# On good morphisms of exact triangles

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Joint with Dan Christensen

New Directions in Group Theory and Triangulated Categories May 3, 2022

## Reference

Christensen and F. On good morphisms of exact triangles. J. Pure  $Appl.\ Algebra\ (2022).$ 

## **Outline**

## Adams spectral sequence

Good morphisms

Examples and non-examples

Questions and results

Homotopy cartesian squares

## Classical Adams spectral sequence

Given finite spectra X and Y, the classical Adams spectral sequence has the form

$$E_2^{s,t} = \operatorname{Ext}_{\mathcal{A}}^s(H^*Y, \Sigma^t H^*X) \Rightarrow [\Sigma^{t-s}X, Y_p^{\wedge}]$$

where  $\mathcal{A}$  denotes the mod p Steenrod algebra.

For E a nice ring spectrum (e.g. MU or BP), the E-based Adams spectral sequence is:

$$E_2^{s,t} = \operatorname{Ext}_{E_*E}^s(\Sigma^t E_* X, E_* Y) \Rightarrow [\Sigma^{t-s} X, L_E Y].$$

# Triangulated version

- Brinkmann (1968): Adams spectral sequence in a triangulated category.
- Franke (1996): Application to E(1)-local (= $KU_{(p)}$ -local) spectra.
- Further work related to Franke's construction (2007 and on): Roitzheim, Barnes, Patchkoria, and others. Applications to E-local spectra and E-module spectra for various E.
- Christensen (1998): Application to ghost lengths and stable module categories.

## Examples of triangulated categories

**Example.** The homotopy category of spectra, a.k.a. the stable homotopy category.

**Example.** The derived category of a ring D(R).

**Example.** The stable module category of a group algebra  $\operatorname{StMod}(kG)$ .

# Projective and injective classes

Eilenberg and Moore (1965) gave a framework for relative homological algebra in any pointed category. When the category is triangulated, their axioms are equivalent to the following.

**Definition.** A **projective class** in  $\mathcal{T}$  is a pair  $(\mathcal{P}, \mathcal{N})$ , where  $\mathcal{P} \subseteq \text{ob } \mathcal{T}$  and  $\mathcal{N} \subseteq \text{mor } \mathcal{T}$ , such that:

- (i)  $\mathcal{P}$  consists of exactly the objects P such that every composite  $P \to X \to Y$  is zero for each  $X \to Y$  in  $\mathcal{N}$ .
- (ii)  $\mathcal{N}$  consists of exactly the maps  $X \to Y$  such that every composite  $P \to X \to Y$  is zero for each P in  $\mathcal{P}$ .
- (iii) For each X in  $\mathcal{T}$ , there is a triangle  $P \to X \to Y$  with P in  $\mathcal{P}$  and  $X \to Y$  in  $\mathcal{N}$ .

An injective class in  $\mathcal{T}$  is a projective class in  $\mathcal{T}^{op}$ .

# **Examples**

#### **Example.** In spectra, take:

$$\mathcal{P} = \text{retracts of wedges of spheres } \bigvee_{i} S^{n_i}$$

 $\mathcal{N} = \text{maps inducing zero on homotopy groups.}$ 

 $(\mathcal{P}, \mathcal{N})$  is the ghost projective class.

**Example.** For E any spectrum, take:

$$\mathcal{I} = \text{retracts of products } \prod_{i} \Sigma^{n_i} E$$

 $\mathcal{N} = \text{maps inducing zero on } E^*(-).$ 

Then  $(\mathcal{I}, \mathcal{N})$  is an injective class.

For  $E = H\mathbb{F}_p$ , this injective class leads to the classical (cohomological) Adams spectral sequence.

# **Examples**

**Example.** For E a homotopy commutative ring spectrum, take:

 $\mathcal{I} = \text{retracts of } E \wedge W$  $\mathcal{N} = \text{maps } f \colon X \to Y \text{ with } E \wedge f \simeq 0 \colon E \wedge X \to E \wedge Y.$ 

The injective class  $(\mathcal{I}, \mathcal{N})$  leads to the *E*-based (homological) Adams spectral sequence.

**Remark.** We always assume that our projective and injective classes are stable, i.e., closed under suspension and desuspension.

# **Examples**

**Example.** Let A be a differential graded (dg) algebra. In dg-A-modules, take:

$$\mathcal{P} = \text{retracts of sums } \bigoplus_{i} A[n_i]$$

 $\mathcal{N} = \text{maps inducing zero on homology.}$ 

 $(\mathcal{P}, \mathcal{N})$  is the ghost projective class.

The Adams spectral sequence relative to  $\mathcal{P}$  is the universal coefficient spectral sequence

$$\operatorname{Ext}_{H_*A}^s(H_*M[t], H_*N) \Rightarrow \operatorname{Ext}_A^s(M[t], N) = D(A)(M[t], N[s])$$

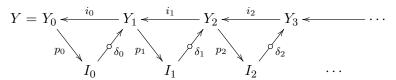
from ordinary Ext to differential Ext.

**Remark.** A monoidal version also recovers the Eilenberg–Moore spectral sequence

$$\operatorname{Tor}_{H_*A}(H_*M, H_*N) \Rightarrow \operatorname{Tor}_A(M, N).$$

#### Adams resolutions

**Definition.** An Adams resolution of an object Y in  $\mathcal{T}$  with respect to an injective class  $(\mathcal{I}, \mathcal{N})$  is a diagram



where each  $I_s$  is injective, each map  $i_s$  is in  $\mathcal{N}$ , and the triangles are triangles.

Axiom (iii) says that you can form such a resolution.

Applying  $\mathcal{T}(X,-)$  leads to an exact couple and therefore a spectral sequence, called the Adams spectral sequence.

$$E_2^{s,t} = \operatorname{Ext}_{\mathcal{I}}^s(\Sigma^t X, Y)$$

## ... But why?

Why work with a triangulated category instead of a stable  $\infty$ -category or stable model category?

- Fewer hypotheses.
- There's a lot we can do using only the triangulated structure.
- Get derived invariants.

## Cofibers in an Adams tower

Starting point: an  $\mathcal{I}$ -Adams tower

$$X = X_0 \stackrel{x_1}{\longleftarrow} X_1 \stackrel{x_2}{\longleftarrow} X_2 \stackrel{\cdots}{\longleftarrow} \cdots$$

For intervals  $[n, m] \leq [n', m']$ , there is a fill-in in the diagram

Question. How convenient can those choices be made?

## **Outline**

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## Mapping cone

**Definition** (Neeman). The mapping cone of a map of triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$f \downarrow \qquad \downarrow g \qquad \downarrow h \qquad \downarrow \Sigma f$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

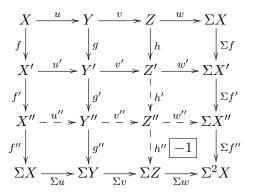
is the sequence

$$X' \oplus Y \xrightarrow{\begin{bmatrix} u' & g \\ 0 & -v \end{bmatrix}} Y' \oplus Z \xrightarrow{\begin{bmatrix} v' & h \\ 0 & -w \end{bmatrix}} Z' \oplus \Sigma X \xrightarrow{\begin{bmatrix} w' & \Sigma f \\ 0 & -\Sigma u \end{bmatrix}} \Sigma X' \oplus \Sigma Y.$$

The map of triangles (f, g, h) is **good** if its mapping cone is an exact triangle.

## Middling good morphisms

**Proposition** (Neeman). If the map of triangles (f, g, h) is good, then it extends to a  $4 \times 4$  diagram



where the first three rows and columns are exact.

**Definition** (Neeman). A map of triangles is middling good if it extends to a  $4 \times 4$  diagram.

# Verdier good morphisms

**Definition.** A map of triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$f \downarrow \qquad \downarrow g \qquad \downarrow h \qquad \downarrow \Sigma f$$

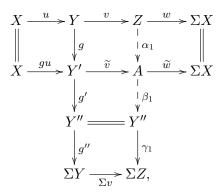
$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

is **Verdier good** if h can be constructed as in Verdier's proof of the  $4 \times 4$  lemma.

Explicitly: ...

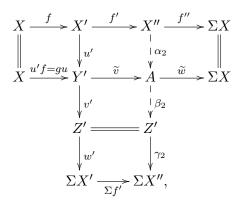
## Verdier good morphisms, cont'd

... There exists an octahedron for the composite  $X \xrightarrow{u} Y \xrightarrow{g} Y'$ :



## Verdier good morphisms, cont'd

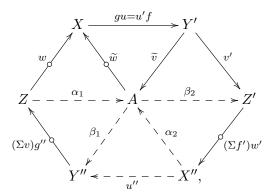
... an octahedron for the composite  $X \xrightarrow{f} X' \xrightarrow{u'} Y'$ :



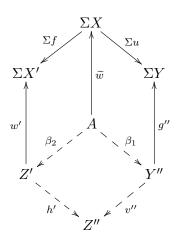
and  $h: Z \to Z'$  is given by  $h = \beta_2 \circ \alpha_1$ .

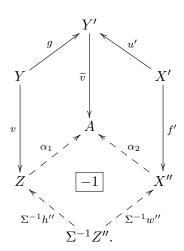
#### Enhanced $4 \times 4$ lemma

**Lemma** (Miller). A map of triangles (f, g, h) is Verdier good if and only if it extends to a  $4 \times 4$  diagram and there is an object A (= cofiber of  $gu: X \to Y'$ ) together with three diagrams:



## Enhanced $4 \times 4$ lemma, cont'd





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#### Fill-in of zero

**Example** (Neeman). The map of triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$0 \downarrow \qquad \downarrow 0 \qquad \qquad \downarrow 0 \qquad \qquad \downarrow 0$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

is good  $\iff h = v'\theta w \text{ for some } \theta \colon \Sigma X \to Y'.$ 

We call such a map h a "lightning flash".

**Proposition** (Christensen–F.). The equivalent conditions above are further equivalent to:

- 1. The map (0,0,h) is Verdier good.
- 2. The map (0,0,h) is middling good.
- 3. The Toda bracket  $\langle w', h, v \rangle \subseteq \mathcal{T}(\Sigma Y, \Sigma X')$  contains zero.

Similarly for maps (0, g, 0) and (f, 0, 0).

# Not middling good

**Example.** In the derived category  $D(\mathbb{Z})$ , the map of triangles

$$\begin{split} \mathbb{Z}[0] & \xrightarrow{n} \mathbb{Z}[0] \xrightarrow{q} \mathbb{Z}/n[0] \xrightarrow{\epsilon} \mathbb{Z}[1] \\ 0 & \downarrow 0 & \downarrow \epsilon & \downarrow 0 \\ \mathbb{Z}[0] & \xrightarrow{q} \mathbb{Z}/n[0] \xrightarrow{\epsilon} \mathbb{Z}[1] \xrightarrow{-n} \mathbb{Z}[1] \end{split}$$

is *not* middling good.

**Example.** In the stable homotopy category, the map of triangles

$$S^{0} \xrightarrow{n} S^{0} \xrightarrow{q} M(n) \xrightarrow{\delta} S^{1}$$

$$\downarrow 0 \qquad \qquad \downarrow \delta \qquad \qquad \downarrow 0$$

$$S^{0} \xrightarrow{q} M(n) \xrightarrow{\delta} S^{1} \xrightarrow{-n} S^{1}$$

is not middling good.

## Chain homotopy

**Definition.** A map of triangles (f, g, h) is **nullhomotopic** if there are maps (F, G, H) as in

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$f \downarrow F \downarrow g G \downarrow h H \downarrow \Sigma f$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

with

$$\begin{cases} f = Fu + (\Sigma^{-1}w')(\Sigma^{-1}H) \\ g = Gv + u'F \\ h = Hw + v'G. \end{cases}$$

Two maps of triangles (f, g, h) and  $(\overline{f}, \overline{g}, \overline{h})$  are **chain homotopic** if their difference is nullhomotopic:

$$(\overline{f} - f, \overline{g} - g, \overline{h} - h) \simeq (0, 0, 0).$$

# Chain homotopy invariance

Fact. Chain homotopic maps have isomorphic mapping cones.

 $\implies$  Goodness is invariant under chain homotopy.

What about Verdier goodness?

**Proposition** (Neeman). If (f, g, h) is Verdier good, then so is  $(f, g, h + v'\theta w)$  for any  $\theta \colon \Sigma X \to Y'$ .

In other words: Adding a lightning flash preserves Verdier goodness.

## Contractible triangles

**Definition.** A triangle  $X \to Y \to Z \to \Sigma X$  is **contractible** if its identity map  $(1_X, 1_Y, 1_Z)$  is nullhomotopic.

**Example.** A split triangle  $X \to Y \to Z \xrightarrow{0} \Sigma X$  is contractible.

**Proposition** (Neeman). If the top row or bottom row of

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$f \downarrow \qquad \downarrow g \qquad \downarrow h \qquad \downarrow \Sigma f$$

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$$

is contractible, then the map of triangles is Verdier good.

## Middling good but not good

**Example** (Neeman). The map of triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$\downarrow u \qquad \qquad \downarrow v \qquad \qquad \downarrow \Sigma u$$

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is always middling good.

It is good  $\iff w = w\theta w \text{ for some } \theta \colon \Sigma X \to Z.$ 

For instance, with  $\mathcal{T}(\Sigma X, Z) = 0$  and  $w \neq 0$ , the map of triangles is not good.

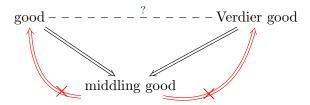
**Fact.** The map (u, v, w) above is Verdier good  $\iff w = w\theta w$  for some  $\theta \colon \Sigma X \to Z$ .

## Middling goodness, cont'd

Corollary. Middling goodness is not invariant under chain homotopy.

Indeed, consider  $(u, v, w) \simeq (0, 0, w)$ .

# **Summary**



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## Main questions

- 1. Is Verdier goodness equivalent to goodness? (Does one imply the other?)
- 2. Is Verdier goodness invariant under chain homotopy? (Does nullhomotopic imply Verdier good?)
- 3. Is Verdier goodness invariant under rotation?

## Warning: Composition

**Proposition** (Neeman+ $\epsilon$ ). Every map of triangles is a composite of two maps that are good and Verdier good.

In particular, a composite of Verdier good maps need not be Verdier good.

## Some special cases

Consider a map of triangles (f, g, h).

- If f and g are (split) monomorphisms, then good  $\iff$  Verdier good.
- If f and g are (split) epimorphisms, then good  $\iff$  Verdier good.
- Case (1, g, h): good  $\implies$  Verdier good (Neeman). More later.
- If one component is zero...

# A zero component

**Lemma.** A map of triangles (f, g, 0) is nullhomotopic  $\iff$  it is nullhomotopic via a nullhomotopy with a single component (F, 0, 0).

## Theorem (Christensen-F.).

- 1. A map of triangles (f, g, 0) is good (if and) only if it is nullhomotopic. Likewise for (f, 0, h) and (0, g, h). (Automatic.)
- 2. For (f,0,h) and (0,g,h), the condition is further equivalent to Verdier goodness.
- 3. If the triangulated structure admits a 3-triangulated enhancement, then (f, g, 0) being good  $\Longrightarrow$  Verdier good.

## Lifting criterion

Corollary. In the diagram with exact rows

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$f \downarrow k / \downarrow g \qquad \downarrow \Sigma f$$

$$X' \xrightarrow{u'} Y' \xrightarrow{u'} Z' \xrightarrow{w'} \Sigma X',$$

there exists a lift  $k: Y \to X'$  satisfying ku = f and u'k = g  $\iff$  The map  $0: Z \to Z'$  is a good fill-in.

**Remark.** Zero being a fill-in is not a sufficient condition. In the map of triangles in  $D(\mathbb{Z})$ 

$$\mathbb{Z} \xrightarrow{q} \mathbb{Z}/2 \xrightarrow{\epsilon} \mathbb{Z}[1] \xrightarrow{-2} \mathbb{Z}[1]$$

$$\downarrow 0 \qquad \qquad \downarrow 0 \qquad \qquad \downarrow 0$$

$$\mathbb{Z}/2 \xrightarrow{\gamma} \mathbb{Z}[1] \xrightarrow{-2} \mathbb{Z}[1] \xrightarrow{-q} \mathbb{Z}/2[1],$$

the fill-in map is zero, but no lift k exists.

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### Homotopy cartesian squares

**Definition.** A commutative square

$$Y \xrightarrow{f} Z$$

$$\downarrow g'$$

$$Y' \xrightarrow{f'} Z'$$

is called **homotopy cartesian** if there is a "Mayer–Vietoris" exact triangle

$$Y \xrightarrow{\left[\begin{smallmatrix} g \\ f \end{smallmatrix}\right]} Y' \oplus Z \xrightarrow{\left[f' - g'\right]} Z' \xrightarrow{\partial} \Sigma Y$$

for some map  $\partial \colon Z' \to \Sigma Y$ , called the **differential**.

### Relationship with good maps

Lemma (Neeman). Consider a homotopy cartesian square

$$Y \xrightarrow{v} Z$$

$$\downarrow g \qquad \qquad \downarrow h$$

$$Y' \xrightarrow{v'} Z'$$

with given differential  $\partial \colon Z' \to \Sigma Y$ . Then the square extends to a good map of triangles of the form (1, g, h)

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$\parallel \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$$

$$X \xrightarrow{v'} Y' \xrightarrow{v'} Z' \xrightarrow{v'} \Sigma X$$

satisfying  $\partial = (\Sigma u)w'$ . Moreover, we may prescribe the top row or the bottom row.

Can we make the relationship more precise?

### Replaceably exact triangle

**Definition** (Vaknin 2001). In a candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

the map u is replaceable with replacing map  $\widehat{u}$  if

$$X \xrightarrow{\widehat{u}} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is exact. Replaceability of v or w is defined similarly.

The candidate triangle is **replaceably exact** if its three maps are replaceable.

## Homotopy cartesian versus goodness

**Proposition.** Consider a map of triangles (1, g, h)

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

$$\parallel \qquad \downarrow^{g} \qquad \downarrow^{h} \qquad \parallel$$

$$X \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$$

and its "Mayer-Vietoris" candidate triangle

$$Y \xrightarrow{\left[\begin{smallmatrix} g \\ v \end{smallmatrix}\right]} Y' \oplus Z \xrightarrow{\left[\begin{smallmatrix} v' & -h \end{smallmatrix}\right]} Z' \xrightarrow{(\Sigma u)w'} \Sigma Y. \tag{MV}$$

- 1. The map (1, g, h) is good if and only if the middle square is homotopy cartesian with differential  $\partial = (\Sigma u)w' \colon Z' \to \Sigma Y$ , i.e., (MV) is exact.
- 2. The middle square is homotopy cartesian if and only if the candidate triangle (MV) is replaceably exact.
- 3. The map (1, g, h) is Verdier good if and only if it extends to an octahedron.

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# Homotopy cartesian squares

Remark. See also Canonaco-Künzer (2011).

#### Pasting lemma

**Proposition** (Christensen–F.). Consider a diagram

$$Y \xrightarrow{v} Z$$

$$g \downarrow \qquad \qquad \downarrow h$$

$$Y' \xrightarrow{v'} Z'$$

$$g' \downarrow \qquad \qquad \downarrow h'$$

$$Y'' \xrightarrow{v''} Z''.$$

If the two squares are homotopy cartesian, then so is their pasting, the big rectangle.

Moreover, given differentials  $\partial_U \colon Z' \to \Sigma Y$  and  $\partial_L \colon Z'' \to \Sigma Y'$  of the upper square and lower square respectively, there exists a differential  $\partial_P \colon Z'' \to \Sigma Y$  for the pasted rectangle satisfying

$$(\Sigma g)\partial_P = \partial_L$$
 and  $\partial_P h' = \partial_U$ .

Strengthens a result of Vaknin (2001).

# Examples in a stable module category

Let  $C_4$  be the cyclic group of order 4, with generator g. Consider the group algebra

$$R = \mathbb{F}_2 C_4 \cong \mathbb{F}_2[x]/x^4$$

with x := q - 1.

We will work in the stable module category StMod(R).

For  $r \in R$ , let

$$\mu_r \colon R/x^i \to R/x^j$$

denote the R-module map sending 1 to r, when it is defined. For example:

$$\mu_x \colon R/x \to R/x^2$$
.

# Examples in $StMod(\mathbb{F}_2C_4)$ (cont'd)

The following map of triangles is good and Verdier good:

$$R/x^{3} \oplus R/x^{2} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & \mu_{x} \end{bmatrix}} R/x^{3} \oplus R/x^{3} \xrightarrow{[0 & \mu_{1}]} R/x \xrightarrow{\begin{bmatrix} 0 & \mu_{1} \end{bmatrix}} R/x \xrightarrow{\begin{bmatrix} 0 \\ \mu_{x} \end{bmatrix}} R/x \oplus R/x^{2}$$

$$\downarrow \begin{bmatrix} \mu_{1} & 0 \end{bmatrix} \qquad \qquad \downarrow \begin{bmatrix} 0 \\ \mu_{x} \end{bmatrix} \qquad \qquad \parallel$$

$$R/x^{3} \oplus R/x^{2} \xrightarrow{[\mu_{1} & 0]} R/x \xrightarrow{[\mu_{1} & 0]} R/x \oplus R/x^{2}.$$

The following map with the same homotopy cartesian middle square is neither good nor Verdier good:

$$R/x^3 \oplus R/x^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & \mu_x \end{bmatrix}} R/x^3 \oplus R/x^3 \xrightarrow{ \begin{bmatrix} 0 & \mu_1 \end{bmatrix}} R/x \xrightarrow{\begin{bmatrix} 0 & \mu_1 \end{bmatrix}} R/x \xrightarrow{\begin{bmatrix} 0 & \mu_1 \end{bmatrix}} R/x \oplus R/x^2$$

$$\downarrow \begin{bmatrix} \mu_1 & 0 \end{bmatrix} \qquad \downarrow \begin{bmatrix} 0 \\ \mu_x \end{bmatrix} \qquad \qquad \downarrow \begin{bmatrix} 0 \\ \mu_x \end{bmatrix}$$

$$R/x^3 \oplus R/x^2 \xrightarrow{ \begin{bmatrix} \mu_1 & 0 \end{bmatrix}} R/x \xrightarrow{ \begin{bmatrix} \mu_2 \\ 0 \end{bmatrix}} R/x \oplus R/x^2.$$

# Examples in $StMod(\mathbb{F}_2C_4)$ (cont'd)

Consider the diagram with exact rows

$$R/x^{3} \oplus R/x^{2} \oplus R/x^{2} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \mu_{x} \end{bmatrix}} R/x^{3} \oplus R/x^{2} \oplus R/x^{3} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} R/x \xrightarrow{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} R/x \xrightarrow{\begin{bmatrix} 0 & 0 \\ \mu_{x} \end{bmatrix}} R/x \oplus R/x^{2} \oplus R/x^{2}$$

$$R/x^{3} \oplus R/x^{2} \oplus R/x^{2} \xrightarrow{\begin{bmatrix} \mu_{1} & 0 & 0 \\ 0 & 0 & \mu_{x} \end{bmatrix}} R/x \oplus R/x^{3} \xrightarrow{\begin{bmatrix} \mu_{1} & 0 & 0 \\ 0 & 0 & \mu_{x} \end{bmatrix}} R/x \oplus R/x^{2} \oplus R/x \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \mu_{x} \end{bmatrix}} R/x \oplus R/x^{2} \oplus R/x^{2} \oplus R/x^{2}$$

$$R/x^{3} \oplus R/x^{2} \oplus R/x^{2} \xrightarrow{\begin{bmatrix} \mu_{1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} R/x \oplus R/x^{2} \oplus R/x^{2}$$

The top map is good.

The bottom map is good.

The composite is **not** good. Its "Mayer–Vietoris" triangle is replaceably exact but not exact.

#### Some future directions

- Search for counterexamples  $\rightsquigarrow$  Computer-assisted calculations in StMod(kG)? See Künzer (2009).
- Exotic triangulated categories? See Muro-Schwede-Strickland (2007), Bergh-Jasso-Thaule (2016).
- *n*-angulated categories? Work in progress by Thaule and Martensen.

Thank you!