Is calculus secretly algebraic topology?

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University of Regina
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No.

Thank you!

Outline

The general Stokes' theorem

The de Rham theorem

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The de Rham theorem

Differential forms

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Why not $\int_b^a f(x)\,dx$? $\rightsquigarrow$ Orientation convention.

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$$\begin{aligned} \int_R f\,dx \wedge dy &= \iint_R f(x, y)\,dA \\ &= \int_a^b \int_c^d f(x, y)\,dy\,dx \overset{\text{Fubini}}{=} \int_c^d \int_a^b f(x, y)\,dx\,dy. \end{aligned}$$

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$$\int_R f\,dy \wedge dx = \int_R f\,(-dx \wedge dy) = -\iint_R f(x,y)\,dA.$$

Exterior derivative

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Rules

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$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n.$$

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$$d(\alpha + \beta) = d\alpha + d\beta.$$

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3. Product rule:
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta.$$

Exterior derivative (cont'd)

4. Squares to zero:

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Exterior derivative (cont'd)

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Key fact

$$\boxed{d^2 = 0}$$

That is: $d(d\omega) = 0$ for any k-form ω.

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&= \frac{\partial P}{\partial y}(-dx \wedge dy) + \frac{\partial Q}{\partial x}dx \wedge dy \\
&= (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy.
\end{aligned}
$$

Example (cont'd)

Taking the vector field $\vec{F} = \langle P, Q \rangle$, we have:

$$d\omega = \text{2d-curl}(\vec{F})\, dx \wedge dy$$

where

$$\text{2d-curl}(\vec{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

The de Rham complex

Definition. Denote

$$\Omega^k(U) = \{k\text{-forms on U}\}.$$

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The **de Rham complex** of U is

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \longrightarrow 0$$

de Rham complex for $n = 1$

Example. For $U \subseteq \mathbb{R}$, the de Rham complex of U is

$$\begin{tikzcd}
\Omega^0(U) \arrow[r, "d"] \arrow[d, equal] & \Omega^1(U) \arrow[r] \arrow[d, "\cong"] & 0 \\
\{\text{functions}\} \arrow[r, "\frac{d}{dx}"] & \{\text{functions}\} \arrow[r] & 0
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Warning! The equality $d^2 = 0$ is saying $d(\frac{df}{dx}\,dx) = 0$, *not* that every second derivative $\frac{d^2 f}{dx^2}$ is zero!

de Rham complex for $n = 2$

Example. For $U \subseteq \mathbb{R}^2$, the de Rham complex of U is

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Example. For $U \subseteq \mathbb{R}^2$, the de Rham complex of U is

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The **gradient** of f is the vector field

$$\operatorname{grad}(f) = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle .$$

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f & \langle P, Q, R \rangle & \langle P, Q, R \rangle & f
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The **curl** of a vector field $\vec{F}$ is the vector field

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}.$$

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The **divergence** of a vector field $\vec{F} = \langle P, Q, R \rangle$ is the function

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

General Stokes' theorem

The general Stokes' theorem

For a $(k-1)$-form ω and a k-dimensional subspace $S \subset U$, we have:

$$\boxed{\int_S d\omega = \int_{\partial S} \omega}$$

where ∂S denotes the boundary of S (with appropriate orientation).

Stokes for $n = 1$

Let $U \subseteq \mathbb{R}$.

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$$\begin{aligned}
\int_{[a,b]} df &= \int_{\partial[a,b]} f \\
\int_a^b \frac{df}{dx}\, dx &= \int_{b-a} f \\
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&= f(b) - f(a).
\end{aligned}$$

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Fundamental theorem of calculus

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Case $k = 1$. For any function f on U and path $\gamma\colon [a, b] \to U$, we have

$$\begin{aligned} \int_\gamma df &= \int_{\partial\gamma} f \\ \int_\gamma f_x\,dx + f_y\,dy &= \int_{\gamma(b)-\gamma(a)} f \\ \int_\gamma \nabla f \cdot d\vec{r} &= f(\gamma(b)) - f(\gamma(a)). \end{aligned}$$

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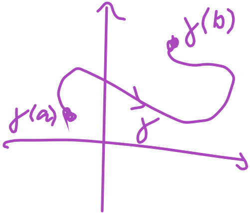
Fundamental theorem of line integrals



Stokes for $n = 2$ (cont'd)

Case $k = 2$. For any vector field $\vec{F} = \langle P, Q \rangle$ on U and region $D \subset U$, we have

$$\iint_D \text{2d-curl}(\vec{F})\, dA = \oint_{\partial D} \vec{F} \cdot d\vec{r}$$
$$\iint_D Q_x - P_y\, dA = \oint_{\partial D} \vec{F} \cdot d\vec{r}.$$

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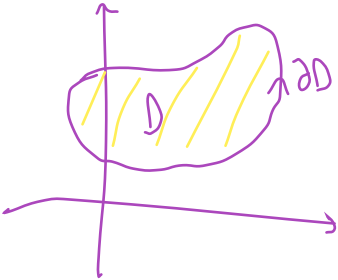
Green's theorem



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Case $k = 2$. For any vector field $\vec{F} = \langle P, Q, R \rangle$ on U and surface $S \subset U$, we have

$$\iint_S \operatorname{curl}(\vec{F}) \cdot \vec{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

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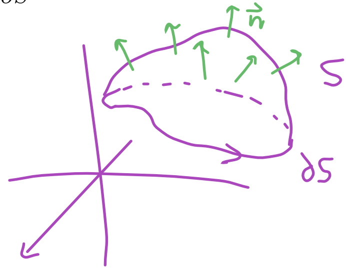
The special Stokes' theorem



Stokes for $n = 3$ (cont'd)

Case $k = 3$. For any vector field $\vec{F} = \langle P, Q, R \rangle$ on U and solid region $D \subset U$, we have

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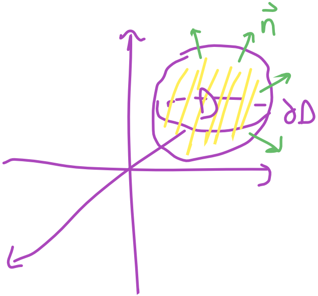
Divergence theorem



Outline

Closed and exact forms

Recall: $d(d\omega) = 0$ for any form ω.

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Answer. It depends on the shape of U!

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The pairing is given by integration:

$$H^k(U;\mathbb{R}) \times H_k(U) \to \mathbb{R}$$

$$\langle[\omega],[C]\rangle = \oint_C \omega.$$

de Rham for $n = 1$

Let $U \subseteq \mathbb{R}$. The cohomology of U is

$$H^k(U;\mathbb{R}) = \begin{cases} \mathbb{R}^{\#\text{components}}, & k = 0 \\ 0, & k \neq 0. \end{cases}$$

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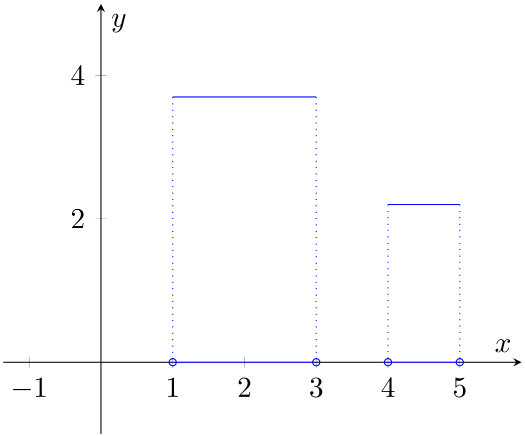
Example. Take the union of intervals $U = (1,3) \cup (4,5)$. Then

$$H^k(U;\mathbb{R}) = \begin{cases} \mathbb{R}^2, & k = 0 \\ 0, & k \neq 0. \end{cases}$$

de Rham for $n = 1$ (cont'd)

Case $k = 0$. A function $f\colon U \to \mathbb{R}$ has derivative $f'(x) = 0 \iff f$ is locally constant, i.e., constant on each component.



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We knew that. On each component, take the integral

$$F(x) = \int_a^x f(t)\,dt.$$

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Example. If U is simply-connected (i.e. “without holes”), then every irrotational vector field is conservative:

$$\pi_1(U) = 0 \implies H_1(U) = 0 \implies H^1(U; \mathbb{R}) = 0.$$

Irrotational versus conservative

Example. Take the punctured plane $U = \mathbb{R}^2 \setminus \{(0,0)\}$, and the vector field on U

$$\vec{F}(x,y) = \frac{1}{x^2+y^2} \langle -y, x \rangle = \frac{1}{r^2} \langle -y, x \rangle = \langle P, Q \rangle .$$

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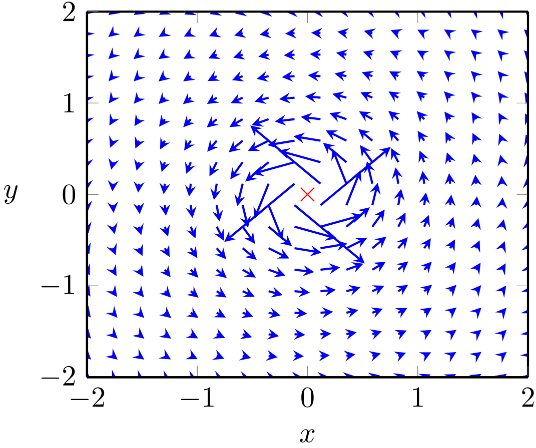

Irrotational versus conservative (cont'd)

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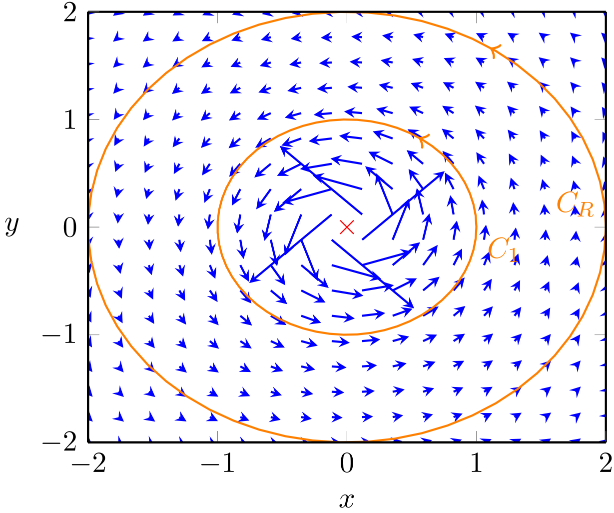

Irrotational versus conservative (cont'd)

Note: Integrating over the circle C_R of radius R yields:

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Both circles represent the same homology class

$$[C_1] = [C_R] \in H_1(U) \cong \mathbb{Z}$$

since their difference is a boundary:

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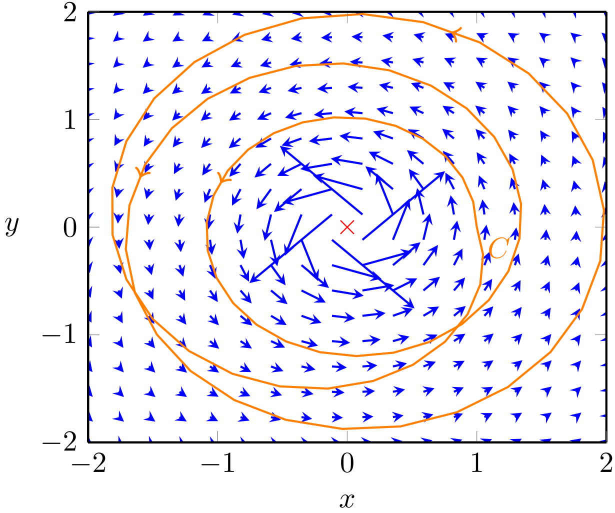
Pairing with the cohomology class $[\vec{F}] \in H^1(U; \mathbb{R})$ yields the same number:

$$\left\langle [\vec{F}], [C_1] \right\rangle = \left\langle [\vec{F}], [C_R] \right\rangle \in \mathbb{R}.$$

A weird curve

What about the this weird curve C?



A weird curve (cont'd)

Since C winds 3 times around the origin, its homology class is

$$[C] = 3[C_1] \in H_1(U).$$

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \left\langle [\vec{F}], [C] \right\rangle \\
&= \left\langle [\vec{F}], 3[C_1] \right\rangle \\
&= 3 \left\langle [\vec{F}], [C_1] \right\rangle \\
&= 3 \oint_{C_1} \vec{F} \cdot d\vec{r} \\
&= 3(2\pi) \\
&= \boxed{6\pi}.
\end{aligned}$$

Another example

Example. Still with $U = \mathbb{R}^2 \setminus \{(0,0)\}$, consider the vector field

$$\vec{F}(x,y) = \frac{1}{x^2+y^2}\left\langle x,y\right\rangle = \frac{1}{r^2}\left\langle x,y\right\rangle = \left\langle P,Q\right\rangle.$$

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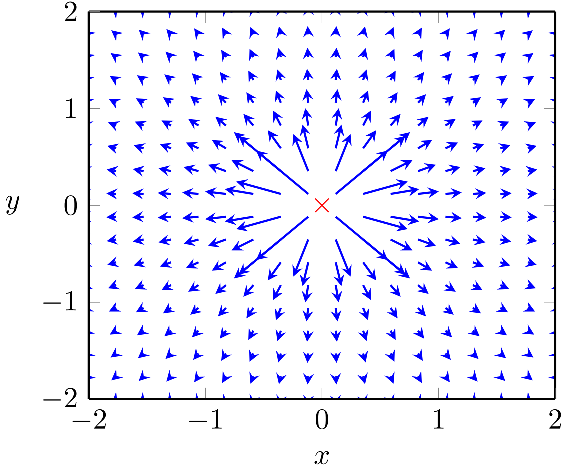

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Hmm... How do we know whether $\vec{F}$ is conservative?

Another example (cont'd)

Since the circle is a generator of the homology

$$[C_1] \in H_1(U) \cong \mathbb{Z}$$

the cohomology class of $\vec{F}$ is trivial:

$$[\vec{F}] = 0 \in H^1(U; \mathbb{R}).$$

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In fact, we can find a potential function:

$$\vec{F} = \operatorname{grad}\left(\frac{1}{2}\ln(x^2 + y^2)\right).$$

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Example. On $U = \mathbb{R}^3$, every divergence-free vector field is a curl:

$$H^2(\mathbb{R}^3; \mathbb{R}) = 0.$$

Electric field

Example. Take the punctured space $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$, and the electric field generated by a charge at the origin

$$\vec{F}(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x,y,z\rangle = \frac{1}{\|\vec{r}\|^3} \langle x,y,z\rangle = \langle P,Q,R\rangle \, .$$

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Hmm... How do we know whether $\vec{F}$ is a curl?

Another example (cont'd)

Since the sphere is a generator of the homology

$$[S_1] \in H_2(U) \cong \mathbb{Z}$$

the cohomology class of $\vec{F}$ is trivial:

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In fact, we can find a vector potential:

$$\vec{F} = \operatorname{curl}\left(\frac{1}{2}\ln(x^2+y^2+z^2)\,\langle 1,1,1\rangle\right).$$

Thank you!