The Euler characteristic

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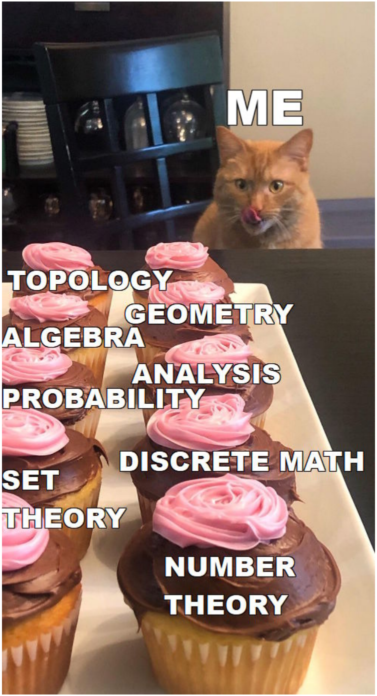
Pi Day
University of Regina
March 14, 2025


ME
TOPOLOGY
GEOMETRY
ALGEBRA
ANALYSIS
PROBABILITY
DISCRETE MATH
SET
THEORY
NUMBER
THEORY

Outline

Discrete math

Topology

Analysis

Differential geometry

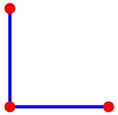
Graphs

Rules of the game:

1. Draw a connected graph on a piece of paper.
2. Count:

   $V = \#$ Vertices
   $E = \#$ Edges
   $F = \#$ Faces (enclosed regions), including the "outer face".
3. Compute $\chi = V - E + F$.

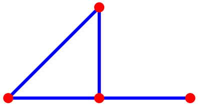
Example.



$$V = 3, \quad E = 2, \quad F = 1 \text{ (the outer face)}$$
$$\implies \chi = 3 - 2 + 1 = \boxed{2}.$$

More graphs

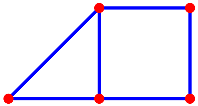
Example.



$$V = 4, \quad E = 4, \quad F = 2$$

$$\Longrightarrow \chi = 4 - 4 + 2 = \boxed{2}.$$

Example.



$$V = 5, \quad E = 6, \quad F = 3$$

$$\Longrightarrow \chi = 5 - 6 + 3 = \boxed{2}.$$

Theorem (Euler's formula)**.** For any connected planar graph:

$$\boxed{\chi = 2}.$$

Polyhedra

Definition. The **Euler characteristic** of a polyhedron is

$$\boxed{\chi = V - E + F}$$

where

$$\begin{cases} V = \#\,\text{Vertices} \\ E = \#\,\text{Edges} \\ F = \#\,\text{Faces.} \end{cases}$$

Example: Regular polyhedra

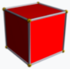
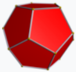
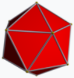
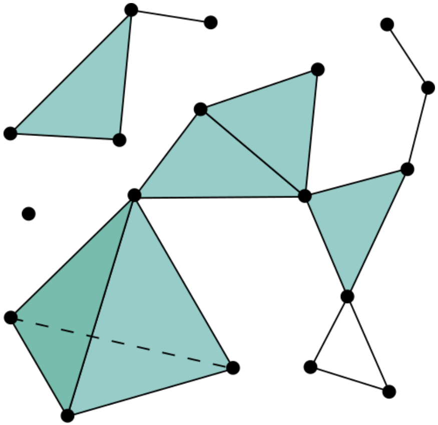
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Image credit: Wikipedia.

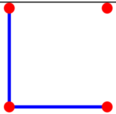
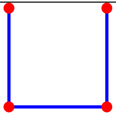
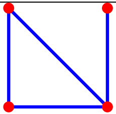
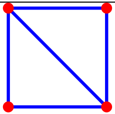
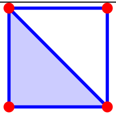
Simplicial complexes

Definition. A (geometric) **simplicial complex** is a union of simplices that intersect in common faces.



A 3-dimensional simplicial complex. Image credit: Wikipedia.

Examples

Complex					
V	4	4	4	4	4
E	2	3	4	5	5
F	0	0	0	0	1
χ	2	1	0	-1	0

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Betti numbers

Question. What is the Euler characteristic χ measuring?

$\rightsquigarrow$ Topological invariant.

Definition. The n^{th} **Betti number** of a space X is

$$b_n(X) \coloneqq \dim H_n(X; \mathbb{Q}).$$

$\approx$ number of "n-dimensional holes" in X.

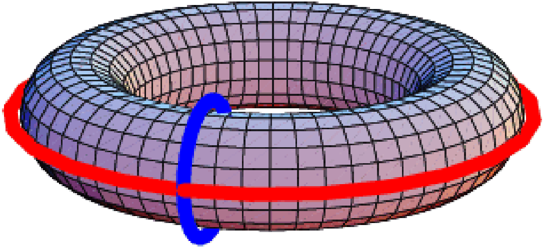
Example. $b_0 =$ number of path components.

$b_1 \approx$ number of "different loops".

Definition. The **Euler characteristic** of a space X is the alternating sum of the Betti numbers:

$$\chi(X) = b_0(X) - b_1(X) + b_2(X) - \cdots = \sum_n (-1)^n b_n(X).$$

Example: Torus



The torus T^2 has Betti numbers

$$b_0 = 1, \quad b_1 = 2, \quad b_2 = 1$$

and Euler characteristic

$$\chi(T^2) = 1 - 2 + 1 = \boxed{0}.$$

Surfaces

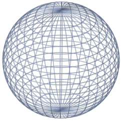
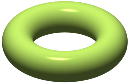
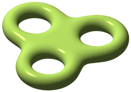
Surface				
Genus g	0	1	2	3
Euler characteristic $\chi = 2 - 2g$	2	0	-2	-4

Image credit: Wikipedia.

Topological invariant

Proposition. Let X be a finite simplicial complex and denote $F_n = \#\, n$-dimensional faces. Then

$$\boxed{\sum_n (-1)^n F_n = \sum_n (-1)^n b_n(X) = \chi(X)}\,.$$

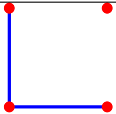
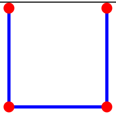
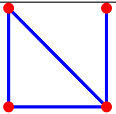
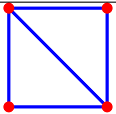
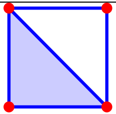
"combinatorial formula = topological formula"

Proof sketch: Boils down to linear algebra! Use the rank-nullity theorem several times.

Example. If X is a 2-dimensional simplicial complex:

$$V - F + E = b_0 - b_1 + b_2.$$

Examples revisited

Complex					
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E	2	3	4	5	5
F	0	0	0	0	1
χ	2	1	0	-1	0
b_0	2	1	1	1	1
b_1	0	0	1	2	1

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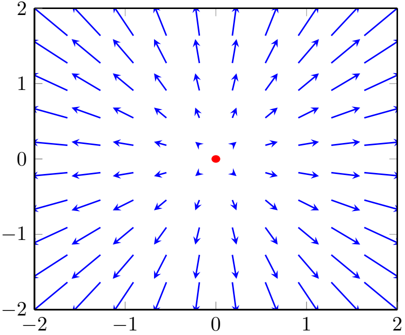
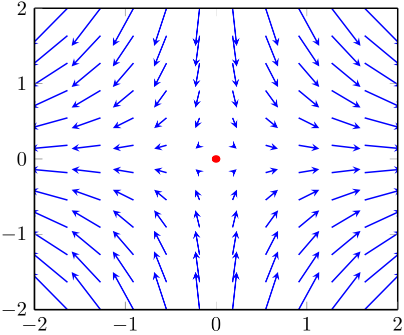
Vector fields

Definition. The **index** of a vector field $\vec{F}$ at a zero P is the winding number of $\vec{F}$ along a small sphere around P.

Example. Some vector fields $\vec{F}(x, y)$ on $\mathbb{R}^2$ with a zero at $P = (0, 0)$:

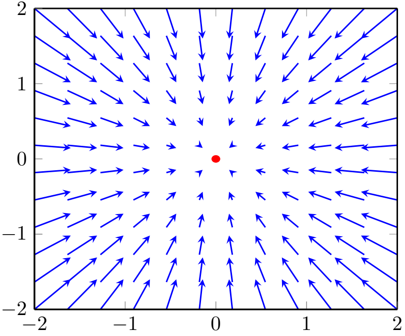


$\mathrm{ind}_P(F) = 1$

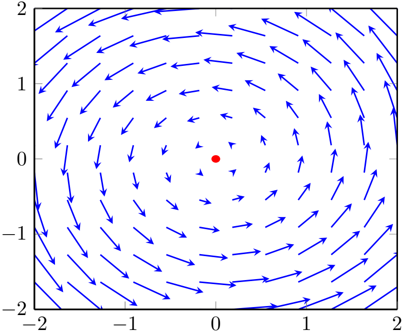


$\mathrm{ind}_P(F) = -1$

More vector fields



$\text{ind}_P(F) = 1$



$\text{ind}_P(F) = 1$

Poincaré–Hopf theorem

Theorem. Let M be a compact manifold without boundary and $\vec{F}$ a vector field on M with isolated zeros. Then the index of $\vec{F}$ is:

$$\boxed{\sum_{\vec{F}(P)=\vec{0}} \mathrm{ind}_P(\vec{F}) = \chi(M)}\,.$$

Given $\chi(S^2) = 2 \neq 0$, this implies:

Corollary (Hairy ball theorem)**.** Every vector field on the sphere S^2 must have a zero.

Hairy ball theorem

Example. On the sphere S^2, consider a vector field $\vec{F}$ that "points south". The zeros of $\vec{F}$ are the North pole and the South pole. The index of $\vec{F}$ is:

$$\operatorname{ind}(\vec{F}) = \operatorname{ind}_N(\vec{F}) + \operatorname{ind}_S(\vec{F}) = 1 + 1 = 2. \quad \checkmark$$

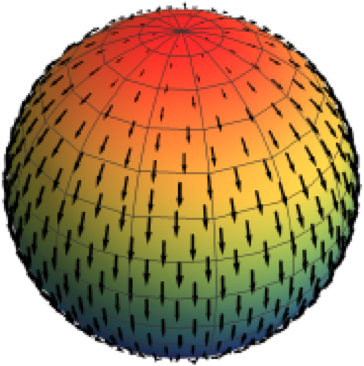


Image credit: Wolfram Demonstrations Project.

Vector fields on a torus

In contrast, there is no “hairy donut theorem”.

The torus T^2 has Euler characteristic $\chi(T^2) = 0$, and indeed there are vector fields on T^2 without zeros.

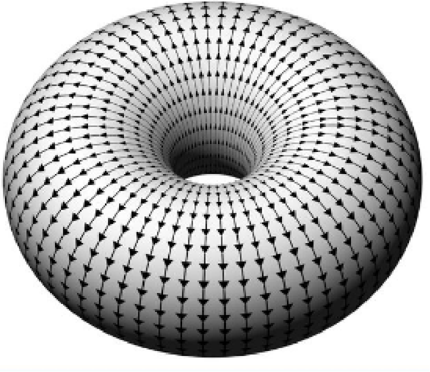


Image credit: Mathematics Stack Exchange.

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Gauss–Bonnet theorem

Theorem. Let M be a compact 2-dimensional Riemannian manifold without boundary. Then:

$$\boxed{\iint_M K\, dA = 2\pi\, \chi(M)}$$

where K denotes the Gaussian curvature.

Example: Torus

The torus T^2 has Euler characteristic $\chi(T^2) = 0$.

$\implies$ A metric on T^2 cannot have curvature that is everywhere positive or everywhere negative.

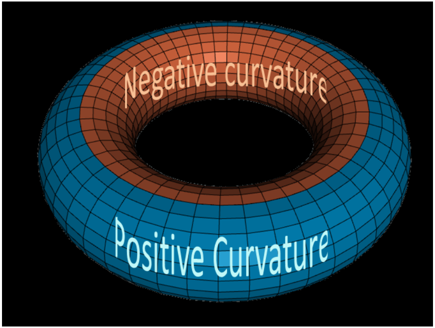


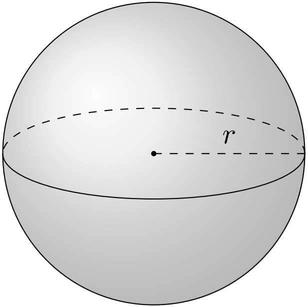
Image credit: Wikipedia.

Remark. There exists a metric on T^2 of constant curvature 0.

Example: Sphere

A standard 2-sphere of radius r has constant curvature $K = \frac{1}{r^2}$.

$$\begin{aligned}
\iint_S K\,dA &= \iint_S \frac{1}{r^2}\,dA \\
&= \frac{1}{r^2}\,\text{Area}(S) \\
&= \frac{1}{r^2}(4\pi r^2) \\
&= \boxed{4\pi} \\
&= 2\pi \cdot 2 \\
&= 2\pi\,\chi(S^2). \quad \checkmark
\end{aligned}$$



Thank you!