The monoidal fibered category of Beck modules

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Outline

Motivation: Quillen (co)homology

Beck modules

Tensor product of Beck modules

Simplicial Beck modules

André–Quillen (co)homology

- Cohomology theory for commutative rings.
- Developed by André and Quillen in the 1960s.
- Non-additive derived functors constructed using simplicial methods.
- Used to solve problems in commutative algebra and algebraic geometry.
- Makes sense for any algebraic structure.

Applications in topology

A sampler of applications in topology.

- Unstable Adams spectral sequence (Miller, Goerss).
- Realization and classification problems (Goerss–Hopkins–Miller, Blanc, Blanc–Dwyer–Goerss, F., Biedermann–Raptis–Stelzer).
- Higher homotopy operations (Baues–Blanc, Blanc–Johnson–Turner).
- Knot theory: Quillen homology of racks and quandles (Szymik, Berest).

Goals

Previous work (F. 2015): Comparing Quillen (co)homology in categories related by an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G.$$

The focus was on $\mathrm{HQ}_*(X)$ and $\mathrm{HQ}^*(X; M)$ for an object X.

Goals

1. Deal with $\mathrm{HQ}_*(X; M)$ for any coefficient module M.
   $\rightsquigarrow$ Need the tensor product of Beck modules.
2. Deal with a simplicial object $X_\bullet$ in $s\mathcal{C}$ and simplicial module $M_\bullet$ over it.

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Setup

Throughout, we will work with an "algebraic" category $\mathcal{C}$.

Definition. An **algebraic theory** is a small category $\mathcal{T}$ with finite products. A **model** for the theory $\mathcal{T}$ is a functor $M\colon \mathcal{T} \to \mathrm{Set}$ that preserves finite products.

Definition. A category is **algebraic** if it is equivalent to the category $\mathrm{Model}(\mathcal{T})$ of models for some algebraic theory $\mathcal{T}$.

Characterization

Theorem (Lawvere 1963 & more)**.** For a category $\mathcal{C}$, the following are equivalent.

1. $\mathcal{C}$ is algebraic.
2. $\mathcal{C}$ is cocomplete, has a set of finitely presentable projective generators, and is exact (in the sense of Barr).
3. $\mathcal{C}$ is a many-sorted finitary variety of algebras, a.k.a. "equational class".
4. $\mathcal{C}$ is the category of algebras for a finitary monad $T\colon \mathrm{Set}^S \to \mathrm{Set}^S$ for some set S.

Example. Your favorite algebraic structures: sets, monoids, groups, abelian groups, rings, commutative rings, R-modules, Lie algebras, chain complexes, DG-algebras, etc.

Beck modules

Definition (Beck 1967)**.** For an object X in $\mathcal{C}$, a **Beck module** over X is an abelian group object in the slice category $\mathcal{C}/X$.

The category of Beck modules is sometimes denoted

$$\mathrm{Mod}(X) := (\mathcal{C}/X)_{\mathrm{ab}}.$$

Definition. The **abelianization** over X

$$Ab_X \colon \mathcal{C}/X \to (\mathcal{C}/X)_{\mathrm{ab}}$$

is the left adjoint of the forgetful functor

$$U_X \colon (\mathcal{C}/X)_{\mathrm{ab}} \to \mathcal{C}/X.$$

Quillen (co)homology

Definition. Let X be an object of $\mathcal{C}$ and M a module over X.

- The **cotangent complex** $\mathbf{L}_X$ of X is the derived abelianization of X, i.e., the simplicial module over X given by

$$\mathbf{L}_X := Ab_X(C_\bullet \to X)$$

  where $C_\bullet \to X$ is a cofibrant replacement of X in $s\mathcal{C}$.
- **Quillen homology** of X is

$$\mathrm{HQ}_n(X) := \pi_n(\mathbf{L}_X).$$

  If the category $\mathrm{Mod}(X)$ has a good notion of tensor product $\otimes$, then Quillen homology with coefficients in M is

$$\mathrm{HQ}_n(X; M) := \pi_n(\mathbf{L}_X \otimes M).$$

- **Quillen cohomology** of X with coefficients in M is the derived functors of derivations:

$$\mathrm{HQ}^n(X; M) := \pi^n \operatorname{Hom}(\mathbf{L}_X, M).$$

Pullback and pushforward

Definition. The pullback functor $f^*: \mathcal{C}/Y \to \mathcal{C}/X$ induces a functor

$$f^*: \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$$

also called the **pullback**. Its left adjoint

$$f_!: \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$$

is called the **pushforward** along f.

pullback = "restriction of scalars"

pushforward = "extension of scalars"

Rings

$\mathcal{C} = \mathrm{Alg}_k$, the category of (associative, unital) k-algebras.

For a k-algebra A:

$$\mathrm{Mod}(A) \cong {}_A\mathrm{Bimod}_A.$$

A Beck module over A is a split extension of A with square zero kernel:

$$0 \longrightarrow M \longrightarrow A \oplus M \underset{s}{\overset{p}{\rightleftarrows}} A \longrightarrow 0.$$

The two actions on M are given by

$$(a, m)(a', m') = (aa', a \cdot m' + m \cdot a')$$

and they coincide for scalars in k.

Rings, cont'd

For a map of k-algebras $f\colon A \to B$, the pushforward functor is

$$\begin{aligned} f_! \colon \mathrm{Mod}(A) &\to \mathrm{Mod}(B) \\ f_!(M) &= B \otimes_A M \otimes_A B. \end{aligned}$$

The A-bimodule $Ab_A A$ is the kernel of the multiplication map:

$$Ab_A A = I_A := \ker(A \otimes_k A \xrightarrow{\mu} A).$$

Proposition (Barr 1967)**.** Quillen cohomology in Alg_k is (up to shift) Shukla cohomology, a.k.a. derived Hochschild cohomology:

$$\mathrm{HQ}^n(A; M) = \begin{cases} \mathrm{Der}_k(A, M) & n = 0 \\ H^{n+1}(A; M) & n > 0. \end{cases}$$

Commutative rings

$\mathcal{C} = \mathrm{Alg}_k$, the category of commutative k-algebras.

For a commutative k-algebra A:

$$\mathrm{Mod}(A) \cong \mathrm{Mod}_A \quad \text{in the usual sense.}$$

Same correspondence as for algebras, except that $A \oplus M$ must be commutative. This forces the two actions to coincide:

$$a \cdot m = m \cdot a.$$

Commutative rings, cont'd

For a map of commutative k-algebras $f\colon A \to B$, the pushforward functor is

$$f_!\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$
$$f_!(M) = B \otimes_A M.$$

The A-module $Ab_A A$ is:

$$Ab_A A = I_A / I_A^2 = \Omega_{A/k},$$

the module of Kähler differentials. It represents k-derivations:

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \cong \mathrm{Der}_k(A, M).$$

Groups

$\mathcal{C} = \text{Gp}$, the category of groups.

For a group G:

$$\begin{aligned}\text{Mod}(G) &\cong G - \text{Mod} \quad \text{in the usual sense} \\ &\cong \mathbb{Z}G - \text{Mod}.\end{aligned}$$

A Beck module over G is a split extension of G with abelian kernel:

$$1 \longrightarrow K \longrightarrow G \ltimes K \underset{e}{\overset{p}{\rightleftarrows}} G \longrightarrow 1.$$

The G-action on K is given by $e(g)k = (g, g \cdot k)$. In other words:

$$(g, k)(g', k') = (gg', k + g \cdot k').$$

Groups, cont'd

For a map of groups $f\colon G \to H$, the pushforward functor is

$$\begin{aligned} f_! \colon \mathrm{Mod}(G) &\to \mathrm{Mod}(H) \\ f_!(M) &= \mathbb{Z}H \otimes_{\mathbb{Z}G} M. \end{aligned}$$

The G-module $Ab_G G$ is the augmentation ideal:

$$Ab_G G = I_G = \ker(\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z}).$$

Proposition (Barr–Beck 1966)**.** Quillen cohomology in Gp is (up to a shift) group cohomology:

$$\mathrm{HQ}^n(G; M) = \begin{cases} \mathrm{Der}(G, M) & n = 0 \\ H^{n+1}(G; M) & n > 0 \end{cases}$$

where $\mathrm{Der}(G, M)$ denotes crossed homomorphisms $G \to M$.

Abelian groups

$\mathcal{C} = \mathrm{Ab}$, the category of abelian groups.

For an abelian group A:

$$\mathrm{Mod}(A) \cong \mathrm{Ab}.$$

Same correspondence as for groups, except that $A \ltimes K$ must be abelian. This forces the A-action on K to be trivial:

$$a \cdot k = k.$$

More generally:

Example (Beck 1967)**.** In an additive category $\mathcal{A}$ with finite limits, Beck modules over any object X are:

$$\begin{gathered} \mathrm{Mod}(X) \cong \mathcal{A} \\ (p\colon E \twoheadrightarrow X) \mapsto \ker(p). \end{gathered}$$

Fibered category

The assignment

$$\text{Mod}(-)\colon \mathcal{C}^{\text{op}} \to \text{AbCat}$$

sending an object X to its category of Beck modules $\text{Mod}(X)$ and a map $f\colon X \to Y$ to the pullback functor $f^*\colon \text{Mod}(Y) \to \text{Mod}(X)$ is a pseudo-functor: $(gf)^* \cong f^* g^*$.

Definition. The Grothendieck construction of the pseudo-functor $\text{Mod}(-)$ yields a fibered category

$$\pi\colon \text{Mod}\mathcal{C} \to \mathcal{C}$$

called the **fibered category of Beck modules** over $\mathcal{C}$, a.k.a. the **tangent category** of $\mathcal{C}$, denoted $T\mathcal{C} \to \mathcal{C}$.

An object of $\text{Mod}\mathcal{C}$ is (X, M), where M is a module over X.

Remark. ∞-categorical analogue using stabilization instead of abelianization (Lurie 2011, Harpaz–Nuiten–Prasma 2019).

Fibered category (cont'd)

Example. 1. For $\mathcal{A}$ additive with finite limits:

$$T\mathcal{A} \cong \mathcal{A} \times \mathcal{A}.$$

2. $T\mathrm{Gp} \cong \Pi\mathrm{Alg}_1^2$, the category of 2-truncated Π-algebras.
3. $T\mathrm{Alg}_k \cong \mathrm{grAlg}_k^{\leq 1}$, the category of graded algebras concentrated in degrees 0 and 1.
4. $T\mathrm{Com}_k \cong \mathrm{grCom}_k^{\leq 1}$.

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Common tensor products

- For a commutative ring R, the tensor product of R-modules is the usual tensor product $M \otimes_R N$.
- For a k-algebra A, the tensor product of A-bimodules is $M \otimes_A N$.
- For a group G, the tensor product of G-modules is $M \otimes_{\mathbb{Z}} N$ with diagonal action

$$g \cdot (m \otimes n) = (gm) \otimes (gn).$$

Goal

Find a categorical construction of the tensor product of Beck modules that recovers those examples (and more).

Remark. Because of the example of A-bimodules, don't expect a *symmetric* monoidal structure.

Commutative theories

Definition. An algebraic theory $\mathcal{T}$ is **commutative** if (roughly) every operation is a homomorphism of the algebraic structure.

Example.

1. The theory $\mathcal{T}_{\mathrm{Gp}}$ of groups is not commutative. The multiplication map
$$G \times G \xrightarrow{\mu} G$$
is a group homomorphism if and only if G is abelian.
2. The theory $\mathcal{T}_{\mathrm{Ab}}$ of abelian groups is commutative.
3. The theory $\mathcal{T}_{\mathrm{Com}}$ of commutative rings is not commutative. The addition map
$$R \times R \xrightarrow{+} R$$
is not a ring homomorphism.
4. For a given commutative ring R, the theory $\mathcal{T}_{\mathrm{Mod}_R}$ of R-modules is commutative.

Tensor product of models

Theorem (Keigher 1978)**.** If $\mathcal{T}$ is a commutative theory, then $\mathrm{Model}(\mathcal{T})$ admits a symmetric monoidal structure characterized by

$$\mathrm{Model}(\mathcal{T})(A \otimes B, C) \cong \mathcal{T} - \mathrm{Bihom}(A \times B, C).$$

Here, a bihomomorphism is a function $f\colon A \times B \to C$ that preserves the operations in each variable.

Example. For the theories $\mathcal{T}_{\mathrm{Ab}}$ and $\mathcal{T}_{\mathrm{Mod}_R}$, this recovers the usual tensor product.

Can be generalized to $\mathcal{V}$-valued models $\mathrm{Model}(\mathcal{T};\mathcal{V})$ where $\mathcal{V}$ is itself symmetric monoidal.

A naive approach

Can we tensor abelian group objects? For an object X in $\mathcal{C}$:

$$\begin{aligned}(\mathcal{C}/X)_{\text{ab}} &\cong \text{Model}(\mathcal{T}_{\text{Ab}}; \mathcal{C}/X) \\ &\cong \text{Model}\left(\mathcal{T}_{\text{Ab}}; \text{Model}(\mathcal{T}_{\mathcal{C}/X})\right) \\ &\cong \text{Model}(\mathcal{T}_{\text{Ab}} \otimes \mathcal{T}_{\mathcal{C}/X}; \text{Set}).\end{aligned}$$

The theory of Beck modules over X

$$\mathcal{T}_{\text{Mod}(X)} = \mathcal{T}_{\text{Ab}} \otimes \mathcal{T}_{\mathcal{C}/X}$$

is not commutative in general.

For a non-commutative theory, the tensor product of models still makes sense, but it imposes too many equations!

Too much commutativity

Example. "Over a non-commutative ring R, you shouldn't tensor two left R-modules." What if I want to:

$$\begin{aligned} M \boxtimes N &:= M \otimes_{\mathbb{Z}} N / \langle (rm) \otimes n - m \otimes (rn) \rangle \\ &= q^* \left((R_{\text{com}} \otimes_R M) \otimes_{R_{\text{com}}} (R_{\text{com}} \otimes_R N) \right) \end{aligned}$$

where $R_{\text{com}} = R/[R, R]$ and $q \colon R \twoheadrightarrow R_{\text{com}}$ is the quotient map.

A similar phenomenon happens with A-bimodules and G-modules.

Pointwise tensor product

For a group G:

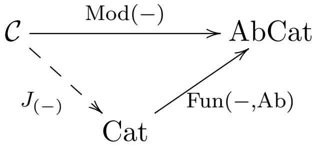
$$G - \mathrm{Mod} \cong \mathrm{Fun}(BG, \mathrm{Ab})$$

where BG denotes the one-object groupoid. The tensor product of G-modules agrees with the pointwise tensor product in $\mathrm{Fun}(BG, \mathrm{Ab})$.

Definition. A category $\mathcal{C}$ has **representable Beck modules** if for all object X in $\mathcal{C}$:

$$\mathrm{Mod}(X) \cong \mathrm{Fun}(J_X, \mathrm{Ab})$$

for some small category J_X, pseudo-functorial in X.



Pointwise tensor product (cont'd)

Example.

- Sets, groups, abelian groups, and monoids have representable Beck modules.
- Rings and commutative rings do not have representable Beck modules.

Desiderata

- If $\mathcal{C}$ has Beck modules $\mathrm{Mod}(X) \cong \mathrm{Fun}(J_X, \mathrm{Ab})$, expect the tensor product to be the pointwise tensor product.
- If $\mathcal{C}$ has Beck modules $\mathrm{Mod}(X) \cong \mathrm{Mod}_{R_X}$ for some (pseudo-functorial) commutative ring R_X, expect the tensor product to be the usual tensor product $M \otimes_{R_X} N$.

Monoidal properties

For commutative rings, extension of scalars $f_! M = S \otimes_R M$ is strong monoidal and (hence) restriction of scalars f^* is lax monoidal.

Don't expect pushforwards $f_! \colon \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$ to be strong monoidal in general, since this is not the case for A-bimodules and G-modules.

Other features to look for:

- Projection formula?

$$f_!(f^* M \otimes N) \to M \otimes f_! N$$

- Wirthmüller context?

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Quillen model structure

Quillen constructed a standard model structure on simplicial objects $s\mathcal{C}$. A map $f_\bullet \colon X_\bullet \to Y_\bullet$ in $s\mathcal{C}$ is a:

- *fibration* (resp. *weak equivalence*) if for every projective object P of $\mathcal{C}$, the map:

$$\mathrm{Hom}_{\mathcal{C}}(P, X_\bullet) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(P, Y_\bullet)$$

  is a fibration (resp. weak equivalence) of simplicial sets.
- *cofibration* if it has the left lifting property with respect to all trivial fibrations.

More concretely: For $\mathcal{C}$ an algebraic category, the model structure is right-induced along the forgetful functor

$$U \colon s\mathcal{C} \to s(\mathrm{Set}^S) = (s\mathrm{Set})^S.$$

For instance, a simplicial ring has an underlying simplicial set.

Nice simplicial objects

Definition. A complete and cocomplete category $\mathcal{C}$ **has nice simplicial objects** if $s\mathcal{C}$ admits Quillen's standard model structure.

Theorem (Quillen 1967)**.** Any quasi-algebraic category has nice simplicial objects.

Proposition. If $\mathcal{C}$ has nice simplicial objects and $s\mathcal{C}$ is cofibrantly generated, then $T\mathcal{C}$ has nice simplicial objects.

Homotopy theory of simplicial modules

Proposition. The category of Beck modules over a simplicial object $X_\bullet$ in $s\mathcal{C}$

$$\mathrm{Mod}(X_\bullet) = (s\mathcal{C}/X_\bullet)_{\mathrm{ab}}$$

admits the model structure right-induced along the forgetful functor

$$U_{X_\bullet} \colon (s\mathcal{C}/X_\bullet)_{\mathrm{ab}} \to s\mathcal{C}/X_\bullet.$$

Recovers the model structure on simplicial modules over a simplicial commutative ring $R_\bullet$ (Quillen 1967, Schwede 1997).

Lemma. There is an equivalence of categories $sT\mathcal{C} \cong T(s\mathcal{C})$ exhibiting $sT\mathcal{C}$ as the tangent category of $\mathcal{C}$:

$$\begin{array}{ccc} sT\mathcal{C} & \overset{\cong}{\longrightarrow} & T(s\mathcal{C}) \\ {\scriptstyle s\pi_{\mathcal{C}}}\big\downarrow & \swarrow{\scriptstyle \pi_{s\mathcal{C}}} & \\ s\mathcal{C}. & & \end{array}$$

Simplicial modules (cont'd)

More explicitly: A module over $X_\bullet$ is the same as a module M_n over X_n for each $n \geq 0$ together with face maps $d_i \colon M_n \to M_{n-1}$ that are maps of modules over the face maps $d_i \colon X_n \to X_{n-1}$, and likewise for degeneracies.

Lemma. The standard model structure on $sT\mathcal{C}$ restricts to each fiber $(s\mathcal{C}/X_\bullet)_{\mathrm{ab}}$ to the model structure induced from that of $s\mathcal{C}/X_\bullet$.

Proposition. Under the identification $sT\mathcal{C} \cong T(s\mathcal{C})$, the standard model structure on $sT\mathcal{C}$ corresponds to the integral model structure on $T(s\mathcal{C})$ in the sense of Harpaz–Prasma (2015).

Future steps

Develop tools to compute $\mathrm{HQ}_*(X_\bullet; M_\bullet)$ and $\mathrm{HQ}^*(X_\bullet; M_\bullet)$ analogous to Quillen's work:

- Transitivity sequence
- Flat base change
- Künneth and universal coefficient spectral sequences.

Thank you!