Toda brackets in n-angulated categories

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New Directions in Group Theory and Triangulated Categories
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Pre-n-angulated categories

Idea: n-angulated $\approx$ "triangulated but with longer triangles".

Introduced by Geiss, Keller, and Oppermann (2013). Motivated by examples in quiver representation theory.

Let $\mathcal{C}$ be an additive category, $\Sigma \colon \mathcal{C} \xrightarrow{\cong} \mathcal{C}$ an automorphism, and $n \geq 3$.

Definition. An ***n*-Σ-sequence** is a diagram in $\mathcal{C}$ of the form

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1.$$

Definition. A **pre-*n*-angulation** of $\mathcal{C}$ is a collection $\mathscr{N}$ of n-Σ-sequences in $\mathcal{C}$, called ***n*-angles**, satisfying the following axioms.

The axioms

(N1)

(a) $\mathscr{N}$ is closed under direct sums, direct summands and isomorphisms of n-Σ-sequences.

(b) For all $X \in \mathcal{C}$, the trivial n-Σ-sequence

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X_1$$

belongs to $\mathscr{N}$.

(c) For each morphism $f\colon X_1 \to X_2$ in $\mathcal{C}$, there exists an n-Σ-sequence in $\mathscr{N}$ whose first morphism is f.

(N2) An n-Σ-sequence belongs to $\mathscr{N}$ if and only if its rotation

$$X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2$$

belongs to $\mathscr{N}$.

The axioms (cont'd)

(N3) Given the solid part of the commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \vdots{\scriptstyle \phi_3} & & & & \vdots{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \Sigma\phi_1} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
\end{array}$$

with rows in $\mathscr{N}$, the dotted morphisms exist and give a morphism of n-Σ-sequences.

Remark. pre-3-angulated = pretriangulated

Getting rid of "pre"

Recall: triangulated = pretriangulated + octahedral axiom.

Definition. A pre-n-angulated category $\mathcal{C}$ is an ***n*-angulated category** if it also satisfies the "mapping cone axiom", i.e., every fill-in problem admits a good fill-in.

Reformulated as a "higher octahedral axiom" by Bergh and Thaule (2013).

Remark. This entire talk is pre-n-angulated.

Non-unique "cofibers"

Main difference: When $n \geq 4$, an n-angle extension of $f\colon X_1 \to X_2$ is not unique up to isomorphism.

Example. In a 4-angulated category $\mathcal{C}$, given a 4-angle

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \Sigma X_1$$

and an object Y, we can add a trivial summand to obtain another 4-angle:

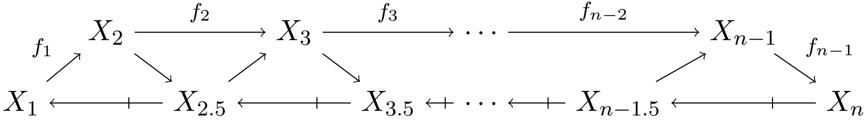
$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{\left[\begin{smallmatrix} f_2 \\ 0 \end{smallmatrix}\right]} X_3 \oplus Y \xrightarrow{\left[\begin{smallmatrix} f_3 & 0 \\ 0 & 1 \end{smallmatrix}\right]} X_4 \oplus Y \xrightarrow{\left[\begin{smallmatrix} f_4 & 0 \end{smallmatrix}\right]} \Sigma X_1.$$

Source of examples

Theorem (GKO 2013)**.** Let $\mathcal{T}$ be a triangulated category with an $(n-2)$-cluster tilting subcategory $\mathcal{C}$ closed under Σ^{n-2} where Σ denotes the suspension in $\mathcal{T}$. Then $(\mathcal{C}, \Sigma^{n-2}, \mathscr{N})$ is an n-angulated category where $\mathscr{N}$ is the class of all n-Σ^{n-2}-sequences

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma^{n-2} X_1$$

in $\mathcal{C}$ such that there exists a diagram in $\mathcal{T}$



with $X_i \in \mathcal{T}$ for all $i \notin \mathbb{Z}$ such that all triangles with base a degree-shifting morphism are triangles in $\mathcal{T}$, and f_n is the composition of the bottom row.

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Triangulated case

Idea: A Toda bracket is constructed by picking nullhomotopies $f_{i+1}f_i \sim 0$ that witness $f_{i+1}f_i = 0$ in the homotopy category.

Definition. Let $\mathcal{T}$ be a triangulated category and let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4$$

be a diagram in $\mathcal{T}$. Consider the following subsets of $\mathcal{T}(\Sigma X_1, X_4)$.

- The **iterated cofiber Toda bracket** $\langle f_3, f_2, f_1 \rangle_{\mathrm{cc}} \subseteq \mathcal{T}(\Sigma X_1, X_4)$ consists of all morphisms $\psi \colon \Sigma X_1 \to X_4$ that appear in a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{y_2} & Y_3 & \xrightarrow{y_3} & \Sigma X_1 \\ \| & & \| & & \downarrow \phi & & \downarrow \psi \\ X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 \end{array}$$

  where the top row is distinguished.

Triangulated case (cont'd)

- The **iterated fiber Toda bracket** $\langle f_3, f_2, f_1 \rangle_{\mathrm{ff}} \subseteq \mathcal{T}(\Sigma X_1, X_4)$ consists of all morphisms $\Sigma\delta \colon \Sigma X_1 \to X_4$ where δ appears in a commutative diagram

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} & & \| & & \| \\
\Sigma^{-1} X_4 & \xrightarrow{w_1} & W_2 & \xrightarrow{w_2} & X_3 & \xrightarrow{f_3} & X_4
\end{array}$$

  where the bottom row is distinguished.

- The **fiber-cofiber Toda bracket** $\langle f_3, f_2, f_1 \rangle_{\mathrm{fc}} \subseteq \mathcal{T}(\Sigma X_1, X_4)$ consists of all composites $\Sigma(\beta_1^2 \beta_1^1) \colon \Sigma X_1 \to X_4$, where β_1^1 and β_1^2 appear in a commutative diagram

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{f_1} & X_2 & & & & \Sigma X_1 \\
\downarrow{\scriptstyle \beta_1^1} & & \| & & & & \downarrow{\scriptstyle \Sigma\beta_1^1} \\
Z_1^2 & \xrightarrow{z_1^2} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{z_3^2} & \Sigma Z_1^2 \\
 & & & & \| & & \downarrow{\scriptstyle \Sigma\beta_1^2} \\
 & & & & X_3 & \xrightarrow{f_3} & X_4.
\end{array}$$

  where the middle row is distinguished.

The constructions agree

Remark. Each bracket is non-empty if and only if $f_{i+1} f_i = 0$.

Proposition. The three definitions of Toda bracket agree.

Classical for triangulated categories. Turns out to be a pretriangulated fact.

In fact, the three brackets are the same coset of the **indeterminacy subgroup**

$$(f_3)_* \mathcal{T}(\Sigma X_1, X_3) + (\Sigma f_1)^* \mathcal{T}(\Sigma X_2, X_4) \subseteq \mathcal{T}(\Sigma X_1, X_4).$$

For higher n

How to generalize Toda brackets to higher n?

The two "extreme" constructions centered at f_1 or f_n have straightforward analogues.

The "balanced" construction centered at f_2 generalizes in two different ways:

- Center the construction at one of the intermediate maps f_i.
- Use **all** the intermediate maps $f_2, \ldots, f_{n-1}$.

Fix a pre-n-angulated category $\mathcal{C}$ and n composable maps in $\mathcal{C}$

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

We will define certain subsets of $\mathcal{C}(\Sigma X_1, X_{n+1})$.

Iterated cofiber

Definition.

- The **iterated cofiber Toda bracket**

$$\langle f_n, \ldots, f_2, f_1 \rangle_{\mathrm{cc}} \subseteq \mathcal{C}(\Sigma X_1, X_{n+1})$$

consists of all morphisms $\psi \colon \Sigma X_1 \to X_{n+1}$ that appear in a commutative diagram

$$\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{y_2} & Y_3 & \xrightarrow{y_3} & Y_4 & \xrightarrow{y_4} & \cdots & \xrightarrow{y_{n-1}} & Y_n & \xrightarrow{y_n} & \Sigma X_1 \\
\| & & \| & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \psi} \\
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_{n+1}
\end{array}$$

where the top row is an n-angle extension of f_1.

Iterated fiber

- Dually, the **iterated fiber Toda bracket**

$$\langle f_n, \ldots, f_2, f_1 \rangle_{\mathrm{ff}} \subseteq \mathcal{C}(\Sigma X_1, X_{n+1})$$

consists of all morphisms $\Sigma\delta \colon \Sigma X_1 \to X_{n+1}$ where δ appears in a commutative diagram

$$\begin{array}{ccccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_{n+1} \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & & & \downarrow{\scriptstyle \gamma_{n-1}} & & \| & & \| \\ \Sigma^{-1} X_{n+1} & \xrightarrow{w_1} & W_2 & \xrightarrow{w_2} & W_3 & \xrightarrow{w_3} & \cdots & \xrightarrow{w_{n-2}} & W_{n-1} & \xrightarrow{w_{n-1}} & X_n & \xrightarrow{f_n} & X_{n+1} \end{array}$$

where the bottom row is an n-angle.

Generalizing those two extreme choices:

Intermediate brackets

- For each $1 \leq i \leq n$ the **[*i*]-intermediate Toda bracket**

$$\langle f_n, \ldots, f_2, f_1 \rangle_{[i]} \subseteq \mathcal{C}(\Sigma X_1, X_{n+1})$$

consists of morphisms $\psi = \Sigma(\beta_1 \alpha_1)\colon \Sigma X_1 \to X_{n+1}$ such that α_1 and β_1 occur in a commutative diagram:

Prepare for a big diagram...

Intermediate brackets (cont'd)

$$
\begin{array}{ccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \\
Z_1 & \xrightarrow{z_1} & Z_2 & \xrightarrow{z_2} & \cdots
\end{array}
$$

$$
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{f_{i-2}} & X_{i-1} & \xrightarrow{f_{i-1}} & X_i & & & & & & \\
 & & \downarrow{\scriptstyle \alpha_{i-1}} & & \| & & & & & & \\
\cdots & \xrightarrow{z_{i-2}} & Z_{i-1} & \xrightarrow{z_{i-1}} & X_i & \xrightarrow{f_i} & X_{i+1} & \xrightarrow{z_{i+1}} & Z_{i+2} & \xrightarrow{z_{i+2}} & \cdots \\
 & & & & & & \| & & \downarrow{\scriptstyle \beta_{i+2}} & & \\
 & & & & & & X_{i+1} & \xrightarrow{f_{i+1}} & X_{i+2} & \xrightarrow{f_{i+2}} & \cdots
\end{array}
$$

$$
\begin{array}{ccccc}
\cdots & \xrightarrow{z_{n-1}} & Z_n & \xrightarrow{z_n} & \Sigma Z_1 \\
 & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \Sigma\beta_1} \\
\cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_{n+1}
\end{array}
$$

where the middle row is some n-angle $Z_\bullet$ in which f_i occurs as the i^{th} morphism.

Fiber-cofiber

- The **fiber-cofiber Toda bracket**

$$\langle f_n, \ldots, f_2, f_1 \rangle_{\text{fc}} \subseteq \mathcal{C}(\Sigma X_1, X_{n+1})$$

consists of all composites

$$\Sigma(\beta_1^{n-1} \cdots \beta_1^2 \beta_1^1) \colon \Sigma X_1 \longrightarrow X_{n+1}$$

where $\beta_1^1, \beta_1^2, \ldots, \beta_1^{n-1}$ appear in a commutative diagram of the following form:

Prepare for an even bigger diagram...

Fiber-cofiber (cont'd)

$$
\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & & & & & & & & & & \Sigma X_1 \\
\downarrow{\scriptstyle \beta_1^1} & & \| & & & & & & & & & & \downarrow{\scriptstyle \Sigma\beta_1^1} \\
Z_1^2 & \xrightarrow{z_1^2} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{z_3^2} & Z_4^2 & \xrightarrow{z_4^2} & \cdots & \xrightarrow{z_{n-1}^2} & Z_n^2 & \xrightarrow{z_n^2} & \Sigma Z_1^2 \\
\downarrow{\scriptstyle \beta_1^2} & & \downarrow{\scriptstyle \beta_2^2} & & \| & & \downarrow{\scriptstyle \beta_4^2} & & & & \downarrow{\scriptstyle \beta_n^2} & & \downarrow{\scriptstyle \Sigma\beta_1^2} \\
Z_1^3 & \xrightarrow{z_1^3} & Z_2^3 & \xrightarrow{z_2^3} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{z_4^3} & \cdots & \xrightarrow{z_{n-1}^3} & Z_n^3 & \xrightarrow{z_n^3} & \Sigma Z_1^3 \\
\downarrow{\scriptstyle \beta_1^3} & & \downarrow{\scriptstyle \beta_2^3} & & \downarrow{\scriptstyle \beta_3^3} & & \| & & & & \downarrow{\scriptstyle \beta_n^3} & & \downarrow{\scriptstyle \Sigma\beta_1^3} \\
Z_1^4 & \xrightarrow{z_1^4} & Z_2^4 & \xrightarrow{z_2^4} & Z_3^4 & \xrightarrow{z_3^4} & X_4 & \xrightarrow{f_4} & \cdots & \xrightarrow{z_{n-1}^4} & Z_n^4 & \xrightarrow{z_n^4} & \Sigma Z_1^4 \\
\downarrow{\scriptstyle \beta_1^4} & & \downarrow{\scriptstyle \beta_2^4} & & \downarrow{\scriptstyle \beta_3^4} & & \downarrow{\scriptstyle \beta_4^4} & & & & \downarrow{\scriptstyle \beta_n^4} & & \downarrow{\scriptstyle \Sigma\beta_1^4} \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow{\scriptstyle \beta_1^{n-2}} & & \downarrow{\scriptstyle \beta_2^{n-2}} & & \downarrow{\scriptstyle \beta_3^{n-2}} & & \downarrow{\scriptstyle \beta_4^{n-2}} & & & & \downarrow{\scriptstyle \beta_n^{n-2}} & & \downarrow{\scriptstyle \Sigma\beta_1^{n-2}} \\
Z_1^{n-1} & \xrightarrow{z_1^{n-1}} & Z_2^{n-1} & \xrightarrow{z_2^{n-1}} & Z_3^{n-1} & \xrightarrow{z_3^{n-1}} & Z_4^{n-1} & \xrightarrow{z_4^{n-1}} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{z_n^{n-1}} & \Sigma Z_1^{n-1} \\
& & & & & & & & & & \| & & \downarrow{\scriptstyle \Sigma\beta_1^{n-1}} \\
& & & & & & & & & & X_n & \xrightarrow{f_n} & X_{n+1}
\end{array}
$$

where row i is an n-angle $Z_\bullet^i$ in which f_i occurs as the i^{th} morphism.

For $n = 4$, an element of the fiber-cofiber Toda bracket $\psi \in \langle f_4, f_3, f_2, f_1 \rangle_{\text{fc}}$ is defined by a diagram

$$
\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & & & & & & \Sigma X_1 \\
\downarrow{\scriptstyle \beta_1^1} & & \| & & & & & & \downarrow{\scriptstyle \Sigma\beta_1^1} \\
Z_1^2 & \xrightarrow{z_1^2} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{z_3^2} & Z_4^2 & \xrightarrow{z_4^2} & \Sigma Z_1^2 \\
\downarrow{\scriptstyle \beta_1^2} & & \downarrow{\scriptstyle \beta_2^2} & & \| & & \downarrow{\scriptstyle \beta_3^2} & & \downarrow{\scriptstyle \Sigma\beta_1^2} \\
Z_1^3 & \xrightarrow{z_1^3} & Z_2^3 & \xrightarrow{z_2^3} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{z_4^3} & \Sigma Z_1^3 \\
& & & & & & \| & & \downarrow{\scriptstyle \Sigma\beta_1^3} \\
& & & & & & X_4 & \xrightarrow{f_4} & X_5.
\end{array}
$$

for some choices of n-angles containing f_2 and f_3 such that

$$\psi = \Sigma(\beta_1^3 \beta_1^2 \beta_1^1).$$

The constructions agree

Theorem (F., Martensen, Thaule)**.** Assuming $f_{i+1} f_i = 0$ for all i, all the given definitions of Toda bracket agree.

Hence, we can safely talk about *the* Toda bracket

$$\langle f_n, \ldots, f_2, f_1 \rangle \subseteq \mathcal{C}(\Sigma X_1, X_{n+1}).$$

In fact, the definitions of the bracket are all the same coset of the **indeterminacy subgroup**

$$(f_n)_* \mathcal{C}(\Sigma X_1, X_n) + (\Sigma f_1)^* \mathcal{C}(\Sigma X_2, X_{n+1}) \subseteq \mathcal{C}(\Sigma X_1, X_{n+1}).$$

Remark. If $f_{i+1} f_i \neq 0$ for some i, then the fiber-cofiber bracket is empty:

$$\langle f_n, \ldots, f_2, f_1 \rangle_{\text{fc}} = \emptyset$$

but the other brackets might be non-empty.

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Exotic example

Consider the category $\mathcal{C} = \mathrm{mod}^{\mathrm{ff}}\mathbb{Z}/p^2$ of finitely generated free modules over $R = \mathbb{Z}/p^2$, with $p = 2$ if n is odd (and p is any prime number if n is even). Take the identity automorphism $\Sigma = \mathrm{Id}$.

Muro, Schwede, and Strickland (2007) constructed an exotic triangulated structure on $\mathrm{mod}^{\mathrm{ff}}\mathbb{Z}/4$.

Bergh, Jasso, and Thaule (2016) constructed an exotic n-angulated structure on $\mathcal{C}$, for any $n \geq 3$.

Example. Taking $n = 4$, the Toda bracket of the diagram

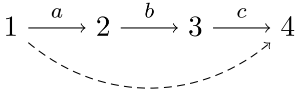
$$R \xrightarrow{p} R \xrightarrow{p} R \xrightarrow{p} R \xrightarrow{p} R$$

as a subset of $\mathcal{C}(\Sigma R, R) = \mathcal{C}(R, R) \cong \mathbb{Z}/p^2$ is

$$\langle p, p, p, p \rangle = \{1 + cp \mid c \in \mathbb{Z}\} = 1 + (p) \subset \mathbb{Z}/p^2.$$

Quiver representation theory

Example. Consider the quiver Q with relations J^3 depicted here:



J = arrow ideal, generated by paths of length 1.

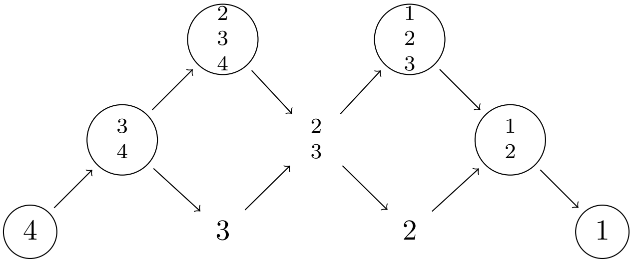
J^3 = ideal generated by all paths of length 3, namely cba.

Path algebra $\Gamma = \Bbbk Q/J^3$.

Take $\mathcal{A} \coloneqq \mathrm{mod}\Gamma$ the category of finitely generated right Γ-modules.

Quiver example (cont'd)

The category $\mathcal{A}$ can be visualized as:



where we denote the quiver representation

$$\begin{smallmatrix}1\\2\end{smallmatrix} = \left(\Bbbk \xrightarrow{1} \Bbbk \longrightarrow 0 \longrightarrow 0\right)$$

and so on.

Take $M :=$ direct sum of the encircled modules.

Quiver example (cont'd)

In the category $\mathcal{A} = \mathrm{mod}\Gamma$, M is a 2-cluster tilting module in the sense of Iyama:

$$\begin{aligned} \operatorname{add} M &= \{X \in \mathcal{A} \mid \operatorname{Ext}^1_\Gamma(X, M) = 0\} \\ &= \{X \in \mathcal{A} \mid \operatorname{Ext}^1_\Gamma(M, X) = 0\}. \end{aligned}$$

Indecomposable projectives in $\mathcal{A}$:

$$P_1 = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \qquad P_2 = \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix} \qquad P_3 = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \qquad P_4 = 4.$$

Indecomposable injectives in $\mathcal{A}$:

$$I_1 = 1 \qquad I_2 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \qquad I_3 = P_1 \qquad I_4 = P_2.$$

Γ is a finite-dimensional $\Bbbk$-algebra of global dimension $\operatorname{gldim} \Gamma = 2$.

Quiver example (cont'd)

$\rightsquigarrow$ Get a 4-angulated structure on the subcategory

$$\mathcal{U} = \operatorname{add}\{M[2i] \mid i \in \mathbb{Z}\} \subseteq \mathrm{D}^b(\mathcal{A}).$$

Some 4-angles in $\mathcal{U}$:

$$P_4 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_3} I_1 \xrightarrow{f_4} \Sigma^2 P_4$$

$$P_4 \xrightarrow{g_1} P_3 \xrightarrow{g_2} P_1 \xrightarrow{g_3} I_2 \xrightarrow{g_4} \Sigma^2 P_4.$$

Toda bracket of the diagram in $\mathcal{U}$:

$$P_4 \xrightarrow{f_1} P_2 \xrightarrow{g_3 f_2} I_2 \xrightarrow{g_4} \Sigma^2 P_4 \xrightarrow{\Sigma^2 g_1} \Sigma^2 P_3$$

$$\langle \Sigma^2 g_1, g_4, g_3 f_2, f_1 \rangle = \operatorname{span}_{\Bbbk}\{\Sigma^2 g_1\} = \mathcal{U}(\Sigma^2 P_4, \Sigma^2 P_3)$$

See the paper for details.

Outline

Juggling formulas

Fix a pre-n-angulated category $\mathcal{C}$ and n composable maps in $\mathcal{C}$

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

satisfying $f_{i+1}f_i = 0$ for all i.

Subadditivity

Relations between brackets and addition:

Proposition. 1. $\langle f_n + f_n', f_{n-1}, \ldots, f_2, f_1 \rangle \subseteq \langle f_n, f_{n-1}, \ldots, f_2, f_1 \rangle + \langle f_n', f_{n-1}, \ldots, f_2, f_1 \rangle$.

2. $\langle f_n, \ldots, f_2, f_1 + f_1' \rangle \subseteq \langle f_n, \ldots, f_2, f_1 \rangle + \langle f_n, f_{n-1}, \ldots, f_2, f_1' \rangle$.
3. For $2 \leq i \leq n-1$, we have the identities

$$\langle f_n, \ldots, f_i + f_i', \ldots, f_2, f_1 \rangle = \langle f_n, \ldots, f_i, \ldots, f_2, f_1 \rangle + \langle f_n, \ldots, f_i', \ldots, f_2, f_1 \rangle.$$

4. For $1 \leq i \leq n$, we have

$$\langle f_n, \ldots, -f_i, \ldots, f_1 \rangle = -\langle f_n, \ldots, f_i, \ldots, f_1 \rangle.$$

Submultiplicativity

Relations between brackets and composition:

Proposition.

1. $f_{n+1}\langle f_n, \ldots, f_2, f_1\rangle \subseteq \langle f_{n+1}f_n, \ldots, f_2, f_1\rangle$.
2. $\langle f_{n+1}, \ldots, f_3, f_2\rangle \Sigma f_1 \subseteq \langle f_{n+1}, \ldots, f_3, f_2 f_1\rangle$.
3. $\langle f_{n+1}, \ldots, f_3, f_2 f_1\rangle \subseteq \langle f_{n+1}, \ldots, f_3 f_2, f_1\rangle$.
4. $\langle f_{n+1}f_n, f_{n-1}, \ldots, f_2, f_1\rangle \subseteq \langle f_{n+1}, f_n f_{n-1}, \ldots, f_2, f_1\rangle$.
5. For all $3 \leq i \leq n-1$ we have the identities

$$\langle f_{n+1}, \ldots, f_{i+1}f_i, f_{i-1}, \ldots, f_2, f_1\rangle = \langle f_{n+1}, \ldots, f_{i+1}, f_i f_{i-1}, \ldots, f_2, f_1\rangle.$$

6. $f_{n+1}\langle f_n, \ldots, f_2, f_1\rangle = \langle f_{n+1}, \ldots, f_3, f_2\rangle (-1)^n \Sigma f_1$.

Remark. Collected reference for 3-fold Toda brackets in (pre)triangulated categories.

Heller's theorem

Lemma (GKO 2013)**.** In a pre-n-angulated category $\mathcal{C}$, every n-angle

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

is **Yoneda exact**, i.e., the sequence of abelian groups

$$\mathcal{C}(A, X_1) \xrightarrow{(f_1)_*} \mathcal{C}(A, X_2) \xrightarrow{(f_2)_*} \cdots \xrightarrow{(f_n)_*} \mathcal{C}(A, \Sigma X_1) \xrightarrow{(\Sigma f_1)_*} \mathcal{C}(A, \Sigma X_2)$$

is exact for every object A of $\mathcal{C}$.

Heller's theorem (cont'd)

Proposition. Let $\mathcal{C}$ be a pre-n-angulated category. An n-Σ-sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

in $\mathcal{C}$ is an n-angle if and only if the following two conditions hold.

1. The n-Σ-sequence $X_\bullet$ is Yoneda exact.
2. The Toda bracket $\langle f_n, \ldots, f_2, f_1 \rangle_{\mathrm{cc}} \subseteq \mathcal{C}(\Sigma X_1, \Sigma X_1)$ contains the identity morphism $1_{\Sigma X_1}$.

The case $n = 3$ is due to Heller (1968).

The proof for $n \geq 3$ is similar, using facts from GKO (2013).

Longer = Higher

In the GKO setup, *higher* Toda brackets in the ambient triangulated category are also available.

Theorem (F., Martensen, Thaule)**.** Let $\mathcal{T}$ be a triangulated category with an $(n-2)$-cluster tilting subcategory $\mathcal{C}$ closed under Σ^{n-2} and let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

be a diagram in $\mathcal{C}$ satisfying $f_{i+1}f_i = 0$ for all i. Then

$$\langle f_n, \ldots, f_2, f_1 \rangle_{n\text{-angulated}} = (-1)^{\sum_{\ell=1}^{n-3} \ell} \langle f_n, \ldots, f_2, f_1 \rangle_{\text{triangulated}}.$$

Massey products

Combining the previous result with a theorem of Jasso and Muro (2023) yields the following in the presence of a DG enhancement.

Corollary. Let $\Bbbk$ be a field and $\mathcal{A}$ a small DG category over $\Bbbk$ such that the homotopy category $H^0(\mathrm{Mod}_{\mathrm{dg}}\,\mathcal{A})$ of the DG category of right DG $\mathcal{A}$-modules is triangulated and has a $(n-2)$-cluster tilting subcategory $\mathcal{U}$ closed with respect to the shift $[n-2]$ such that $\mathcal{U}$ is n-angulated. For n composable maps f_i in $\mathcal{U}$, we have

$$\langle\!\langle f_n, \ldots, f_2, f_1 \rangle\!\rangle[n-2] = -\langle f_n, \ldots, f_2, f_1 \rangle$$

where the left side is the Massey product computed using the DG category $\mathcal{A}$ and the right side is the Toda bracket in the n-angulated category $\mathcal{U}$.

Future directions

A few ideas for related research.

- Computations and applications.
- Do n-angulated categories have a homotopical interpretation? Are Toda brackets obstructions to something?
- Higher **and** longer. Toda brackets of m composable maps f_i in an n-angulated category, for $m \geq n$.
- Adams spectral sequence in n-angulated categories, cf. Beligiannis (2015). Relation between differentials and Toda brackets?

Thank you!