *,\ n \geq 0.$$

$\mathrm{Com}^{\mathrm{un}}$-algebra = commutative monoid.

Example. The (non-unital) commutative operad:

$$\mathrm{Com}(n) = \begin{cases} *, & n \geq 1 \\ \emptyset, & n = 0. \end{cases}$$

Com-algebra = commutative semigroup.

Operads in vector spaces

What if we want $\Bbbk$-vector spaces equipped with operations?

Instead of working in $(\mathrm{Set}, \times)$, work in $(\mathrm{Vect}_\Bbbk, \otimes_\Bbbk)$.

$\rightsquigarrow$ Replace the Cartesian product $\times$ with the tensor product $\otimes_\Bbbk$ everywhere.

Definition. A (symmetric) **operad** in $\Bbbk$-vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ is a sequence of $\Bbbk$-vector spaces $\mathcal{P}(n)$ with a right action of the symmetric group Σ_n together with composition maps

$$\circ\colon \mathcal{P}(k) \otimes_\Bbbk \mathcal{P}(n_1) \otimes_\Bbbk \cdots \otimes_\Bbbk \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k)$$

and a unit $\eta \in \mathcal{P}(1)$ such that composition is associative, unital, and equivariant.

Warning. This forces operations $\mu \in \mathcal{P}(n)$ to be $\Bbbk$-multilinear.

Examples

Taking the $\mathbb{k}$-vector space spanned by an operad in sets yields an operad in $\mathbb{k}$-vector spaces.

Example. The (unital) associative operad:

$$\mathrm{As}^{\mathrm{un}}(n) = \mathbb{k}\Sigma_n,\ n \geq 0.$$

$\mathrm{As}^{\mathrm{un}}$-algebra = (unital) $\mathbb{k}$-algebra.

Example. The (non-unital) associative operad:

$$\mathrm{As}(n) = \begin{cases} \mathbb{k}\Sigma_n, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

As-algebra = (non-unital) $\mathbb{k}$-algebra.

Examples (cont'd)

Example. The (unital) commutative operad:

$$\mathrm{Com}^{\mathrm{un}}(n) = \mathbb{k},\ n \geq 0.$$

$\mathrm{Com}^{\mathrm{un}}$-algebra = (unital) commutative $\mathbb{k}$-algebra.

Example. The (non-unital) commutative operad:

$$\mathrm{Com}(n) = \begin{cases} \mathbb{k}, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

Com-algebra = (non-unital) commutative $\mathbb{k}$-algebra.

Examples (cont'd)

Example. The Lie operad Lie is generated by an operation

$$[-,-] \in \mathrm{Lie}(2) \quad \text{Lie bracket } [x,y]$$

subject to relations

$$\begin{cases} [-,-] = -[-,-]\cdot(12) \in \mathrm{Lie}(2) \\ [x,y] = -[y,x] \quad \text{skew-symmetry} \\ [-,[-,-]] + [-,[-,-]]\cdot(132) + [-,[-,-]]\cdot(123) = 0 \in \mathrm{Lie}(3) \\ [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \text{Jacobi identity.} \end{cases}$$

Lie-algebra = Lie algebra if $\mathrm{char}(\Bbbk) \neq 2$.

Warning. In the case $\mathrm{char}(\Bbbk) = 2$, Lie-algebra = "quasi-Lie algebra". Can't encode the equation $[x,x] = 0$ with an operad.

Quillen cohomology of $\mathcal{P}$-algebra

For the category $\mathcal{C} = \mathcal{P}\text{-Alg}$ of $\mathcal{P}$-algebras, the “package” has been worked out. Nice account in Loday–Vallette (2012).

Outline

Free $\mathcal{P}$-algebras

Setup: $\Bbbk$ = field of characteristic $p > 0$.

$\mathcal{P}$ = operad in $\Bbbk$-vector spaces that is *reduced*, i.e., $\mathcal{P}(0) = 0$.

Free $\mathcal{P}$-algebra functor (forget-of-free monad)

$$S(\mathcal{P})\colon \mathrm{Vect}_{\Bbbk} \to \mathrm{Vect}_{\Bbbk}$$
$$S(\mathcal{P})(V) = \bigoplus_{n\geq 1} \big(\mathcal{P}(n) \otimes V^{\otimes n}\big)_{\Sigma_n}$$

Example: Tensor algebra

Example. Associative operad:

$$\begin{aligned} S(\mathrm{As})(V) &= \bigoplus_{n\geq 1} \left(\mathrm{As}(n) \otimes V^{\otimes n}\right)_{\Sigma_n} \\ &= \bigoplus_{n\geq 1} \left(\Bbbk\Sigma_n \otimes V^{\otimes n}\right)_{\Sigma_n} \\ &\cong \bigoplus_{n\geq 1} V^{\otimes n} \\ &= T(V) \end{aligned}$$

the (non-unital) tensor algebra.

Example: Symmetric algebra

Example. Commutative operad:

$$\begin{aligned} S(\mathrm{Com})(V) &= \bigoplus_{n\geq 1} \big(\mathrm{Com}(n) \otimes V^{\otimes n}\big)_{\Sigma_n} \\ &= \bigoplus_{n\geq 1} \big(\Bbbk \otimes V^{\otimes n}\big)_{\Sigma_n} \\ &\cong \bigoplus_{n\geq 1} \mathrm{Sym}^n(V) \\ &= \mathrm{Sym}(V) \end{aligned}$$

the (non-unital) symmetric algebra.

Divided power algebras

Instead of taking coinvariants, take *invariants*. Norm map:

$$\begin{array}{ccc} \bigoplus_{n\geq 1} \left(\mathcal{P}(n) \otimes V^{\otimes n}\right)_{\Sigma_n} & \longrightarrow & \bigoplus_{n\geq 1} \left(\mathcal{P}(n) \otimes V^{\otimes n}\right)^{\Sigma_n} \\ \| & & \|\,\text{def} \\ S(\mathcal{P})(V) & & \Gamma(\mathcal{P})(V) \end{array}$$

Proposition. $\Gamma(\mathcal{P})\colon \mathrm{Vect}_{\Bbbk} \to \mathrm{Vect}_{\Bbbk}$ is a monad.

Definition. A **divided power $\mathcal{P}$-algebra** is an algebra for the monad $\Gamma(\mathcal{P})$.

$\Gamma(\mathcal{P})$-algebra $\approx$ $\mathcal{P}$-algebra + extra operations, usually non-additive.

Examples

Example. If $\mathrm{char}(\Bbbk) = 0$, then $S(\mathcal{P}) \xrightarrow{\cong} \Gamma(\mathcal{P})$.

$\rightsquigarrow$ No extra operations in that case.

Divided power operations are a positive characteristic phenomenon.

Theorem (Fresse 2000)**.**
1. Associative operad: $S(\mathrm{As}) \xrightarrow{\cong} \Gamma(\mathrm{As})$, so a $\Gamma(\mathrm{As})$-algebra is just an As-algebra, i.e., a $\Bbbk$-algebra.
2. Commutative operad: $\Gamma(\mathrm{Com})$-algebra = classical divided power algebra.
3. Lie operad: $\Gamma(\mathrm{Lie})$-algebra = restricted Lie algebra.

Results

Theorem (Ikonicoff 2020)**.** Equational presentation for $\Gamma(\mathcal{P})$-algebras.

$\rightsquigarrow$ List of operations satisfying a (big) list of equations.

Theorem (Dokas–F.–Ikonicoff)**.** Worked out the “package” for $\Gamma(\mathcal{P})$-algebras.

Reality check: Recovered the examples of classical divided power algebras ($\mathcal{P} = \mathrm{Com}$) and restricted Lie algebras ($\mathcal{P} = \mathrm{Lie}$).

Results (cont'd)

Comparison maps involving Quillen cohomology of $\Gamma(\mathcal{P})$-algebras.

A map of operads $f\colon \mathcal{P} \to \mathcal{Q}$ induces adjunctions:

$$\begin{array}{ccc}
\mathcal{P}\text{-Alg} & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & \mathcal{Q}\text{-Alg} \\
\text{free} \downarrow \uparrow \text{forget} & & \text{free} \downarrow \uparrow \text{forget} \\
\Gamma(\mathcal{P})\text{-Alg} & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & \Gamma(\mathcal{Q})\text{-Alg}.
\end{array}$$

$\rightsquigarrow$ Comparison maps between Quillen cohomology in those different categories.

Example. Inclusion of operads Lie $\to$ As.

Merci!