Multiparameter persistence modules in the large scale

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Outline

Persistence modules

Localized persistence modules

Classification of indecomposables

Which subcategories are we quotienting out?

Rank invariant

Persistence modules

Fix a ground field $\mathbb{k}$.

Definition. For $m \geq 1$, an ***m*-parameter persistence module** is a diagram

$$\mathbb{N}^m \to \mathrm{Vect}_{\mathbb{k}}.$$

$\cong$ graded module over the graded polynomial algebra

$$R \coloneqq \mathbb{k}[t_1, \ldots, t_m],$$

which is $\mathbb{N}^m$-graded with multigrading

$$|t_i| = \vec{e}_i = (0, \ldots, \overbrace{\;1\;}^{i}, \ldots, 0).$$

Write $M(\vec{d})$ for the $\mathbb{k}$-vector space in multidegree $\vec{d} \in \mathbb{N}^m$.

Goal / Dream

Work with *finitely generated* R-modules:

$$R\text{-mod} \coloneqq R\text{-Mod}^{\text{fin.gen.}} \subset R\text{-Mod}.$$

Goal

Classify the indecomposable objects in R-mod.

One-parameter case

For $m = 1$, finitely generated $\Bbbk[t]$-modules decompose into **interval modules**

$$
\begin{aligned}
[a, b) &:= t^a \Bbbk[t] / t^b \Bbbk[t] \\
&= \operatorname{coker}\left(t^a \Bbbk[t] \xrightarrow{t^{b-a}} t^a \Bbbk[t] \right).
\end{aligned}
$$

Example.

$$
\begin{aligned}
M &= t^4 \Bbbk[t] \oplus t^2 \Bbbk[t] / t^7 \Bbbk[t] \\
&= [4, \infty) \oplus [2, 7).
\end{aligned}
$$

$\rightsquigarrow$ Barcodes.

Multiparameter case

Not available for $m \geq 2$, because $\Bbbk[t_1, t_2]$ has wild representation type.

How to deal with that?

One approach: Extract invariants that are both computable and significant. Rank invariant and various refinements. Many authors: [Botnan, Oppermann, Oudot], etc.

Another approach: Focus on certain families of modules admitting a nice decomposition, such as rectangle-decomposable modules. [Botnan, Lebovici, Oudot], [Asashiba, Buchet, Escolar, Nakashima, Yoshiwaki], etc.

Approach: localize

Our approach: We localize R-mod until the resulting category admits a classification of indecomposables, or at least a partial classification.

Related work: [Harrington, Otter, Schenck, Tillmann] and [Bauer, Botnan, Oppermann, Steen].

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Inverting some variables

Fact. The homogeneous prime ideals of R are those of the form $(t_{i_1}, \cdots, t_{i_k})$.

The various localizations of a module M fit together.

Example. For M a module over $R = \Bbbk[t_1, t_2]$:

$$\begin{tikzcd}
\Bbbk[t_1, t_2^{\pm}] \otimes_R M \arrow[r, "\text{invert } t_1"] & \Bbbk[t_1^{\pm}, t_2^{\pm}] \otimes_R M \\
M \arrow[u, "\text{invert } t_2"] \arrow[r, "\text{invert } t_1"] & \Bbbk[t_1^{\pm}, t_2] \otimes_R M. \arrow[u, "\text{invert } t_2"']
\end{tikzcd}$$

Inverting some variables (cont'd)

Notation. 1. $[m] \coloneqq \{1, 2, \ldots, m\}$

2. For a subset $\sigma \subseteq [m]$, denote the localization of rings

$$R_\sigma \coloneqq R[t_i^{-1} \mid i \in \sigma],$$

   which is $\sigma^{-1}\mathbb{N}^m$-graded.

3. $\varphi_i \coloneqq$ the localization map "invert t_i".

Example. With this notation, the previous square becomes:

$$\begin{tikzcd}
R_{\{2\}} \otimes_R M \arrow[r, "\varphi_1"] & R_{\{1,2\}} \otimes_R M \\
M \arrow[u, "\varphi_2"] \arrow[r, "\varphi_1"] & R_{\{1\}} \otimes_R M. \arrow[u, "\varphi_2"']
\end{tikzcd}$$

K-localized persistence modules

Idea: Forget the module M but keep some of its localizations.

If we keep a localization $R_\sigma \otimes_R M$, we should keep all further localizations $R_\tau \otimes_R M$ for $\sigma \subseteq \tau$.

Definition. Let K be a simplicial complex on the vertex set $[m]$. A ***K*-localized persistence module** M consists of:

1. For each missing face $\sigma \notin K$, a finitely generated R_σ-module M_σ.
2. For each $\sigma \subseteq \tau$ with $\sigma \notin K$ (and hence $\tau \notin K$), a map of R_σ-modules $\varphi_{\sigma,\tau} \colon M_\sigma \to M_\tau$ such that the induced map of R_τ-modules
$$R_\tau \otimes_{R_\sigma} M_\sigma \xrightarrow{\cong} M_\tau$$
is an isomorphism.

Let $\mathcal{D}(K)$ denote the category of K-localized persistence modules.

The role of K

Small $K \rightsquigarrow$ Localize a little.

Big $K \rightsquigarrow$ Localize a lot.

Example. Extreme cases:

1. $K = \{\} = \mathrm{sk}_{-2}\,\Delta^{m-1} \rightsquigarrow$ Don't localize:
$$\mathcal{D}(K) \cong R\text{-mod}.$$

2. $K = \partial\Delta^{m-1} = \mathrm{sk}_{m-2}\,\Delta^{m-1} \rightsquigarrow$ Invert all the t_i:
$$\mathcal{D}(K) = R_{[m]}\text{-mod} \cong \mathrm{vect}_{\Bbbk}.$$

3. Even more extreme! $K = \Delta^{m-1} \rightsquigarrow$ Localize everything into oblivion:
$$\mathcal{D}(K) = 0.$$

Example: $m = 2$

Take $m = 2$ and $K = \{\emptyset\} = \operatorname{sk}_{-1} \Delta^1$.

A K-localized persistence module M consists of modules

$$\begin{tikzcd}
M_{\{2\}} \arrow[r, "\varphi_1"] \arrow[d, dash, dashed] & M_{\{1,2\}} \\
{} \arrow[r, dash, dashed] & M_{\{1\}} \arrow[u, "\varphi_2"']
\end{tikzcd}$$

where φ_i inverts t_i.

Example: $m = 3$

Take $m = 3$ and $K = \operatorname{sk}_0 \Delta^2 = \{\emptyset, \{1\}, \{2\}, \{3\}\}$.

A K-localized persistence module M consists of modules

$$\begin{tikzcd}
M_{\sigma_1} \arrow[rr, "\varphi_1"] & & M_{[3]} \\
& M_{\sigma_2} \arrow[ur, "\varphi_2"] & \\
& & M_{\sigma_3} \arrow[uu, "\varphi_3"']
\end{tikzcd}$$

where $\quad \sigma_i \coloneqq [m] \setminus \{i\} \quad \rightsquigarrow$ "all but t_i have been inverted"

$\varphi_i \coloneqq \varphi_{[m]\setminus\{i\},[m]} \colon M_{[m]\setminus\{i\}} \to M_{[m]} \quad \rightsquigarrow$ "invert t_i".

A Serre quotient

Consider the canonical functor

$$L_K\colon R\text{-mod} \to \mathcal{D}(K)$$

that keeps the relevant localizations of M:

$$L_K(M)_\sigma = R_\sigma \otimes_R M.$$

Lemma. L_K is exact.

Proposition. L_K is a Serre quotient functor:

$$\begin{tikzcd}
R\text{-mod} \arrow[r, "L_K"] \arrow[d, "q"'] & \mathcal{D}(K). \\
R\text{-mod}/\ker(L_K) \arrow[ur, dashed, "\cong"'] &
\end{tikzcd}$$

A hopeless dream?

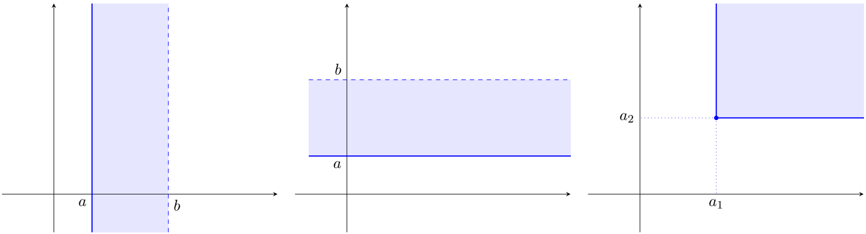
Denote $K_m \coloneqq \mathrm{sk}_{m-3}\,\Delta^{m-1} \rightsquigarrow$ Allow **at most one** non-inverted t_i.

Proposition. For any smaller simplicial complex $K \subset K_m$, $\mathcal{D}(K)$ has wild representation type.

Proof. $\mathcal{D}(K)$ contains a copy of $\mathbb{k}[x, y]$-mod as a retract. □

In this section, focus on $\mathcal{D}(K_m)$.

Some indecomposables



$[a, b)_1$ $[a, b)_2$ $[\vec{a}, \infty)$

Some indecomposable objects in $\mathcal{D}(K_2)$.

Theorem (F.–Stanley)**.** Every object in $\mathcal{D}(K_2)$ decomposes (in a unique way) as a direct sum of:

- "vertical strips" $[a, b)_1$
- "horizontal strips" $[a, b)_2$
- "quadrants" $[\vec{a}, \infty)$.

A torsion pair

Consider two full subcategories of $\mathcal{D}(K_m)$:

$$\begin{aligned}\mathcal{T} &= \{M \mid M_{\sigma_i} \text{ is a torsion } R_{\sigma_i}\text{-module for all } i \in [m]\} \\ &= \{M \mid M_{[m]} = 0\}\end{aligned}$$

$$\mathcal{F} = \{M \mid M_{\sigma_i} \text{ is a torsion-free } R_{\sigma_i}\text{-module for all } i \in [m]\}.$$

Proposition. 1. $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\mathcal{D}(K_m)$.

2. For all M in $\mathcal{D}(K_m)$, the natural short exact sequence

$$0 \to T(M) \to M \to F(M) \to 0$$

splits.

$\Longrightarrow M \cong T(M) \oplus F(M)$

Torsion objects

For $a < b < \infty$, consider the "interval $[a, b)$ in the i^{th} direction":

$$[a,b)_i = L_{K_m}\left(t_i^a R / t_i^b R\right).$$

Lemma. $[a,b)_i$ is indecomposable in $\mathcal{D}(K_m)$.

Proposition. Each torsion object $M \in \mathcal{T}$ decomposes as a direct sum of objects of the form $[a,b)_i$.

Dimension count arguments

After inverting all variables, we are left with a vector space:

$$\begin{aligned} R_{[m]}\text{-mod} &\xrightarrow{\cong} \mathrm{vect}_{\Bbbk} \\ M &\mapsto M(\vec{0}). \end{aligned}$$

Remark. The grading is crucial here. An *ungraded* $\Bbbk[t^{\pm}]$-module corresponds to a $\Bbbk$-vector space V equipped with an automorphism $\mu_t \colon V \xrightarrow{\cong} V$.

Definition.

1. For $\sigma \subseteq [m]$ and an R_σ-module M, the **rank** of M is
$$\operatorname{rank} M \coloneqq \dim_{\Bbbk}\left(R_{[m]} \otimes_{R_\sigma} M\right).$$
2. The **rank** of an object M in $\mathcal{D}(K)$ is
$$\begin{aligned} \operatorname{rank} M &\coloneqq \operatorname{rank} M_{[m]} \\ &= \operatorname{rank} M_\sigma \quad \text{for any missing face } \sigma \notin K. \end{aligned}$$

Torsion-free objects

Notation. For a multidegree $\vec{a} \in \mathbb{N}^m$, consider the object of $\mathcal{D}(K_m)$

$$[\vec{a}, \infty) = L_{K_m}(t^{\vec{a}} R),$$

where $t^{\vec{a}} := t_1^{a_1} \cdots t_m^{a_m}$.

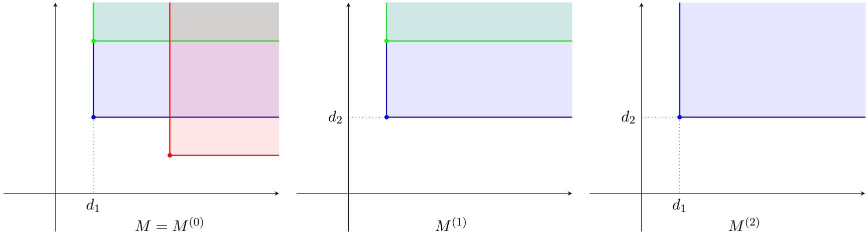
"quadrant module starting at $\vec{a}$"

Lemma.
1. $[\vec{a}, \infty)$ is torsion-free, of rank 1, and indecomposable.
2. Any torsion-free object of rank 1 is of the form $[\vec{a}, \infty)$.

Proposition. Every torsion-free object M in $\mathcal{D}(K_2)$ decomposes as a direct sum of modules of the form $[\vec{a}, \infty)$.

Proof sketch...

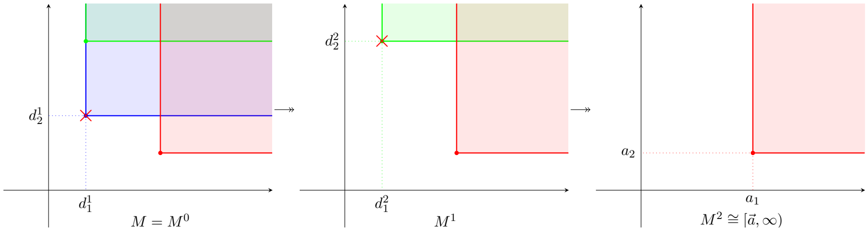
Scanning process



Scan for an element x of lowest degree $\vec{d} \in \mathbb{N}^2$ in lexicographic order.

The quotient $M/\langle x \rangle$ is still torsion-free.

Battleship game



Scan, mod out, repeat $r - 1$ times, where $r = \operatorname{rank} M$.

The composite epimorphism $M \twoheadrightarrow [\vec{a}, \infty)$ admits a section, splitting off a rank 1 summand from M.

In higher dimension $m \geq 3$

Warning: $[\vec{a}, \infty)$ is **not** projective in $\mathcal{D}(K_2)$.

Example. Consider the map in $\mathcal{D}(K_2)$

$$\begin{tikzcd} {[(1,0),\infty) \oplus [(0,1),\infty)} \arrow[r, "{f=[\,\mathrm{inc}\ \mathrm{inc}\,]}"] & {[(0,0),\infty).} \arrow[l, dashed, bend left, "\nexists s"] \end{tikzcd}$$

The map f is an epimorphism but does not admit a section.

Proposition. For any $m \geq 3$, there exists a torsion-free object in $\mathcal{D}(K_m)$ that is of rank 2 and indecomposable.

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Rank invariant

Tensor ideals

Recall: Serre quotient

$$\mathcal{D}(K) \cong R\text{-mod}/\ker(L_K).$$

The subcategory $\ker(L_K) \subseteq R$-mod is a "**tensor ideal**": a Serre subcategory closed under tensoring with a $\mathbb{Z}^m$-graded R-module as long as the result is still $\mathbb{N}^m$-graded.

$\rightsquigarrow$ Allow shifting the degrees *down*, but not below 0.

Classification of tensor ideals

Recall: The **support** of an R-module M is the set of homogeneous prime ideals $P \subset R$ for which the localization $M_P \neq 0$.

For $P = (t_{i_1}, \cdots, t_{i_k})$, we will record the *complement* $[m] \setminus \{i_1, \ldots, i_k\}$, the variables that *may* be inverted.

Proposition. [F.–(Don)Stanley] There is a bijection

$$\begin{tikzcd} \text{“tensor ideals” of } R\text{-mod} \arrow[r, bend left, "\text{complemented support}"] & \text{simplicial complexes on } [m]. \arrow[l, bend left, "\text{(Richard)Stanley–Reisner}"] \end{tikzcd}$$

Morever $\ker(L_k) =$ the “tensor ideal” generated by $\Bbbk[K]$.

Stanley–Reisner ring

Definition. The **Stanley–Reisner ring** or *face ring* of a simplicial complex K is the polynomial ring modulo the monomials corresponding to missing faces:

$$\Bbbk[K] \coloneqq R/(t_\sigma \colon \sigma \notin K).$$

Example. $K = \operatorname{sk}_{-1} \Delta^{m-1} = \{\emptyset\}$

$$\begin{aligned}
&\Longrightarrow \Bbbk[K] = R/(t_1, \ldots, t_m) = \Bbbk \\
&\Longrightarrow \operatorname{Supp} \Bbbk[K] = \{(t_1, \ldots, t_m)\} \\
&\Longrightarrow \text{complemented } \operatorname{Supp} \Bbbk[K] = \{\emptyset\} = K.
\end{aligned}$$

Simple objects

What about $\ker(L_{K_m})$?

Proposition. $\mathcal{D}(K_m)$ is obtained from R-mod by quotienting out the Serre subcategory generated by the simple objects $m-1$ times successively.

Corollary. $\mathcal{D}(K_2)$ is the category of 2-parameter persistence modules up to finite diagrams:

$$\mathcal{D}(K_2) \cong \Bbbk[t_1, t_2]\text{-mod}/\{\text{finite modules}\}.$$

$\rightsquigarrow$ Large-scale behavior of the persistence module.

A link with toric geometry

Coherent sheaves on the projective line $\mathbb{P}^1_{\Bbbk}$:

$$\mathrm{Coh}(\mathbb{P}^1) \cong \mathbb{Z}\text{-graded } \Bbbk[s,t]\text{-mod}/\{\text{finite modules}\}.$$

The $\mathbb{Z}^2$-graded variant of our category $\mathcal{D}(K_2)$ is the bigraded analogue of the right-hand side:

$$\mathcal{D}(K_2)_{\mathbb{Z}^2} \cong \mathbb{Z}^2\text{-graded } \Bbbk[s,t]\text{-mod}/\{\text{finite modules}\}.$$

Colin Ingalls pointed out that this category is equivalent to *torus-equivariant* coherent sheaves on $\mathbb{P}^1$.

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Definition. Let M be an R-module. The **rank invariant** of M is the function assigning to each pair of multidegrees $\vec{a}, \vec{b} \in \mathbb{N}^m$ with $\vec{a} \leq \vec{b}$ the integer

$$r_M(\vec{a}, \vec{b}) = \operatorname{rank}\left(M(\vec{a}) \xrightarrow{t^{\vec{b}-\vec{a}}\cdot} M(\vec{b}) \right).$$

Widely studied invariant of multiparameter persistence modules.

Rank invariant is not enough

The rank invariant of an R-module does not determine $L_K(M)$.

Example. In the case $m = 2$:

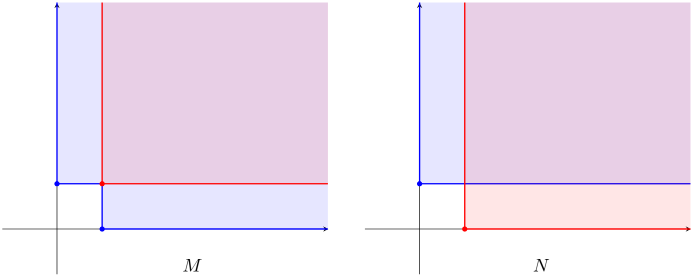
$$M = (t_1, t_2) \oplus t_1 t_2 R$$
$$N = t_1 R \oplus t_2 R$$

have the same rank invariant.

However, they have different K_2-localizations in $\mathcal{D}(K_2)$:

$$L_{K_2}(M) \cong [(0,0), \infty) \oplus [(1,1), \infty)$$
$$L_{K_2}(N) \cong [(1,0), \infty) \oplus [(0,1), \infty).$$

Same rank invariant



Rank invariant is sometimes enough

Proposition. For $K = K_m$, the rank invariant of an R-module M determines the R_{σ_i}-modules M_{σ_i} and the $\Bbbk$-vector space $M_{[m]}$.

Proposition. If M lies in the image of the right adjoint (delocalization functor)

$$\rho_{K_2} \colon \mathcal{D}(K_2) \to \Bbbk[t_1, t_2]\text{-mod},$$

then the rank invariant of M determines the localization $L_{K_2}(M)$.

Some questions

Question. Which refinement of the rank invariant of a $\Bbbk[t_1, t_2]$-module M determines the torsion-free part of $L_{K_2}(M)$ in $\mathcal{D}(K_2)$?

Question. Does $\mathcal{D}(K_3)$ have wild representation type?

Thank you!