On good morphisms of exact triangles

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New Directions in Group Theory and Triangulated Categories
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Reference

Christensen and F. On good morphisms of exact triangles. *J. Pure Appl. Algebra* (2022).

Outline

Classical Adams spectral sequence

Given finite spectra X and Y, the classical Adams spectral sequence has the form

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(H^*Y, \Sigma^t H^*X) \Rightarrow [\Sigma^{t-s}X, Y_p^\wedge]$$

where $\mathcal{A}$ denotes the mod p Steenrod algebra.

For E a nice ring spectrum (e.g. MU or BP), the E-based Adams spectral sequence is:

$$E_2^{s,t} = \mathrm{Ext}^s_{E_*E}(\Sigma^t E_*X, E_*Y) \Rightarrow [\Sigma^{t-s}X, L_E Y].$$

Triangulated version

- Brinkmann (1968): Adams spectral sequence in a triangulated category.
- Franke (1996): Application to $E(1)$-local ($=KU_{(p)}$-local) spectra.
- Further work related to Franke's construction (2007 and on): Roitzheim, Barnes, Patchkoria, and others. Applications to E-local spectra and E-module spectra for various E.
- Christensen (1998): Application to ghost lengths and stable module categories.

Examples of triangulated categories

Example. The homotopy category of spectra, a.k.a. the stable homotopy category.

Example. The derived category of a ring $D(R)$.

Example. The stable module category of a group algebra $\mathrm{StMod}(kG)$.

Projective and injective classes

Eilenberg and Moore (1965) gave a framework for relative homological algebra in any pointed category. When the category is triangulated, their axioms are equivalent to the following.

Definition. A **projective class** in $\mathcal{T}$ is a pair $(\mathcal{P}, \mathcal{N})$, where $\mathcal{P} \subseteq \operatorname{ob} \mathcal{T}$ and $\mathcal{N} \subseteq \operatorname{mor} \mathcal{T}$, such that:

(i) $\mathcal{P}$ consists of exactly the objects P such that every composite $P \to X \to Y$ is zero for each $X \to Y$ in $\mathcal{N}$.

(ii) $\mathcal{N}$ consists of exactly the maps $X \to Y$ such that every composite $P \to X \to Y$ is zero for each P in $\mathcal{P}$.

(iii) For each X in $\mathcal{T}$, there is a triangle $P \to X \to Y$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{N}$.

An **injective class** in $\mathcal{T}$ is a projective class in $\mathcal{T}^{\mathrm{op}}$.

Examples

Example. In spectra, take:

$$\mathcal{P} = \text{retracts of wedges of spheres } \bigvee_i S^{n_i}$$

$$\mathcal{N} = \text{maps inducing zero on homotopy groups.}$$

$(\mathcal{P}, \mathcal{N})$ is the ghost projective class.

Example. For E any spectrum, take:

$$\mathcal{I} = \text{retracts of products } \prod_i \Sigma^{n_i} E$$

$$\mathcal{N} = \text{maps inducing zero on } E^*(-).$$

Then $(\mathcal{I}, \mathcal{N})$ is an injective class.

For $E = H\mathbb{F}_p$, this injective class leads to the classical (cohomological) Adams spectral sequence.

Examples

Example. For E a homotopy commutative ring spectrum, take:

$$\mathcal{I} = \text{retracts of } E \wedge W$$
$$\mathcal{N} = \text{maps } f \colon X \to Y \text{ with } E \wedge f \simeq 0 \colon E \wedge X \to E \wedge Y.$$

The injective class $(\mathcal{I}, \mathcal{N})$ leads to the E-based (homological) Adams spectral sequence.

Remark. We always assume that our projective and injective classes are stable, i.e., closed under suspension and desuspension.

Examples

Example. Let A be a differential graded (dg) algebra. In dg-A-modules, take:

$$\mathcal{P} = \text{retracts of sums } \bigoplus_i A[n_i]$$

$$\mathcal{N} = \text{maps inducing zero on homology.}$$

$(\mathcal{P}, \mathcal{N})$ is the ghost projective class.

The Adams spectral sequence relative to $\mathcal{P}$ is the universal coefficient spectral sequence

$$\mathrm{Ext}^s_{H_*A}(H_*M[t], H_*N) \Rightarrow \mathrm{Ext}^s_A(M[t], N) = D(A)(M[t], N[s])$$

from ordinary Ext to differential Ext.

Remark. A monoidal version also recovers the Eilenberg–Moore spectral sequence

$$\mathrm{Tor}_{H_*A}(H_*M, H_*N) \Rightarrow \mathrm{Tor}_A(M, N).$$

Definition. An **Adams resolution** of an object Y in $\mathcal{T}$ with respect to an injective class $(\mathcal{I}, \mathcal{N})$ is a diagram

$$
\begin{tikzcd}
Y = Y_0 \arrow[dr, "p_0"'] & & Y_1 \arrow[ll, "i_0"'] \arrow[dr, "p_1"'] & & Y_2 \arrow[ll, "i_1"'] \arrow[dr, "p_2"'] & & Y_3 \arrow[ll, "i_2"'] & & \cdots \arrow[ll] \\
& I_0 \arrow[ur, "\delta_0"'] & & I_1 \arrow[ur, "\delta_1"'] & & I_2 \arrow[ur, "\delta_2"'] & & \cdots &
\end{tikzcd}
$$

where each I_s is injective, each map i_s is in $\mathcal{N}$, and the triangles are triangles.

Axiom (iii) says that you can form such a resolution.

Applying $\mathcal{T}(X, -)$ leads to an exact couple and therefore a spectral sequence, called the Adams spectral sequence.

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{I}}(\Sigma^t X, Y)$$

... But why?

Why work with a triangulated category instead of a stable ∞-category or stable model category?

- Fewer hypotheses.
- There's a lot we can do using only the triangulated structure.
- Get derived invariants.

Cofibers in an Adams tower

Starting point: an $\mathcal{I}$-Adams tower

$$X = X_0 \xleftarrow{x_1} X_1 \xleftarrow{x_2} X_2 \longleftarrow \cdots$$

For intervals $[n, m] \leq [n', m']$, there is a fill-in in the diagram

$$\begin{tikzcd}
X_{m'} \arrow[r] \arrow[d] & X_{n'} \arrow[r] \arrow[d] & X_{n'}/X_{m'} \arrow[r] \arrow[d, dashed] & \Sigma X_{m'} \arrow[d] \\
X_m \arrow[r] & X_n \arrow[r] & X_n/X_m \arrow[r] & \Sigma X_m.
\end{tikzcd}$$

Question. How convenient can those choices be made?

Outline

Adams spectral sequence

Good morphisms

Examples and non-examples

Questions and results

Homotopy cartesian squares

Mapping cone

Definition (Neeman)**.** The **mapping cone** of a map of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

is the sequence

$$X' \oplus Y \xrightarrow{\left[\begin{smallmatrix} u' & g \\ 0 & -v \end{smallmatrix}\right]} Y' \oplus Z \xrightarrow{\left[\begin{smallmatrix} v' & h \\ 0 & -w \end{smallmatrix}\right]} Z' \oplus \Sigma X \xrightarrow{\left[\begin{smallmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{smallmatrix}\right]} \Sigma X' \oplus \Sigma Y.$$

The map of triangles (f, g, h) is **good** if its mapping cone is an exact triangle.

Middling good morphisms

Proposition (Neeman)**.** If the map of triangles (f, g, h) is good, then it extends to a 4×4 diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] \arrow[d, "f'"'] & Y' \arrow[r, "v'"] \arrow[d, "g'"] & Z' \arrow[r, "w'"] \arrow[d, dashed, "h'"] & \Sigma X' \arrow[d, "\Sigma f'"] \\
X'' \arrow[r, dashed, "u''"] \arrow[d, "f''"'] & Y'' \arrow[r, dashed, "v''"] \arrow[d, "g''"] & Z'' \arrow[r, dashed, "w''"] \arrow[d, dashed, "h''"] \arrow[dr, phantom, "\boxed{-1}"] & \Sigma X'' \arrow[d, "\Sigma f''"] \\
\Sigma X \arrow[r, "\Sigma u"'] & \Sigma Y \arrow[r, "\Sigma v"'] & \Sigma Z \arrow[r, "\Sigma w"'] & \Sigma^2 X
\end{tikzcd}$$

where the first three rows and columns are exact.

Definition (Neeman)**.** A map of triangles is **middling good** if it extends to a 4×4 diagram.

Verdier good morphisms

Definition. A map of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

is **Verdier good** if h can be constructed as in Verdier's proof of the 4×4 lemma.

Explicitly: ...

Verdier good morphisms, cont'd

... There exists an octahedron for the composite $X \xrightarrow{u} Y \xrightarrow{g} Y'$:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "\alpha_1"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "gu"] & Y' \arrow[r, "\widetilde{v}"] \arrow[d, "g'"] & A \arrow[r, "\widetilde{w}"] \arrow[d, dashed, "\beta_1"] & \Sigma X \\
& Y'' \arrow[r, equal] \arrow[d, "g''"] & Y'' \arrow[d, "\gamma_1"] & \\
& \Sigma Y \arrow[r, "\Sigma v"'] & \Sigma Z, &
\end{tikzcd}$$

Verdier good morphisms, cont'd

... an octahedron for the composite $X \xrightarrow{f} X' \xrightarrow{u'} Y'$:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & X' \arrow[r, "f'"] \arrow[d, "u'"] & X'' \arrow[r, "f''"] \arrow[d, dashed, "\alpha_2"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "u'f=gu"] & Y' \arrow[r, "\widetilde{v}"] \arrow[d, "v'"] & A \arrow[r, "\widetilde{w}"] \arrow[d, dashed, "\beta_2"] & \Sigma X \\
 & Z' \arrow[r, equal] \arrow[d, "w'"] & Z' \arrow[d, "\gamma_2"] & \\
 & \Sigma X' \arrow[r, "\Sigma f'"'] & \Sigma X'', &
\end{tikzcd}
$$

and $h\colon Z \to Z'$ is given by $h = \beta_2 \circ \alpha_1$.

Enhanced 4×4 lemma

Lemma (Miller)**.** A map of triangles (f, g, h) is Verdier good if and only if it extends to a 4×4 diagram and there is an object A ($=$ cofiber of $gu\colon X \to Y'$) together with three diagrams:

$$
\begin{tikzcd}
 & X \arrow[rr, "gu=u'f"] & & Y' \arrow[dl, "\widetilde{v}"'] \arrow[dr, "v'"] & \\
Z \arrow[ur, "w"] \arrow[rr, dashed, "\alpha_1"] & & A \arrow[ul, "\widetilde{w}"'] \arrow[rr, dashed, "\beta_2"] \arrow[dl, dashed, "\beta_1"'] & & Z' \arrow[dl, "(\Sigma f')w'"] \\
 & Y'' \arrow[ul, "(\Sigma v)g''"] & & X'', \arrow[ul, dashed, "\alpha_2"'] \arrow[ll, dashed, "u''"] &
\end{tikzcd}
$$

Enhanced 4×4 lemma, cont'd

$$
\begin{tikzcd}
 & \Sigma X \arrow[dl, "\Sigma f"'] \arrow[dr, "\Sigma u"] & \\
\Sigma X' & & \Sigma Y \\
 & A \arrow[uu, "\widetilde{w}"'] \arrow[dl, dashed, "\beta_2"'] \arrow[dr, dashed, "\beta_1"] & \\
Z' \arrow[uu, "w'"] \arrow[dr, dashed, "h'"'] & & Y'' \arrow[uu, "g''"'] \arrow[dl, dashed, "v''"] \\
 & Z'' &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y' \arrow[dd, "\widetilde{v}"'] & \\
Y \arrow[ur, "g"] \arrow[dd, "v"'] & & X' \arrow[ul, "u'"'] \arrow[dd, "f'"] \\
 & A & \\
Z \arrow[ur, dashed, "\alpha_1"] \arrow[rr, phantom, "\boxed{-1}"] & & X'' \arrow[ul, dashed, "\alpha_2"'] \\
 & \Sigma^{-1} Z''. \arrow[ul, dashed, "\Sigma^{-1} h''"] \arrow[ur, dashed, "\Sigma^{-1} w''"'] &
\end{tikzcd}
$$

Outline

Fill-in of zero

Example (Neeman)**.** The map of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "0"'] & Y \arrow[r, "v"] \arrow[d, "0"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "0"] \arrow[dll, dashed, "\theta"'] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

is good $\iff$ $h = v'\theta w$ for some $\theta \colon \Sigma X \to Y'$.

We call such a map h a "**lightning flash**".

Proposition (Christensen–F.)**.** The equivalent conditions above are further equivalent to:

1. The map $(0, 0, h)$ is Verdier good.
2. The map $(0, 0, h)$ is middling good.
3. The Toda bracket $\langle w', h, v\rangle \subseteq \mathcal{T}(\Sigma Y, \Sigma X')$ contains zero.

Similarly for maps $(0, g, 0)$ and $(f, 0, 0)$.

Not middling good

Example. In the derived category $D(\mathbb{Z})$, the map of triangles

$$\begin{tikzcd}
\mathbb{Z}[0] \arrow[r, "n"] \arrow[d, "0"'] & \mathbb{Z}[0] \arrow[r, "q"] \arrow[d, "0"] & \mathbb{Z}/n[0] \arrow[r, "\epsilon"] \arrow[d, "\epsilon"] & \mathbb{Z}[1] \arrow[d, "0"] \\
\mathbb{Z}[0] \arrow[r, "q"'] & \mathbb{Z}/n[0] \arrow[r, "\epsilon"'] & \mathbb{Z}[1] \arrow[r, "-n"'] & \mathbb{Z}[1]
\end{tikzcd}$$

is *not* middling good.

Example. In the stable homotopy category, the map of triangles

$$\begin{tikzcd}
S^0 \arrow[r, "n"] \arrow[d, "0"'] & S^0 \arrow[r, "q"] \arrow[d, "0"] & M(n) \arrow[r, "\delta"] \arrow[d, "\delta"] & S^1 \arrow[d, "0"] \\
S^0 \arrow[r, "q"'] & M(n) \arrow[r, "\delta"'] & S^1 \arrow[r, "-n"'] & S^1
\end{tikzcd}$$

is *not* middling good.

Chain homotopy

Definition. A map of triangles (f, g, h) is **nullhomotopic** if there are maps (F, G, H) as in

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dl, "F"'] & Z \arrow[r, "w"] \arrow[d, "h"] \arrow[dl, "G"'] & \Sigma X \arrow[d, "\Sigma f"] \arrow[dl, "H"'] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

with

$$\begin{cases} f = Fu + (\Sigma^{-1}w')(\Sigma^{-1}H) \\ g = Gv + u'F \\ h = Hw + v'G. \end{cases}$$

Two maps of triangles (f, g, h) and $(\overline{f}, \overline{g}, \overline{h})$ are **chain homotopic** if their difference is nullhomotopic:

$$(\overline{f} - f, \overline{g} - g, \overline{h} - h) \simeq (0, 0, 0).$$

Chain homotopy invariance

Fact. Chain homotopic maps have isomorphic mapping cones.

$\implies$ Goodness is invariant under chain homotopy.

What about Verdier goodness?

Proposition (Neeman)**.** If (f, g, h) is Verdier good, then so is $(f, g, h + v'\theta w)$ for any $\theta\colon \Sigma X \to Y'$.

In other words: Adding a lightning flash preserves Verdier goodness.

Contractible triangles

Definition. A triangle $X \to Y \to Z \to \Sigma X$ is **contractible** if its identity map $(1_X, 1_Y, 1_Z)$ is nullhomotopic.

Example. A split triangle $X \to Y \to Z \xrightarrow{0} \Sigma X$ is contractible.

Proposition (Neeman)**.** If the top row or bottom row of

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X'
\end{tikzcd}$$

is contractible, then the map of triangles is Verdier good.

Middling good but not good

Example (Neeman)**.** The map of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "u"'] & Y \arrow[r, "v"] \arrow[d, "v"] & Z \arrow[r, "w"] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\ Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & \Sigma X \arrow[r, "-\Sigma u"'] & \Sigma Y \end{tikzcd}$$

is always middling good.

It is good $\iff$ $w = w\theta w$ for some $\theta\colon \Sigma X \to Z$.

For instance, with $\mathcal{T}(\Sigma X, Z) = 0$ and $w \neq 0$, the map of triangles is *not* good.

Fact. The map (u, v, w) above is Verdier good $\iff$ $w = w\theta w$ for some $\theta\colon \Sigma X \to Z$.

Middling goodness, cont'd

Corollary. Middling goodness is *not* invariant under chain homotopy.

Indeed, consider $(u, v, w) \simeq (0, 0, w)$.

Summary

$$
\begin{tikzcd}
\text{good} \arrow[rr, dashed, no head, "?"] \arrow[dr, Rightarrow] & & \text{Verdier good} \arrow[dl, Rightarrow] \\
& \text{middling good} \arrow[ul, Rightarrow, bend left, "\times" marking] \arrow[ur, Rightarrow, bend right, "\times" marking] &
\end{tikzcd}
$$

Outline

Main questions

1. Is Verdier goodness equivalent to goodness?
   (Does one imply the other?)
2. Is Verdier goodness invariant under chain homotopy?
   (Does nullhomotopic imply Verdier good?)
3. Is Verdier goodness invariant under rotation?

Warning: Composition

Proposition (Neeman+ϵ)**.** Every map of triangles is a composite of two maps that are good and Verdier good.

In particular, a composite of Verdier good maps need not be Verdier good.

Some special cases

Consider a map of triangles (f, g, h).

- If f and g are (split) monomorphisms, then good $\iff$ Verdier good.
- If f and g are (split) epimorphisms, then good $\iff$ Verdier good.
- Case $(1, g, h)$: good $\implies$ Verdier good (Neeman). More later.
- If one component is zero...

A zero component

Lemma. A map of triangles $(f, g, 0)$ is nullhomotopic $\iff$ it is nullhomotopic via a nullhomotopy with a single component $(F, 0, 0)$.

Theorem (Christensen–F.)**.**

1. A map of triangles $(f, g, 0)$ is good (if and) only if it is nullhomotopic.
   Likewise for $(f, 0, h)$ and $(0, g, h)$. (Automatic.)
2. For $(f, 0, h)$ and $(0, g, h)$, the condition is further equivalent to Verdier goodness.
3. **If** the triangulated structure admits a 3-triangulated enhancement, then $(f, g, 0)$ being good $\implies$ Verdier good.

Lifting criterion

Corollary. In the diagram with exact rows

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dl, dashed, "k"', "{\color{red}?}"] & Z \arrow[r, "w"] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X', \end{tikzcd}$$

there exists a lift $k\colon Y \to X'$ satisfying $ku = f$ and $u'k = g$
$\iff$ The map $0\colon Z \to Z'$ is a good fill-in.

Remark. Zero being a fill-in is not a sufficient condition. In the map of triangles in $D(\mathbb{Z})$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "q"] \arrow[d, "0"] & \mathbb{Z}/2 \arrow[r, "\epsilon"] \arrow[d, "\epsilon"] \arrow[dl, dashed, "{\color{red}\nexists\, k}"'] & \mathbb{Z}[1] \arrow[r, "-2"] \arrow[d, "0"] & \mathbb{Z}[1] \arrow[d, "0"] \\ \mathbb{Z}/2 \arrow[r, "\gamma"] & \mathbb{Z}[1] \arrow[r, "-2"] & \mathbb{Z}[1] \arrow[r, "-q"] & \mathbb{Z}/2[1], \end{tikzcd}$$

the fill-in map is zero, but no lift k exists.

Outline

Adams spectral sequence

Good morphisms

Examples and non-examples

Questions and results

Homotopy cartesian squares

Homotopy cartesian squares

Definition. A commutative square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\ Y' \arrow[r, "f'"'] & Z' \end{tikzcd}$$

is called **homotopy cartesian** if there is a "Mayer–Vietoris" exact triangle

$$Y \xrightarrow{\begin{bmatrix} g \\ f \end{bmatrix}} Y' \oplus Z \xrightarrow{[\, f' \ -g' \,]} Z' \xrightarrow{\partial} \Sigma Y$$

for some map $\partial \colon Z' \to \Sigma Y$, called the **differential**.

Relationship with good maps

Lemma (Neeman)**.** Consider a homotopy cartesian square

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ Y' \arrow[r, "v'"'] & Z' \end{tikzcd}$$

with given differential $\partial\colon Z' \to \Sigma Y$. Then the square extends to a good map of triangles of the form $(1, g, h)$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X \end{tikzcd}$$

satisfying $\partial = (\Sigma u)w'$. Moreover, we may prescribe the top row or the bottom row.

Can we make the relationship more precise?

Replaceably exact triangle

Definition (Vaknin 2001)**.** In a candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X,$$

the map u is **replaceable** with **replacing map** $\widehat{u}$ if

$$X \xrightarrow{\widehat{u}} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is exact. Replaceability of v or w is defined similarly.

The candidate triangle is **replaceably exact** if its three maps are replaceable.

Homotopy cartesian versus goodness

Proposition. Consider a map of triangles $(1, g, h)$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X
\end{tikzcd}$$

and its "Mayer–Vietoris" candidate triangle

$$Y \xrightarrow{\left[\begin{smallmatrix} g \\ v \end{smallmatrix}\right]} Y' \oplus Z \xrightarrow{[\, v' \ -h \,]} Z' \xrightarrow{(\Sigma u)w'} \Sigma Y. \tag{MV}$$

1. The map $(1, g, h)$ is good if and only if the middle square is homotopy cartesian with differential $\partial = (\Sigma u)w' \colon Z' \to \Sigma Y$, i.e., (MV) is exact.
2. The middle square is homotopy cartesian if and only if the candidate triangle (MV) is replaceably exact.
3. The map $(1, g, h)$ is Verdier good if and only if it extends to an octahedron.

Homotopy cartesian squares

Remark. See also Canonaco–Künzer (2011).

Pasting lemma

Proposition (Christensen–F.)**.** Consider a diagram

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ Y' \arrow[r, "v'"] \arrow[d, "g'"'] & Z' \arrow[d, "h'"] \\ Y'' \arrow[r, "v''"] & Z''. \end{tikzcd}$$

If the two squares are homotopy cartesian, then so is their pasting, the big rectangle.

Moreover, given differentials $\partial_U \colon Z' \to \Sigma Y$ and $\partial_L \colon Z'' \to \Sigma Y'$ of the upper square and lower square respectively, there exists a differential $\partial_P \colon Z'' \to \Sigma Y$ for the pasted rectangle satisfying

$$(\Sigma g)\partial_P = \partial_L \qquad \text{and} \qquad \partial_P h' = \partial_U .$$

Strengthens a result of Vaknin (2001).

Examples in a stable module category

Let C_4 be the cyclic group of order 4, with generator g. Consider the group algebra

$$R = \mathbb{F}_2 C_4 \cong \mathbb{F}_2[x]/x^4$$

with $x := g - 1$.

We will work in the stable module category $\mathrm{StMod}(R)$.

For $r \in R$, let

$$\mu_r \colon R/x^i \to R/x^j$$

denote the R-module map sending 1 to r, when it is defined. For example:

$$\mu_x \colon R/x \to R/x^2.$$

Examples in $\mathrm{StMod}(\mathbb{F}_2 C_4)$ (cont'd)

The following map of triangles is good and Verdier good:

$$\begin{tikzcd}
R/x^3 \oplus R/x^2 \arrow[r, "\begin{bmatrix}1 & 0\\ 0 & \mu_x\end{bmatrix}"] \arrow[d, equal] & R/x^3 \oplus R/x^3 \arrow[r, "{[0\ \mu_1]}"] \arrow[d, "{[\mu_1\ 0]}"] & R/x \arrow[r, "\begin{bmatrix}0\\ \mu_x\end{bmatrix}"] \arrow[d, "\begin{bmatrix}0\\ \mu_x\end{bmatrix}"] & R/x \oplus R/x^2 \arrow[d, equal] \\
R/x^3 \oplus R/x^2 \arrow[r, "{[\mu_1\ 0]}"'] & R/x \arrow[r, "\begin{bmatrix}\mu_x\\ 0\end{bmatrix}"'] & R/x^2 \oplus R/x^2 \arrow[r, "\begin{bmatrix}\mu_1 & 0\\ 0 & 1\end{bmatrix}"'] & R/x \oplus R/x^2.
\end{tikzcd}$$

The following map with the same homotopy cartesian middle square is neither good nor Verdier good:

$$\begin{tikzcd}
R/x^3 \oplus R/x^2 \arrow[r, "\begin{bmatrix}1 & 0\\ 0 & \mu_x\end{bmatrix}"] \arrow[d, equal] & R/x^3 \oplus R/x^3 \arrow[r, "{[0\ \mu_1]}"] \arrow[d, "{[\mu_1\ 0]}"] & R/x \arrow[r, "\begin{bmatrix}0\\ \mu_x\end{bmatrix}"] \arrow[d, "\begin{bmatrix}0\\ \mu_x\end{bmatrix}"] & R/x \oplus R/x^2 \arrow[d, equal] \\
R/x^3 \oplus R/x^2 \arrow[r, "{[\mu_1\ 0]}"'] & R/x \arrow[r, "\begin{bmatrix}\mu_x\\ 0\end{bmatrix}"'] & R/x^2 \oplus R/x^2 \arrow[r, "\begin{bmatrix}\mu_1 & \mu_1\\ 0 & 1\end{bmatrix}"'] & R/x \oplus R/x^2.
\end{tikzcd}$$

Examples in $\mathrm{StMod}(\mathbb{F}_2 C_4)$ (cont'd)

Consider the diagram with exact rows

$$\begin{tikzcd}
R/x^3 \oplus R/x^2 \oplus R/x^2 \arrow[r, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_x \end{bmatrix}"] \arrow[d, equal] & R/x^3 \oplus R/x^2 \oplus R/x^3 \arrow[r, "{[0\ 0\ \mu_1]}"] \arrow[d, "\begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}"] & R/x \arrow[r, "\begin{bmatrix} 0 \\ 0 \\ \mu_x \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}"] & R/x \oplus R/x^2 \oplus R/x^2 \arrow[d, equal] \\
R/x^3 \oplus R/x^2 \oplus R/x^2 \arrow[r, "\begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & 0 & \mu_x \end{bmatrix}"] \arrow[d, equal] & R/x \oplus R/x^3 \arrow[r, "\begin{bmatrix} \mu_x & 0 \\ 0 & 0 \\ 0 & \mu_1 \end{bmatrix}"] \arrow[d, "{[1\ 0]}"] & R/x^2 \oplus R/x^2 \oplus R/x \arrow[r, "\begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_x \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_x \end{bmatrix}"] & R/x \oplus R/x^2 \oplus R/x^2 \arrow[d, equal] \\
R/x^3 \oplus R/x^2 \oplus R/x^2 \arrow[r, "{[\mu_1\ 0\ 0]}"'] & R/x \arrow[r, "\begin{bmatrix} \mu_x \\ 0 \\ 0 \end{bmatrix}"'] & R/x^2 \oplus R/x^2 \oplus R/x^2 \arrow[r, "\begin{bmatrix} \mu_1 & 0 & \mu_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}"'] & R/x \oplus R/x^2 \oplus R/x^2.
\end{tikzcd}$$

The top map is good.

The bottom map is good.

The composite is not good. Its "Mayer–Vietoris" triangle is replaceably exact but not exact.

Some future directions

- Search for counterexamples $\rightsquigarrow$ Computer-assisted calculations in $\mathrm{StMod}(kG)$? See Künzer (2009).
- Exotic triangulated categories? See Muro–Schwede–Strickland (2007), Bergh–Jasso–Thaule (2016).
- n-angulated categories? Work in progress by Thaule and Martensen.

Thank you!