

Quillen cohomology of divided power algebras over an operad

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Outline

Introduction

Quillen cohomology

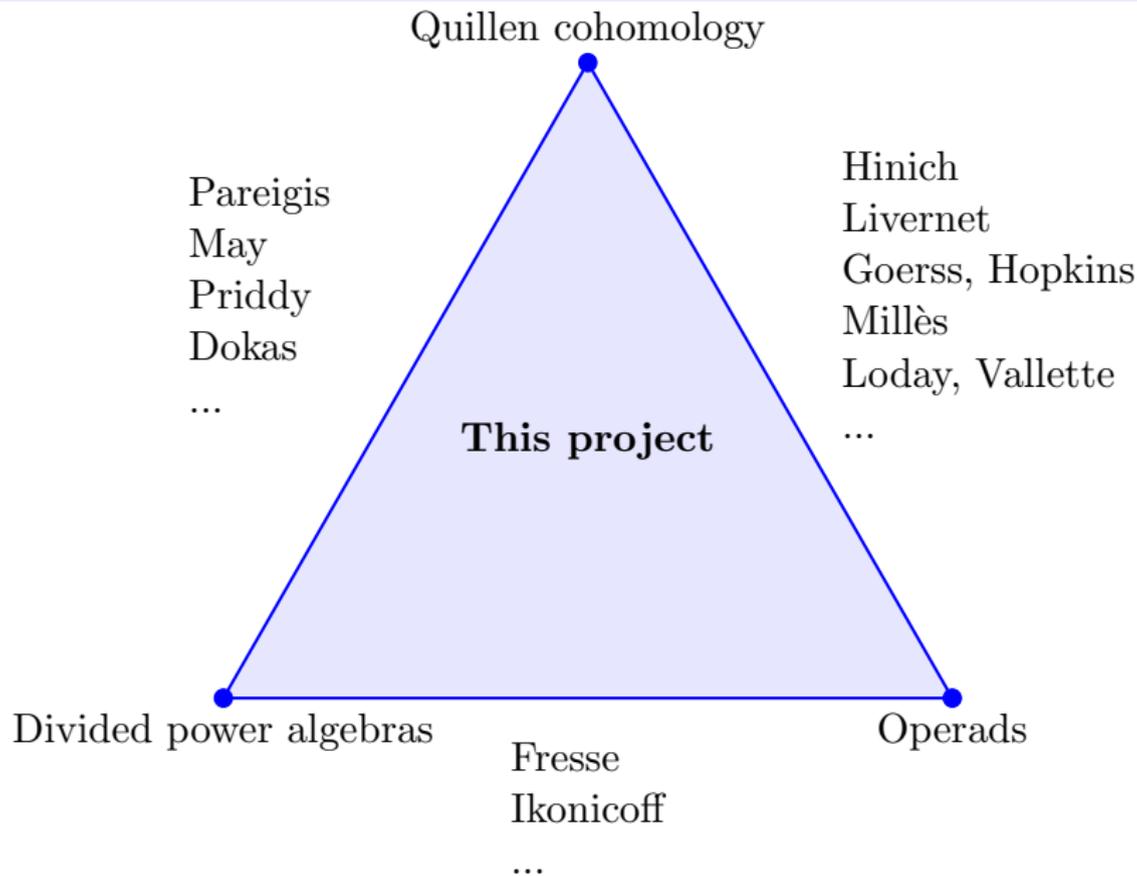
Divided power algebras (classical)

Restricted Lie algebras

Operads

Divided power algebras (operadic)

Overview



André–Quillen (co)homology

- Cohomology theory for commutative rings.
- Developed by André and Quillen in the 1960s.
- Non-additive derived functors constructed using simplicial methods.
- Used to solve problems in commutative algebra and algebraic geometry.
- Makes sense for any algebraic structure.

Applications in topology

A sampler of applications in topology.

- Unstable Adams spectral sequence (Miller, Goerss).
- Obstruction theory for ring spectra (Goerss–Hopkins–Miller, Lurie)
- Realization and classification problems (Blanc, Blanc–Dwyer–Goerss, F., Biedermann–Raptis–Stelzer).
- Higher homotopy operations (Baues–Blanc, Blanc–Johnson–Turner).
- Knot theory: Quillen homology of racks and quandles (Szymik, Berest).

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Cohomology in algebra

Cohomology theories for algebraic structures:

- Group cohomology
- Lie algebra cohomology
- Hochschild cohomology of associative algebras
- André–Quillen cohomology of commutative rings
- etc.

Unified approach: Barr–Beck triple cohomology, based on simplicial resolutions (1969).

Put into the framework of model categories by Quillen (1967).

Idea: cohomology \approx derived functors of derivations.

Beck modules

Setup: An “algebraic” category \mathcal{C} , e.g., groups, abelian groups, rings, commutative rings, R -modules, Lie algebras, etc.

Need a good notion of *coefficient module* M over X :

$$H^*(X; M).$$

Definition (Beck 1967). For an object X in a category \mathcal{C} , a **Beck module** over X is an abelian group object in the slice category \mathcal{C}/X .

Denote the category of Beck modules over X

$$\text{Mod}(X) := (\mathcal{C}/X)_{\text{ab}}.$$

Pullback and pushforward

Definition. The pullback functor $f^*: \mathcal{C}/Y \rightarrow \mathcal{C}/X$ induces a functor

$$f^*: \text{Mod}(Y) \rightarrow \text{Mod}(X)$$

also called the **pullback**.

Its left adjoint

$$f_!: \text{Mod}(X) \rightarrow \text{Mod}(Y)$$

is called the **pushforward** along f .

pullback = “restriction of scalars”

pushforward = “extension of scalars”

Groups

$\mathcal{C} = \text{Gp}$, the category of groups.

For a group G :

$$\begin{aligned}\text{Mod}(G) &\cong G\text{-modules in the usual sense} \\ &\cong \mathbb{Z}G\text{-Mod.}\end{aligned}$$

A Beck module over G is a split extension of G with abelian kernel:

$$1 \longrightarrow K \longrightarrow G \ltimes K \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{e} \end{array} G \longrightarrow 1.$$

The G -action on K is given by $e(g)k = (g, g \cdot k)$. In other words:

$$(g, k)(g', k') = (gg', k + g \cdot k').$$

Groups, cont'd

For a map of groups $f: G \rightarrow H$, the pushforward functor is

$$\begin{aligned} f_! : \text{Mod}(G) &\rightarrow \text{Mod}(H) \\ f_!(M) &= \mathbb{Z}H \otimes_{\mathbb{Z}G} M. \end{aligned}$$

Rings

$\mathcal{C} = \text{Alg}_{\mathbb{k}}$, the category of (unital) algebras over a commutative ring \mathbb{k} .

For a \mathbb{k} -algebra A :

$$\text{Mod}(A) \cong {}_A\text{Bimod}_A.$$

A Beck module over A is a split extension of A with square zero kernel:

$$0 \longrightarrow M \longrightarrow A \oplus M \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} A \longrightarrow 0.$$

The two actions on M are given by

$$(a, m)(a', m') = (aa', a \cdot m' + m \cdot a')$$

and they coincide for scalars in \mathbb{k} .

Rings, cont'd

For a map of \mathbb{k} -algebras $f: A \rightarrow B$, the pullback functor

$$f^*: \text{Mod}(B) \rightarrow \text{Mod}(A)$$

is the usual restriction of scalars:

$$a \cdot m \cdot a' := f(a) \cdot m \cdot f(a').$$

The pushforward functor is

$$\begin{aligned} f_! : \text{Mod}(A) &\rightarrow \text{Mod}(B) \\ f_!(M) &= B \otimes_A M \otimes_A B. \end{aligned}$$

Commutative rings

$\mathcal{C} = \text{Com}_{\mathbb{k}}$, the category of commutative \mathbb{k} -algebras.

For a commutative \mathbb{k} -algebra A :

$$\text{Mod}(A) \cong \text{Mod}_A \quad \text{in the usual sense.}$$

Same correspondence as for algebras, except that $A \oplus M$ must be commutative. This forces the two actions to coincide:

$$a \cdot m = m \cdot a.$$

For a map of commutative k -algebras $f: A \rightarrow B$, the pushforward functor is extension of scalars:

$$\begin{aligned} f_! : \text{Mod}(A) &\rightarrow \text{Mod}(B) \\ f_!(M) &= B \otimes_A M. \end{aligned}$$

Remark. The notion of Beck module depends on the ambient category! A Beck module over A in $\text{Com}_{\mathbb{k}}$ is **not** the same as in $\text{Alg}_{\mathbb{k}}$.

Lie algebras

\mathbb{k} = a field.

$\mathcal{C} = \text{Lie}_{\mathbb{k}}$, the category of Lie algebras over \mathbb{k} .

For a Lie algebra L :

$$\begin{aligned}\text{Mod}(L) &\cong L\text{-modules in the usual sense} \\ &\cong U(L)\text{-Mod}\end{aligned}$$

where $U(L)$ is the universal enveloping algebra of L .

A Beck module over L is a split extension of L with abelian kernel:

$$0 \longrightarrow M \longrightarrow L \oplus M \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} L \longrightarrow 0$$

with $[m, m'] = 0$ for all $m, m' \in M$.

Lie algebras (cont'd)

The action of L on M is given by

$$[(\ell, 0), (0, m)] = (0, \ell \cdot m),$$

which satisfies

$$[\ell, \ell'] \cdot m = \ell \cdot (\ell' \cdot m) - \ell' \cdot (\ell \cdot m).$$

Universal enveloping algebra

Usually, the category of Beck modules over X is equivalent to (left) modules over a ring $\mathbb{U}_{\mathcal{C}}(X)$, called the **universal enveloping algebra** of X :

$$\text{Mod}(X) \cong \mathbb{U}_{\mathcal{C}}(X)\text{-Mod.}$$

Examples of universal enveloping algebras:

1. For a group G , the group ring $\mathbb{U}_{\text{Gp}}(G) = \mathbb{Z}G$.
2. For a ring A , the enveloping ring $\mathbb{U}_{\text{Alg}_{\mathbb{Z}}}(A) = A \otimes A^{\text{op}}$.
3. For a commutative ring R , the ring itself $\mathbb{U}_{\text{Com}_{\mathbb{Z}}}(R) = R$.
4. For a Lie algebra L over \mathbb{k} , the (classical) universal enveloping algebra $\mathbb{U}_{\text{Lie}_{\mathbb{k}}}(L) = U(L)$.

Derivations and differentials

Definition. A **derivation** of X with coefficients in a Beck module $p: E \rightarrow X$ is a section of p :

$$\begin{array}{ccc} X & \xrightarrow{s} & E \\ & \searrow & \downarrow p \\ & & X. \end{array}$$

The abelian group of derivations is

$$\begin{aligned} \text{Der}(X, E) &:= \text{Hom}_{\mathcal{C}/X}(X \xrightarrow{\text{id}} X, E \xrightarrow{p} X) \\ &\cong \text{Hom}_{(\mathcal{C}/X)_{\text{ab}}}(Ab_X(X \xrightarrow{\text{id}} X), E \xrightarrow{p} X), \end{aligned}$$

where $Ab_X: \mathcal{C}/X \rightarrow (\mathcal{C}/X)_{\text{ab}}$ is the **abelianization** functor, i.e., the left adjoint to the forgetful functor $(\mathcal{C}/X)_{\text{ab}} \rightarrow \mathcal{C}/X$.

The module of **Kähler differentials** is $\Omega_{\mathcal{C}}(X) = Ab_X X$, which represents derivations.

Example: Commutative rings

Take $\mathcal{C} = \text{Com}_{\mathbb{k}}$. Given a commutative \mathbb{k} -algebra A and an A -module M , a \mathbb{k} -linear map

$$(\text{id}, f): A \rightarrow A \oplus M$$

is a (unital) ring homomorphism if and only if

$$\begin{cases} f(1_A) = 0 \\ f(ab) = a \cdot f(b) + f(a) \cdot b \quad \rightsquigarrow \text{Leibniz rule.} \end{cases}$$

$\text{Der}(A, M)$ = derivations in the usual sense.

The A -module $Ab_A A$ is:

$$Ab_A A = I_A / I_A^2 = \Omega_{A/k},$$

where $I_A = \ker(\mu: A \otimes_{\mathbb{k}} A \rightarrow A)$. Classical module of Kähler differentials, which represents k -derivations:

$$\text{Hom}_A(\Omega_{A/k}, M) \cong \text{Der}_k(A, M).$$

Geometric interpretation

Analogy:

Kähler differentials in algebraic geometry



differential 1-forms in differential geometry

Example. 1. $A = \mathbb{k}[x, y]$

$$\begin{aligned}\Omega_{A/\mathbb{k}} &\cong \{p dx + q dy \mid p, q \in \mathbb{k}[x, y]\} \\ &\cong A \langle dx, dy \rangle \quad \text{free } A\text{-module.}\end{aligned}$$

2. $A = \mathbb{k}[x, y]/(f)$

$$\Omega_{A/\mathbb{k}} \cong A \langle dx, dy \rangle / \left\langle \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right\rangle.$$

Quillen (co)homology

Quillen homology of $X \approx$ (simplicially) derived functors of Kähler differentials.

Quillen cohomology of X with coefficients in a module $M \approx$ (simplicially) derived functors of derivations with coefficients in M .

The “package”:

1. Beck modules
2. Universal enveloping algebra
3. Derivations
4. Kähler differentials

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Divided power algebras

Introduced by Cartan in the 1950s. Appear in positive characteristic.

Divided power operations \approx operations γ_n that look like $\gamma_n(x) = \frac{x^n}{n!}$. More precisely...

Divided power algebras (cont'd)

Definition. Let A be a commutative \mathbb{k} -algebra. A **system of divided powers** on an ideal $I \subseteq A$ is a collection of maps $\gamma_i: I \rightarrow A$ for $i \geq 0$ satisfying:

$$\gamma_0(a) = 1$$

$$\gamma_1(a) = a$$

$$\gamma_i(a) \in I, \quad i \geq 1$$

$$\gamma_i(a+b) = \sum_{k=0}^{i} \gamma_k(a)\gamma_{i-k}(b), \quad a, b \in I, \quad i \geq 0$$

$$\gamma_i(ab) = a^i \gamma_i(b), \quad a \in A, \quad i \geq 0$$

$$\gamma_i(a)\gamma_j(a) = \frac{(i+j)!}{i!j!} \gamma_{i+j}(a), \quad a \in I, \quad i, j \geq 0$$

$$\gamma_i(\gamma_j(a)) = \frac{(ij)!}{i!(j!)^i} \gamma_{ij}(a), \quad a \in I, \quad i \geq 0, \quad j \geq 1.$$

Usually A has an augmentation $\epsilon: A \rightarrow \mathbb{k}$ and $I = \ker(\epsilon)$.

Examples

Example. The free divided power algebra over \mathbb{Z} on one generator x is

$$\mathbb{Z}[x, \frac{x^2}{2}, \dots, \frac{x^n}{n!}, \dots] \subset \mathbb{Q}[x].$$

Example. Over a field \mathbb{k} of characteristic 0, divided powers must in fact be given by

$$\gamma_n(x) = \frac{x^n}{n!}.$$

\rightsquigarrow No additional structure in that case.

Divided powers are a **positive characteristic** phenomenon.

In positive characteristic

Proposition (Soublin 1987). Over a field \mathbb{k} of characteristic $p > 0$, a system of divided power operations is the same data as just the operation $\pi = \gamma_p: I \rightarrow I$ satisfying:

$$a^p = 0, \quad a \in I$$

$$\pi(a + b) = \pi(a) + \pi(b) + \sum_{k=1}^{k=p-1} \frac{(-1)^k}{k} a^k b^{p-k}, \quad a, b \in I$$

$$\pi(ab) = 0, \quad a, b \in I$$

$$\pi(\lambda a) = \lambda^p \pi(a), \quad a \in I, \lambda \in \mathbb{k}.$$

Note: The operation π is non-additive!

Theorem (Dokas 2009). Worked out the “package” for divided power algebras over a field \mathbb{k} of characteristic $p > 0$.

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Restricted Lie algebras

Over a field \mathbb{k} of characteristic $p > 0$, Lie algebras usually come with extra structure, that of a *restricted* Lie algebra.

Textbook account in Jacobson (1962).

Important in topology. See work of Milnor–Moore, May, Rector, Bousfield, Curtis, Priddy, etc. in the 1960s.

Restricted Lie algebras (cont'd)

Definition. A **restricted Lie algebra** L over \mathbb{k} is a Lie algebra together with a map $(-)^{[p]}: L \rightarrow L$ called the p -map satisfying:

$$(\alpha x)^{[p]} = \alpha^p x^{[p]}, \quad \alpha \in \mathbb{k}$$

$$[x, y^{[p]}] = [\underbrace{\cdots [x, y], y}_{p}, \cdots, y]$$

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$$

where $s_i(x, y)$ is the coefficient of λ^{i-1} in $\text{ad}_{\lambda x + y}^{p-1}(x)$.

Here $\text{ad}_x: L \rightarrow L$ denotes the adjoint representation

$$\text{ad}_x(y) := [y, x], \quad x, y \in L.$$

Example. For $p = 2$, the last equation becomes:

$$(x + y)^{[2]} = x^{[2]} + [x, y] + y^{[2]}.$$

Examples

Example. For a \mathbb{k} -algebra A , the underlying restricted Lie algebra of A has the commutator bracket

$$[x, y] = xy - yx$$

and the p^{th} power map

$$x^{[p]} = x^p.$$

Example. The free restricted Lie algebra on a \mathbb{k} -vector space V is the subspace

$$L^r(V) \subseteq T(V)$$

obtained by taking V and closing it under commutators and p^{th} powers.

Theorem (Dokas 2004). Worked out the “package” for restricted Lie algebras over a field \mathbb{k} of characteristic $p > 0$.

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Operads in sets

Idea: Encode a type of algebraic structure as

$$\mathcal{P}(n) = \{\text{the allowed } n\text{-ary operations}\}.$$

Definition. A (symmetric) **operad** in sets $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ is a sequence of sets $\mathcal{P}(n)$ with a right action of the symmetric group Σ_n together with composition maps

$$\circ: \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \rightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

and a unit $\eta \in \mathcal{P}(1)$ such that composition is associative, unital, and equivariant.

Algebras over operads

Example. For any set X , the **endomorphism operad** of X is:

$$\left\{ \begin{array}{l} \text{End}_X(n) := \text{Hom}(X^n, X) \\ \circ = \text{usual composition} \\ \eta = \text{id}_X \in \text{Hom}(X, X) \\ \Sigma_n \text{ acts by permuting the factors of } X^n. \end{array} \right.$$

Definition. A **\mathcal{P} -algebra** is a set X equipped with action maps

$$\alpha_n : \mathcal{P}(n) \rightarrow \text{Hom}(X^n, X)$$

compatible with composition, unit, and equivariance. In other words, an operad map

$$\alpha : \mathcal{P} \rightarrow \text{End}_X .$$

Each n -ary operation symbol $\mu \in \mathcal{P}(n)$ yields an actual n -ary operation on X .

Examples

Example. The (unital) associative operad:

$$\text{As}^{\text{un}}(n) = \Sigma_n, \quad n \geq 0.$$

As^{un} -algebra = monoid.

Example. The (non-unital) associative operad:

$$\text{As}(n) = \begin{cases} \Sigma_n, & n \geq 1 \\ \emptyset, & n = 0. \end{cases}$$

As -algebra = semigroup.

Examples (cont'd)

Example. The (unital) commutative operad:

$$\text{Com}^{\text{un}}(n) = *, \quad n \geq 0.$$

Com^{un} -algebra = commutative monoid.

Example. The (non-unital) commutative operad:

$$\text{Com}(n) = \begin{cases} *, & n \geq 1 \\ \emptyset, & n = 0. \end{cases}$$

Com -algebra = commutative semigroup.

Operads in vector spaces

What if we want \mathbb{k} -vector spaces equipped with operations?

Instead of working in (Set, \times) , work in $(\text{Vect}_{\mathbb{k}}, \otimes_{\mathbb{k}})$.

\rightsquigarrow Replace the Cartesian product \times with the tensor product $\otimes_{\mathbb{k}}$ everywhere.

Definition. A (symmetric) **operad** in \mathbb{k} -vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$ is a sequence of \mathbb{k} -vector spaces $\mathcal{P}(n)$ with a right action of the symmetric group Σ_n together with composition maps

$$\circ: \mathcal{P}(k) \otimes_{\mathbb{k}} \mathcal{P}(n_1) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} \mathcal{P}(n_k) \rightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

and a unit $\eta \in \mathcal{P}(1)$ such that composition is associative, unital, and equivariant.

Warning. This forces operations $\mu \in \mathcal{P}(n)$ to be **\mathbb{k} -multilinear**.

Examples

Taking the \mathbb{k} -vector space spanned by an operad in sets yields an operad in \mathbb{k} -vector spaces.

Example. The (unital) associative operad:

$$\text{As}^{\text{un}}(n) = \mathbb{k}\Sigma_n, \quad n \geq 0.$$

As^{un} -algebra = (unital) \mathbb{k} -algebra.

Example. The (non-unital) associative operad:

$$\text{As}(n) = \begin{cases} \mathbb{k}\Sigma_n, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

As -algebra = (non-unital) \mathbb{k} -algebra.

Examples (cont'd)

Example. The (unital) commutative operad:

$$\text{Com}^{\text{un}}(n) = \mathbb{k}, n \geq 0.$$

Com^{un} -algebra = (unital) commutative \mathbb{k} -algebra.

Example. The (non-unital) commutative operad:

$$\text{Com}(n) = \begin{cases} \mathbb{k}, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

Com -algebra = (non-unital) commutative \mathbb{k} -algebra.

Examples (cont'd)

Example. The Lie operad Lie is generated by an operation

$$[-, -] \in \text{Lie}(2) \quad \text{Lie bracket } [x, y]$$

subject to relations

$$\left\{ \begin{array}{l} [-, -] = -[-, -] \cdot (12) \in \text{Lie}(2) \\ [x, y] = -[y, x] \quad \text{skew-symmetry} \\ [-, [-, -]] + [-, [-, -]] \cdot (132) + [-, [-, -]] \cdot (123) = 0 \in \text{Lie}(3) \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{Jacobi identity.} \end{array} \right.$$

Lie-algebra = Lie algebra if $\text{char}(\mathbb{k}) \neq 2$.

Warning. In the case $\text{char}(\mathbb{k}) = 2$, Lie-algebra = “quasi-Lie algebra”. Can't encode the equation $[x, x] = 0$ with an operad.

Quillen cohomology of \mathcal{P} -algebra

For the category $\mathcal{C} = \mathcal{P}\text{-Alg}$ of \mathcal{P} -algebras, the “package” has been worked out. Nice account in Loday–Vallette (2012).

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Free \mathcal{P} -algebras

Setup: \mathbb{k} = field of characteristic $p > 0$.

\mathcal{P} = operad in \mathbb{k} -vector spaces that is *reduced*, i.e., $\mathcal{P}(0) = 0$.

Free \mathcal{P} -algebra functor (forget-of-free monad)

$$S(\mathcal{P}): \text{Vect}_{\mathbb{k}} \rightarrow \text{Vect}_{\mathbb{k}}$$

$$S(\mathcal{P})(V) = \bigoplus_{n \geq 1} (\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n}$$

Example: Tensor algebra

Example. Associative operad:

$$\begin{aligned} S(\text{As})(V) &= \bigoplus_{n \geq 1} (\text{As}(n) \otimes V^{\otimes n})_{\Sigma_n} \\ &= \bigoplus_{n \geq 1} (\mathbb{k}\Sigma_n \otimes V^{\otimes n})_{\Sigma_n} \\ &\cong \bigoplus_{n \geq 1} V^{\otimes n} \\ &= T(V) \end{aligned}$$

the (non-unital) tensor algebra.

Example: Symmetric algebra

Example. Commutative operad:

$$\begin{aligned} S(\text{Com})(V) &= \bigoplus_{n \geq 1} (\text{Com}(n) \otimes V^{\otimes n})_{\Sigma_n} \\ &= \bigoplus_{n \geq 1} (\mathbb{k} \otimes V^{\otimes n})_{\Sigma_n} \\ &\cong \bigoplus_{n \geq 1} \text{Sym}^n(V) \\ &= \text{Sym}(V) \end{aligned}$$

the (non-unital) symmetric algebra.

Divided power algebras

Instead of taking coinvariants, take *invariants*. Norm map:

$$\begin{array}{ccc} \bigoplus_{n \geq 1} (\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n} & \longrightarrow & \bigoplus_{n \geq 1} (\mathcal{P}(n) \otimes V^{\otimes n})^{\Sigma_n} \\ \parallel & & \parallel \text{def} \\ S(\mathcal{P})(V) & & \Gamma(\mathcal{P})(V) \end{array}$$

Proposition. $\Gamma(\mathcal{P}): \text{Vect}_{\mathbb{k}} \rightarrow \text{Vect}_{\mathbb{k}}$ is a monad.

Definition. A **divided power \mathcal{P} -algebra** is an algebra for the monad $\Gamma(\mathcal{P})$.

$\Gamma(\mathcal{P})$ -algebra \approx \mathcal{P} -algebra + extra operations, usually **non-additive**.

Examples

Example. If $\text{char}(\mathbb{k}) = 0$, then $S(\mathcal{P}) \xrightarrow{\cong} \Gamma(\mathcal{P})$.

\rightsquigarrow No extra operations in that case.

Divided power operations are a **positive characteristic** phenomenon.

Theorem (Fresse 2000). 1. Associative operad: $S(\text{As}) \xrightarrow{\cong} \Gamma(\text{As})$,
so a $\Gamma(\text{As})$ -algebra is just an As-algebra, i.e., a \mathbb{k} -algebra.

2. Commutative operad: $\Gamma(\text{Com})$ -algebra = classical divided power algebra.

3. Lie operad: $\Gamma(\text{Lie})$ -algebra = restricted Lie algebra.

Results

Theorem (Ikonicoff 2020). Equational presentation for $\Gamma(\mathcal{P})$ -algebras.

\rightsquigarrow List of operations satisfying a (big) list of equations.

Theorem (Dokas–F.–Ikonicoff). Worked out the “package” for $\Gamma(\mathcal{P})$ -algebras.

Reality check: Recovered the examples of classical divided power algebras ($\mathcal{P} = \text{Com}$) and restricted Lie algebras ($\mathcal{P} = \text{Lie}$).

Results (cont'd)

Comparison maps involving Quillen cohomology of $\Gamma(\mathcal{P})$ -algebras.

A map of operads $f: \mathcal{P} \rightarrow \mathcal{Q}$ induces adjunctions:

$$\begin{array}{ccc} \mathcal{P}\text{-Alg} & \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \end{array} & \mathcal{Q}\text{-Alg} \\ \begin{array}{c} \text{free} \downarrow \\ \uparrow \text{forget} \end{array} & & \begin{array}{c} \text{free} \downarrow \\ \uparrow \text{forget} \end{array} \\ \Gamma(\mathcal{P})\text{-Alg} & \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \end{array} & \Gamma(\mathcal{Q})\text{-Alg} \end{array}$$

\rightsquigarrow Comparison maps between Quillen cohomology in those different categories.

Example. Inclusion of operads $\text{Lie} \rightarrow \text{As}$.

Thank you!