Quillen cohomology of divided power algebras over an operad

Martin Frankland
University of Regina

Joint with Ioannis Dokas and Sacha Ikonicoff

Prairie Mathematics Colloquium
November 7, 2024

Outline

Introduction

Quillen cohomology

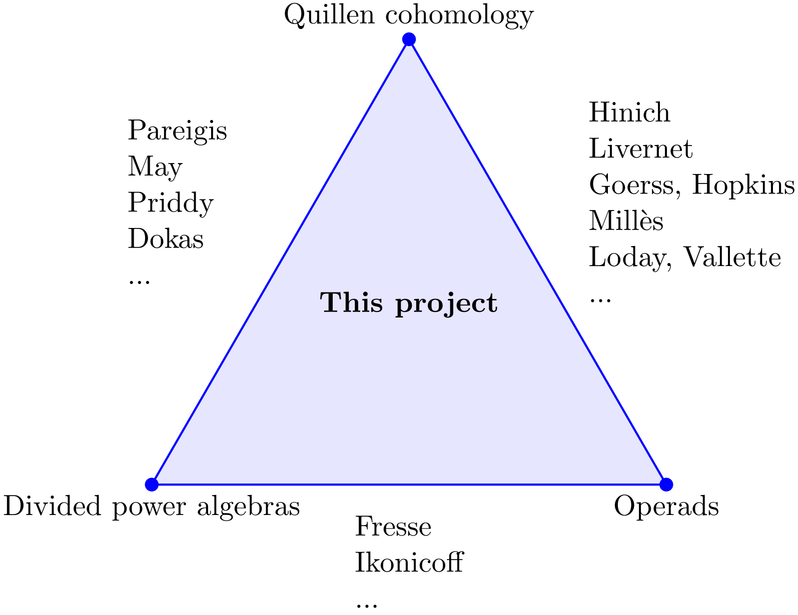
Divided power algebras (classical)

Restricted Lie algebras

Operads

Divided power algebras (operadic)

Overview


Quillen cohomology
Pareigis
May
Priddy
Dokas
...
Hinich
Livernet
Goerss, Hopkins
Millès
Loday, Vallette
...
This project
Divided power algebras
Operads
Fresse
Ikonicoff
...

André–Quillen (co)homology

- Cohomology theory for commutative rings.
- Developed by André and Quillen in the 1960s.
- Non-additive derived functors constructed using simplicial methods.
- Used to solve problems in commutative algebra and algebraic geometry.
- Makes sense for any algebraic structure.

Applications in topology

A sampler of applications in topology.

- Unstable Adams spectral sequence (Miller, Goerss).
- Obstruction theory for ring spectra (Goerss–Hopkins–Miller, Lurie)
- Realization and classification problems (Blanc, Blanc–Dwyer–Goerss, F., Biedermann–Raptis–Stelzer).
- Higher homotopy operations (Baues–Blanc, Blanc–Johnson–Turner).
- Knot theory: Quillen homology of racks and quandles (Szymik, Berest).

Outline

Cohomology in algebra

Cohomology theories for algebraic structures:

- Group cohomology
- Lie algebra cohomology
- Hochschild cohomology of associative algebras
- André–Quillen cohomology of commutative rings
- etc.

Unified approach: Barr–Beck triple cohomology, based on simplicial resolutions (1969).

Put into the framework of model categories by Quillen (1967).

Idea: cohomology $\approx$ derived functors of derivations.

Beck modules

Setup: An "algebraic" category $\mathcal{C}$, e.g., groups, abelian groups, rings, commutative rings, R-modules, Lie algebras, etc.

Need a good notion of *coefficient module* M over X:

$$H^*(X; M).$$

Definition (Beck 1967)**.** For an object X in a category $\mathcal{C}$, a **Beck module** over X is an abelian group object in the slice category $\mathcal{C}/X$.

Denote the category of Beck modules over X

$$\mathrm{Mod}(X) \coloneqq (\mathcal{C}/X)_{\mathrm{ab}}.$$

Pullback and pushforward

Definition. The pullback functor $f^*\colon \mathcal{C}/Y \to \mathcal{C}/X$ induces a functor

$$f^*\colon \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$$

also called the **pullback**.

Its left adjoint

$$f_!\colon \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$$

is called the **pushforward** along f.

$$\text{pullback} = \text{“restriction of scalars”}$$

$$\text{pushforward} = \text{“extension of scalars”}$$

Groups

$\mathcal{C} = \mathrm{Gp}$, the category of groups.

For a group G:

$$\begin{aligned} \mathrm{Mod}(G) &\cong G\text{-modules in the usual sense} \\ &\cong \mathbb{Z}G\text{-Mod}. \end{aligned}$$

A Beck module over G is a split extension of G with abelian kernel:

$$1 \longrightarrow K \longrightarrow G \ltimes K \underset{e}{\overset{p}{\rightleftarrows}} G \longrightarrow 1.$$

The G-action on K is given by $e(g)k = (g, g \cdot k)$. In other words:

$$(g, k)(g', k') = (gg', k + g \cdot k').$$

Groups, cont'd

For a map of groups $f\colon G \to H$, the pushforward functor is

$$\begin{aligned} f_! \colon \mathrm{Mod}(G) &\to \mathrm{Mod}(H) \\ f_!(M) &= \mathbb{Z}H \otimes_{\mathbb{Z}G} M. \end{aligned}$$

Rings

$\mathcal{C} = \mathrm{Alg}_{\Bbbk}$, the category of (unital) algebras over a commutative ring $\Bbbk$.

For a $\Bbbk$-algebra A:

$$\mathrm{Mod}(A) \cong {}_A\mathrm{Bimod}_A.$$

A Beck module over A is a split extension of A with square zero kernel:

$$0 \longrightarrow M \longrightarrow A \oplus M \underset{s}{\overset{p}{\rightleftarrows}} A \longrightarrow 0.$$

The two actions on M are given by

$$(a, m)(a', m') = (aa', a \cdot m' + m \cdot a')$$

and they coincide for scalars in $\Bbbk$.

Rings, cont'd

For a map of $\Bbbk$-algebras $f\colon A \to B$, the pullback functor

$$f^*\colon \mathrm{Mod}(B) \to \mathrm{Mod}(A)$$

is the usual restriction of scalars:

$$a \cdot m \cdot a' \coloneqq f(a) \cdot m \cdot f(a').$$

The pushforward functor is

$$\begin{aligned} f_!\colon \mathrm{Mod}(A) &\to \mathrm{Mod}(B) \\ f_!(M) &= B \otimes_A M \otimes_A B. \end{aligned}$$

Commutative rings

$\mathcal{C} = \mathrm{Com}_{\Bbbk}$, the category of commutative $\Bbbk$-algebras.

For a commutative $\Bbbk$-algebra A:

$$\mathrm{Mod}(A) \cong \mathrm{Mod}_A \quad \text{in the usual sense.}$$

Same correspondence as for algebras, except that $A \oplus M$ must be commutative. This forces the two actions to coincide:

$$a \cdot m = m \cdot a.$$

For a map of commutative k-algebras $f \colon A \to B$, the pushforward functor is extension of scalars:

$$f_! \colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$
$$f_!(M) = B \otimes_A M.$$

Remark. The notion of Beck module depends on the ambient category! A Beck module over A in $\mathrm{Com}_{\Bbbk}$ is **not** the same as in $\mathrm{Alg}_{\Bbbk}$.

Lie algebras

$\Bbbk$ = a field.

$\mathcal{C} = \mathrm{Lie}_{\Bbbk}$, the category of Lie algebras over $\Bbbk$.

For a Lie algebra L:

$$\begin{aligned} \mathrm{Mod}(L) &\cong L\text{-modules in the usual sense} \\ &\cong U(L)\text{-Mod} \end{aligned}$$

where $U(L)$ is the universal enveloping algebra of L.

A Beck module over L is a split extension of L with abelian kernel:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & L \oplus M \arrow[r, shift left, "p"] & L \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

with $[m, m'] = 0$ for all $m, m' \in M$.

Lie algebras (cont'd)

The action of L on M is given by

$$[(\ell, 0), (0, m)] = (0, \ell \cdot m),$$

which satisfies

$$[\ell, \ell'] \cdot m = \ell \cdot (\ell' \cdot m) - \ell' \cdot (\ell \cdot m).$$

Universal enveloping algebra

Usually, the category of Beck modules over X is equivalent to (left) modules over a ring $\mathbb{U}_{\mathcal{C}}(X)$, called the **universal enveloping algebra** of X:

$$\mathrm{Mod}(X) \cong \mathbb{U}_{\mathcal{C}}(X)\text{-Mod}.$$

Examples of universal enveloping algebras:

1. For a group G, the group ring $\mathbb{U}_{\mathrm{Gp}}(G) = \mathbb{Z}G$.
2. For a ring A, the enveloping ring $\mathbb{U}_{\mathrm{Alg}_{\mathbb{Z}}}(A) = A \otimes A^{\mathrm{op}}$.
3. For a commutative ring R, the ring itself $\mathbb{U}_{\mathrm{Com}_{\mathbb{Z}}}(R) = R$.
4. For a Lie algebra L over $\Bbbk$, the (classical) universal enveloping algebra $\mathbb{U}_{\mathrm{Lie}_{\Bbbk}}(L) = U(L)$.

Derivations and differentials

Definition. A **derivation** of X with coefficients in a Beck module $p\colon E \to X$ is a section of p:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, equal] & E \arrow[d, "p"] \\ & X. \end{tikzcd}$$

The abelian group of derivations is

$$\begin{aligned} \mathrm{Der}(X, E) &\coloneqq \mathrm{Hom}_{\mathcal{C}/X}(X \xrightarrow{\mathrm{id}} X, E \xrightarrow{p} X) \\ &\cong \mathrm{Hom}_{(\mathcal{C}/X)_{\mathrm{ab}}}(Ab_X(X \xrightarrow{\mathrm{id}} X), E \xrightarrow{p} X), \end{aligned}$$

where $Ab_X\colon \mathcal{C}/X \to (\mathcal{C}/X)_{\mathrm{ab}}$ is the **abelianization** functor, i.e., the left adjoint to the forgetful functor $(\mathcal{C}/X)_{\mathrm{ab}} \to \mathcal{C}/X$.

The module of **Kähler differentials** is $\Omega_{\mathcal{C}}(X) = Ab_X X$, which represents derivations.

Example: Commutative rings

Take $\mathcal{C} = \mathrm{Com}_{\Bbbk}$. Given a commutative $\Bbbk$-algebra A and an A-module M, a $\Bbbk$-linear map

$$(\mathrm{id}, f) \colon A \to A \oplus M$$

is a (unital) ring homomorphism if and only if

$$\begin{cases} f(1_A) = 0 \\ f(ab) = a \cdot f(b) + f(a) \cdot b \quad \rightsquigarrow \text{Leibniz rule.} \end{cases}$$

$$\mathrm{Der}(A, M) = \text{derivations in the usual sense.}$$

The A-module $Ab_A A$ is:

$$Ab_A A = I_A / I_A^2 = \Omega_{A/k},$$

where $I_A = \ker(\mu \colon A \otimes_{\Bbbk} A \to A)$. Classical module of Kähler differentials, which represents k-derivations:

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \cong \mathrm{Der}_k(A, M).$$

Geometric interpretation

Analogy:

$$\begin{array}{c}\text{Kähler differentials in algebraic geometry}\\ \Big\updownarrow\\ \text{differential 1-forms in differential geometry}\end{array}$$

Example. 1. $A = \Bbbk[x, y]$

$$\begin{aligned}\Omega_{A/\Bbbk} &\cong \{p\,dx + q\,dy \mid p, q \in \Bbbk[x, y]\}\\ &\cong A\,\langle dx, dy\rangle \quad \text{free } A\text{-module.}\end{aligned}$$

2. $A = \Bbbk[x, y]/(f)$

$$\Omega_{A/\Bbbk} \cong A\,\langle dx, dy\rangle \,/\left\langle \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \right\rangle.$$

Quillen (co)homology

Quillen homology of $X \approx$ (simplicially) derived functors of Kähler differentials.

Quillen cohomology of X with coefficients in a module $M \approx$ (simplicially) derived functors of derivations with coefficients in M.

The "package":

1. Beck modules
2. Universal enveloping algebra
3. Derivations
4. Kähler differentials

Outline

Divided power algebras

Introduced by Cartan in the 1950s. Appear in positive characteristic.

Divided power operations $\approx$ operations γ_n that look like $\gamma_n(x) = \frac{x^n}{n!}$. More precisely...

Divided power algebras (cont'd)

Definition. Let A be a commutative $\Bbbk$-algebra. A **system of divided powers** on an ideal $I \subseteq A$ is a collection of maps $\gamma_i \colon I \to A$ for $i \geq 0$ satisfying:

$$\begin{aligned}
\gamma_0(a) &= 1 \\
\gamma_1(a) &= a \\
\gamma_i(a) &\in I,\ i \geq 1 \\
\gamma_i(a+b) &= \sum_{k=0}^{k=i} \gamma_k(a)\gamma_{i-k}(b),\ a,b \in I,\ i \geq 0 \\
\gamma_i(ab) &= a^i \gamma_i(b),\ a \in A,\ i \geq 0 \\
\gamma_i(a)\gamma_j(a) &= \frac{(i+j)!}{i!j!}\gamma_{i+j}(a),\ a \in I,\ i,j \geq 0 \\
\gamma_i(\gamma_j(a)) &= \frac{(ij)!}{i!(j!)^i}\gamma_{ij}(a),\ a \in I,\ i \geq 0,\ j \geq 1.
\end{aligned}$$

Usually A has an augmentation $\epsilon \colon A \to \Bbbk$ and $I = \ker(\epsilon)$.

Examples

Example. The free divided power algebra over $\mathbb{Z}$ on one generator x is

$$\mathbb{Z}[x, \frac{x^2}{2}, \dots, \frac{x^n}{n!}, \dots] \subset \mathbb{Q}[x].$$

Example. Over a field $\Bbbk$ of characteristic 0, divided powers must in fact be given by

$$\gamma_n(x) = \frac{x^n}{n!}.$$

$\rightsquigarrow$ No additional structure in that case.

In positive characteristic

Proposition (Soublin 1987)**.** Over a field $\Bbbk$ of characteristic $p > 0$, a system of divided power operations is the same data as just the operation $\pi = \gamma_p \colon I \to I$ satisfying:

$$\begin{aligned} a^p &= 0,\ a \in I \\ \pi(a+b) &= \pi(a) + \pi(b) + \sum_{k=1}^{k=p-1} \frac{(-1)^k}{k} a^k b^{p-k},\ a, b \in I \\ \pi(ab) &= 0,\ a, b \in I \\ \pi(\lambda a) &= \lambda^p \pi(a),\ a \in I, \lambda \in \Bbbk. \end{aligned}$$

Note: The operation π is non-additive!

Theorem (Dokas 2009)**.** Worked out the "package" for divided power algebras over a field $\Bbbk$ of characteristic $p > 0$.

Outline

Restricted Lie algebras

Over a field $\Bbbk$ of characteristic $p > 0$, Lie algebras usually come with extra structure, that of a *restricted* Lie algebra.

Textbook account in Jacobson (1962).

Important in topology. See work of Milnor–Moore, May, Rector, Bousfield, Curtis, Priddy, etc. in the 1960s.

Restricted Lie algebras (cont'd)

Definition. A **restricted Lie algebra** L over $\Bbbk$ is a Lie algebra together with a map $(-)^{[p]}\colon L \to L$ called the p-map satisfying:

$$(\alpha x)^{[p]} = \alpha^p \, x^{[p]}, \ \alpha \in \Bbbk$$

$$[x, y^{[p]}] = [\cdots [x, \underbrace{y], y], \cdots, y}_{p}]$$

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$$

where $is_i(x,y)$ is the coefficient of λ^{i-1} in $\mathrm{ad}_{\lambda x+y}^{p-1}(x)$.

Here $\mathrm{ad}_x\colon L \to L$ denotes the adjoint representation

$$\mathrm{ad}_x(y) \coloneqq [y,x], \ x, y \in L.$$

Example. For $p = 2$, the last equation becomes:

$$(x+y)^{[2]} = x^{[2]} + [x,y] + y^{[2]}.$$

Examples

Example. For a $\Bbbk$-algebra A, the underlying restricted Lie algebra of A has the commutator bracket

$$[x, y] = xy - yx$$

and the p^{th} power map

$$x^{[p]} = x^p.$$

Example. The free restricted Lie algebra on a $\Bbbk$-vector space V is the subspace

$$L^r(V) \subseteq T(V)$$

obtained by taking V and closing it under commutators and p^{th} powers.

Theorem (Dokas 2004)**.** Worked out the "package" for restricted Lie algebras over a field $\Bbbk$ of characteristic $p > 0$.

Outline

Operads in sets

Idea: Encode a type of algebraic structure as

$$\mathcal{P}(n) = \{\text{the allowed } n\text{-ary operations}\}.$$

Definition. A (symmetric) **operad** in sets $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ is a sequence of sets $\mathcal{P}(n)$ with a right action of the symmetric group Σ_n together with composition maps

$$\circ \colon \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k)$$

and a unit $\eta \in \mathcal{P}(1)$ such that composition is associative, unital, and equivariant.

Algebras over operads

Example. For any set X, the **endomorphism operad** of X is:

$$\begin{cases} \mathrm{End}_X(n) \coloneqq \mathrm{Hom}(X^n, X) \\ \circ = \text{ usual composition} \\ \eta = \mathrm{id}_X \in \mathrm{Hom}(X, X) \\ \Sigma_n \text{ acts by permuting the factors of } X^n. \end{cases}$$

Definition. A **$\mathcal{P}$-algebra** is a set X equipped with action maps

$$\alpha_n \colon \mathcal{P}(n) \to \mathrm{Hom}(X^n, X)$$

compatible with composition, unit, and equivariance. In other words, an operad map

$$\alpha \colon \mathcal{P} \to \mathrm{End}_X .$$

Each n-ary operation symbol $\mu \in \mathcal{P}(n)$ yields an actual n-ary operation on X.

Examples

Example. The (unital) associative operad:

$$\mathrm{As}^{\mathrm{un}}(n) = \Sigma_n,\ n \geq 0.$$

$\mathrm{As}^{\mathrm{un}}$-algebra = monoid.

Example. The (non-unital) associative operad:

$$\mathrm{As}(n) = \begin{cases} \Sigma_n, & n \geq 1 \\ \emptyset, & n = 0. \end{cases}$$

As-algebra = semigroup.

Examples (cont'd)

Example. The (unital) commutative operad:

$$\mathrm{Com}^{\mathrm{un}}(n) = *,\ n \geq 0.$$

$\mathrm{Com}^{\mathrm{un}}$-algebra = commutative monoid.

Example. The (non-unital) commutative operad:

$$\mathrm{Com}(n) = \begin{cases} *, & n \geq 1 \\ \emptyset, & n = 0. \end{cases}$$

Com-algebra = commutative semigroup.

Operads in vector spaces

What if we want $\Bbbk$-vector spaces equipped with operations?

Instead of working in $(\mathrm{Set}, \times)$, work in $(\mathrm{Vect}_{\Bbbk}, \otimes_{\Bbbk})$.

$\rightsquigarrow$ Replace the Cartesian product $\times$ with the tensor product $\otimes_{\Bbbk}$ everywhere.

Definition. A (symmetric) **operad** in $\Bbbk$-vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ is a sequence of $\Bbbk$-vector spaces $\mathcal{P}(n)$ with a right action of the symmetric group Σ_n together with composition maps

$$\circ\colon \mathcal{P}(k) \otimes_{\Bbbk} \mathcal{P}(n_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k)$$

and a unit $\eta \in \mathcal{P}(1)$ such that composition is associative, unital, and equivariant.

Warning. This forces operations $\mu \in \mathcal{P}(n)$ to be $\Bbbk$-multilinear.

Examples

Taking the $\mathbb{k}$-vector space spanned by an operad in sets yields an operad in $\mathbb{k}$-vector spaces.

Example. The (unital) associative operad:

$$\mathrm{As}^{\mathrm{un}}(n) = \mathbb{k}\Sigma_n,\ n \geq 0.$$

$\mathrm{As}^{\mathrm{un}}$-algebra = (unital) $\mathbb{k}$-algebra.

Example. The (non-unital) associative operad:

$$\mathrm{As}(n) = \begin{cases} \mathbb{k}\Sigma_n, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

As-algebra = (non-unital) $\mathbb{k}$-algebra.

Examples (cont'd)

Example. The (unital) commutative operad:

$$\mathrm{Com}^{\mathrm{un}}(n) = \mathbb{k},\ n \geq 0.$$

$\mathrm{Com}^{\mathrm{un}}$-algebra = (unital) commutative $\mathbb{k}$-algebra.

Example. The (non-unital) commutative operad:

$$\mathrm{Com}(n) = \begin{cases} \mathbb{k}, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

Com-algebra = (non-unital) commutative $\mathbb{k}$-algebra.

Examples (cont'd)

Example. The Lie operad Lie is generated by an operation

$$[-,-] \in \mathrm{Lie}(2) \quad \text{Lie bracket } [x,y]$$

subject to relations

$$\begin{cases} [-,-] = -[-,-]\cdot(12) \in \mathrm{Lie}(2) \\ [x,y] = -[y,x] \quad \text{skew-symmetry} \\ [-,[-,-]] + [-,[-,-]]\cdot(132) + [-,[-,-]]\cdot(123) = 0 \in \mathrm{Lie}(3) \\ [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \text{Jacobi identity.} \end{cases}$$

Lie-algebra = Lie algebra if $\mathrm{char}(\Bbbk) \neq 2$.

Warning. In the case $\mathrm{char}(\Bbbk) = 2$, Lie-algebra = "quasi-Lie algebra". Can't encode the equation $[x,x] = 0$ with an operad.

Quillen cohomology of $\mathcal{P}$-algebra

For the category $\mathcal{C} = \mathcal{P}$-Alg of $\mathcal{P}$-algebras, the "package" has been worked out. Nice account in Loday–Vallette (2012).

Outline

Free $\mathcal{P}$-algebras

Setup: $\Bbbk$ = field of characteristic $p > 0$.

$\mathcal{P}$ = operad in $\Bbbk$-vector spaces that is *reduced*, i.e., $\mathcal{P}(0) = 0$.

Free $\mathcal{P}$-algebra functor (forget-of-free monad)

$$\begin{aligned} S(\mathcal{P})&\colon \mathrm{Vect}_{\Bbbk} \to \mathrm{Vect}_{\Bbbk} \\ S(\mathcal{P})(V) &= \bigoplus_{n\geq 1} \big(\mathcal{P}(n) \otimes V^{\otimes n}\big)_{\Sigma_n} \end{aligned}$$

Example: Tensor algebra

Example. Associative operad:

$$
\begin{aligned}
S(\mathrm{As})(V) &= \bigoplus_{n\geq 1} \left(\mathrm{As}(n) \otimes V^{\otimes n}\right)_{\Sigma_n} \\
&= \bigoplus_{n\geq 1} \left(\Bbbk\Sigma_n \otimes V^{\otimes n}\right)_{\Sigma_n} \\
&\cong \bigoplus_{n\geq 1} V^{\otimes n} \\
&= T(V)
\end{aligned}
$$

the (non-unital) tensor algebra.

Example: Symmetric algebra

Example. Commutative operad:

$$\begin{aligned} S(\mathrm{Com})(V) &= \bigoplus_{n\geq 1} \left(\mathrm{Com}(n) \otimes V^{\otimes n}\right)_{\Sigma_n} \\ &= \bigoplus_{n\geq 1} \left(\Bbbk \otimes V^{\otimes n}\right)_{\Sigma_n} \\ &\cong \bigoplus_{n\geq 1} \mathrm{Sym}^n(V) \\ &= \mathrm{Sym}(V) \end{aligned}$$

the (non-unital) symmetric algebra.

Divided power algebras

Instead of taking coinvariants, take *invariants*. Norm map:

$$
\begin{tikzcd}
\bigoplus_{n\geq 1} \left(\mathcal{P}(n)\otimes V^{\otimes n}\right)_{\Sigma_n} \arrow[r] \arrow[d, equal] & \bigoplus_{n\geq 1} \left(\mathcal{P}(n)\otimes V^{\otimes n}\right)^{\Sigma_n} \arrow[d, equal, "\text{def}"] \\
S(\mathcal{P})(V) & \Gamma(\mathcal{P})(V)
\end{tikzcd}
$$

Proposition. $\Gamma(\mathcal{P})\colon \mathrm{Vect}_{\Bbbk} \to \mathrm{Vect}_{\Bbbk}$ is a monad.

Definition. A **divided power $\mathcal{P}$-algebra** is an algebra for the monad $\Gamma(\mathcal{P})$.

$\Gamma(\mathcal{P})$-algebra $\approx$ $\mathcal{P}$-algebra + extra operations, usually non-additive.

Examples

Example. If $\mathrm{char}(\Bbbk) = 0$, then $S(\mathcal{P}) \xrightarrow{\cong} \Gamma(\mathcal{P})$. No extra operations in that case.

$\rightsquigarrow$ Divided power operations are a *positive characteristic* phenomenon.

Theorem (Fresse 2000)**.**

1. Associative operad: $S(\mathrm{As}) \xrightarrow{\cong} \Gamma(\mathrm{As})$, so a $\Gamma(\mathrm{As})$-algebra is just an As-algebra, i.e., a $\Bbbk$-algebra.
2. Commutative operad: $\Gamma(\mathrm{Com})$-algebra = classical divided power algebra.
3. Lie operad: $\Gamma(\mathrm{Lie})$-algebra = restricted Lie algebra.

Results

Theorem (Ikonicoff 2020)**.** Equational presentation for $\Gamma(\mathcal{P})$-algebras.

$\rightsquigarrow$ List of operations satisfying a (big) list of equations.

Theorem (Dokas–F.–Ikonicoff)**.** Worked out the "package" for $\Gamma(\mathcal{P})$-algebras.

Reality check: Recovered the examples of classicial divided power algebras ($\mathcal{P} = \mathrm{Com}$) and restricted Lie algebras ($\mathcal{P} = \mathrm{Lie}$).

Comparison maps involving Quillen cohomology of $\Gamma(\mathcal{P})$-algebras.

Thank you!