The homotopy theory of simplicial Beck modules

Martin Frankland
University of Regina

Mathematical Congress of the Americas
Session on Categories and Topology
July 21, 2021

Outline

Motivation: Quillen (co)homology

Beck modules

Simplicial Beck modules

André–Quillen (co)homology

- Cohomology theory for commutative rings.
- Developed by André and Quillen in the 1960s.
- Non-additive derived functors constructed using simplicial methods.
- Used to solve problems in commutative algebra and algebraic geometry.
- Makes sense for any algebraic structure.

Applications in topology

A sampler of applications in topology.

- Unstable Adams spectral sequence (Miller, Goerss).
- Realization and classification problems (Goerss–Hopkins–Miller, Blanc, Blanc–Dwyer–Goerss, F., Biedermann–Raptis–Stelzer).
- Higher homotopy operations (Baues–Blanc, Blanc–Johnson–Turner).
- Knot theory: Quillen homology of racks and quandles (Szymik, Berest).

Goal

Previous work (F. 2015): Comparing Quillen (co)homology in categories related by an adjunction

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G.$$

The focus was on $\mathrm{HQ}_*(X)$ and $\mathrm{HQ}^*(X;M)$ for an object X.

Goal

Deal with a simplicial object $X_\bullet$ in $s\mathcal{C}$ and simplicial module $M_\bullet$ over it.

Outline

Motivation: Quillen (co)homology

Beck modules

Simplicial Beck modules

Setup

Throughout, we will work with an "algebraic" category $\mathcal{C}$.

Definition. An **algebraic theory** is a small category $\mathcal{T}$ with finite products. A **model** for the theory $\mathcal{T}$ is a functor $M\colon \mathcal{T} \to \mathrm{Set}$ that preserves finite products.

Definition. A category is **algebraic** if it is equivalent to the category $\mathrm{Model}(\mathcal{T})$ of models for some algebraic theory $\mathcal{T}$.

Characterization

Theorem (Lawvere 1963 & more)**.** For a category $\mathcal{C}$, the following are equivalent.

1. $\mathcal{C}$ is algebraic.
2. $\mathcal{C}$ is cocomplete, has a set of finitely presentable projective generators, and is exact (in the sense of Barr).
3. $\mathcal{C}$ is a many-sorted finitary variety of algebras, a.k.a. "equational class".
4. $\mathcal{C}$ is the category of algebras for a finitary monad $T \colon \mathrm{Set}^S \to \mathrm{Set}^S$ for some set S.

Example. Your favorite algebraic structures: sets, monoids, groups, abelian groups, rings, commutative rings, R-modules, Lie algebras, chain complexes, DG-algebras, etc.

Beck modules

Definition (Beck 1967)**.** For an object X in $\mathcal{C}$, a **Beck module** over X is an abelian group object in the slice category $\mathcal{C}/X$.

The category of Beck modules is sometimes denoted

$$\mathrm{Mod}(X) := (\mathcal{C}/X)_{\mathrm{ab}}.$$

Definition. The **abelianization** over X

$$Ab_X \colon \mathcal{C}/X \to (\mathcal{C}/X)_{\mathrm{ab}}$$

is the left adjoint of the forgetful functor

$$U_X \colon (\mathcal{C}/X)_{\mathrm{ab}} \to \mathcal{C}/X.$$

Quillen (co)homology

Definition. Let X be an object of $\mathcal{C}$ and M a module over X.

- The **cotangent complex** $\mathbf{L}_X$ of X is the derived abelianization of X, i.e., the simplicial module over X given by
$$\mathbf{L}_X := Ab_X(C_\bullet \to X)$$
where $C_\bullet \to X$ is a cofibrant replacement of X in $s\mathcal{C}$.
- **Quillen homology** of X is
$$\mathrm{HQ}_n(X) := \pi_n(\mathbf{L}_X).$$
If the category $\mathrm{Mod}(X)$ has a good notion of tensor product $\otimes$, then Quillen homology with coefficients in M is
$$\mathrm{HQ}_n(X; M) := \pi_n(\mathbf{L}_X \otimes M).$$
- **Quillen cohomology** of X with coefficients in M is the derived functors of derivations:
$$\mathrm{HQ}^n(X; M) := \pi^n \, \mathrm{Hom}(\mathbf{L}_X, M).$$

Pullback and pushforward

Definition. Given a map $f\colon X \to Y$, the pullback functor $f^*\colon \mathcal{C}/Y \to \mathcal{C}/X$ induces a functor

$$f^*\colon \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$$

also called the **pullback**. Its left adjoint

$$f_!\colon \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$$

is called the **pushforward** along f.

$$\text{pullback} = \text{“restriction of scalars”}$$

$$\text{pushforward} = \text{“extension of scalars”}$$

Rings

$\mathcal{C} = \mathrm{Alg}_k$, the category of (associative, unital) k-algebras.

For a k-algebra A:

$$\mathrm{Mod}(A) \cong {}_A\mathrm{Bimod}_A.$$

A Beck module over A is a split extension of A with square zero kernel:

$$0 \longrightarrow M \longrightarrow A \oplus M \underset{s}{\overset{p}{\rightleftarrows}} A \longrightarrow 0.$$

The two actions on M are given by

$$(a, m)(a', m') = (aa', a \cdot m' + m \cdot a')$$

and they coincide for scalars in k.

Rings, cont'd

For a map of k-algebras $f\colon A \to B$, the pushforward functor is

$$\begin{aligned} f_! \colon \operatorname{Mod}(A) &\to \operatorname{Mod}(B) \\ f_!(M) &= B \otimes_A M \otimes_A B. \end{aligned}$$

The A-bimodule $Ab_A A$ is the kernel of the multiplication map:

$$Ab_A A = I_A := \ker(A \otimes_k A \xrightarrow{\mu} A).$$

Proposition (Barr 1967)**.** Quillen cohomology in Alg_k is (up to shift) Shukla cohomology, a.k.a. derived Hochschild cohomology:

$$\mathrm{HQ}^n(A; M) = \begin{cases} \mathrm{Der}_k(A, M) & n = 0 \\ H^{n+1}(A; M) & n > 0. \end{cases}$$

Commutative rings

$\mathcal{C} = \mathrm{Com}_k$, the category of commutative k-algebras.

For a commutative k-algebra A:

$$\mathrm{Mod}(A) \cong \mathrm{Mod}_A \quad \text{in the usual sense.}$$

Same correspondence as for algebras, except that $A \oplus M$ must be commutative. This forces the two actions to coincide:

$$a \cdot m = m \cdot a.$$

Commutative rings, cont'd

For a map of commutative k-algebras $f\colon A \to B$, the pushforward functor is

$$f_!\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$
$$f_!(M) = B \otimes_A M.$$

The A-module $Ab_A A$ is:

$$Ab_A A = I_A/I_A^2 = \Omega_{A/k},$$

the module of Kähler differentials. It represents k-derivations:

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \cong \mathrm{Der}_k(A, M).$$

Groups

$\mathcal{C} = \mathrm{Gp}$, the category of groups.

For a group G:

$$\begin{aligned} \mathrm{Mod}(G) &\cong G - \mathrm{Mod} \quad \text{in the usual sense} \\ &\cong \mathbb{Z}G - \mathrm{Mod}. \end{aligned}$$

A Beck module over G is a split extension of G with abelian kernel:

$$1 \longrightarrow K \longrightarrow G \ltimes K \underset{e}{\overset{p}{\rightleftarrows}} G \longrightarrow 1.$$

The G-action on K is given by $e(g)k = (g, g \cdot k)$. In other words:

$$(g,k)(g',k') = (gg', k + g \cdot k').$$

Groups, cont'd

For a map of groups $f\colon G \to H$, the pushforward functor is

$$\begin{aligned} f_! \colon \mathrm{Mod}(G) &\to \mathrm{Mod}(H) \\ f_!(M) &= \mathbb{Z}H \otimes_{\mathbb{Z}G} M. \end{aligned}$$

The G-module $Ab_G G$ is the augmentation ideal:

$$Ab_G G = I_G = \ker(\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z}).$$

Proposition (Barr–Beck 1966)**.** Quillen cohomology in Gp is (up to a shift) group cohomology:

$$\mathrm{HQ}^n(G; M) = \begin{cases} \mathrm{Der}(G, M) & n = 0 \\ H^{n+1}(G; M) & n > 0 \end{cases}$$

where $\mathrm{Der}(G, M)$ denotes crossed homomorphisms $G \to M$.

Abelian groups

$\mathcal{C} = \text{Ab}$, the category of abelian groups.

For an abelian group A:

$$\text{Mod}(A) \cong \text{Ab}.$$

Same correspondence as for groups, except that $A \ltimes K$ must be abelian. This forces the A-action on K to be trivial:

$$a \cdot k = k.$$

More generally:

Example (Beck 1967)**.** In an additive category $\mathcal{A}$ with finite limits, Beck modules over any object X are:

$$\begin{aligned} \text{Mod}(X) &\cong \mathcal{A} \\ (p \colon E \twoheadrightarrow X) &\mapsto \ker(p). \end{aligned}$$

Fibered category

The assignment

$$\mathrm{Mod}(-)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{AbCat}$$

sending an object X to its category of Beck modules $\mathrm{Mod}(X)$ and a map $f\colon X \to Y$ to the pullback functor $f^*\colon \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$ is a pseudo-functor: $(gf)^* \cong f^* g^*$.

Definition. The Grothendieck construction of the pseudo-functor $\mathrm{Mod}(-)$ yields a fibered category

$$\pi\colon \mathrm{Mod}\mathcal{C} \to \mathcal{C}$$

called the **fibered category of Beck modules** over $\mathcal{C}$, a.k.a. the **tangent category** of $\mathcal{C}$, denoted $T\mathcal{C} \to \mathcal{C}$.

An object of $T\mathcal{C}$ is (X, M), where M is a module over X.

Remark. ∞-categorical analogue using stabilization instead of abelianization (Lurie 2011, Harpaz–Nuiten–Prasma 2019; building on Schwede 1997, Basterra–Mandell 2005).

Fibered category (cont'd)

Example. 1. For $\mathcal{A}$ additive with finite limits:

$$T\mathcal{A} \cong \mathcal{A} \times \mathcal{A}.$$

2. $T\mathrm{Gp} \cong \Pi\mathrm{Alg}_1^2$, the category of 2-truncated Π-algebras.
3. $T\mathrm{Alg}_k \cong \mathrm{grAlg}_k^{\leq 1}$, the category of graded k-algebras concentrated in degrees 0 and 1.
4. $T\mathrm{Com}_k \cong \mathrm{grCom}_k^{\leq 1}$.

Outline

Motivation: Quillen (co)homology

Beck modules

Simplicial Beck modules

Quillen model structure

Quillen constructed a standard model structure on simplicial objects $s\mathcal{C}$. A map $f_\bullet \colon X_\bullet \to Y_\bullet$ in $s\mathcal{C}$ is a:

- *fibration* (resp. *weak equivalence*) if for every projective object P of $\mathcal{C}$, the map:

$$\operatorname{Hom}_{\mathcal{C}}(P, X_\bullet) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{C}}(P, Y_\bullet)$$

  is a fibration (resp. weak equivalence) of simplicial sets.
- *cofibration* if it has the left lifting property with respect to all trivial fibrations.

More concretely: For $\mathcal{C}$ an algebraic category, the model structure is right-induced along the forgetful functor

$$U \colon s\mathcal{C} \to s(\mathrm{Set}^S) = (s\mathrm{Set})^S.$$

For instance, a simplicial ring has an underlying simplicial set.

Nice simplicial objects

Definition. A complete and cocomplete category $\mathcal{C}$ **has nice simplicial objects** if $s\mathcal{C}$ admits Quillen's standard model structure.

Theorem (Quillen 1967)**.** Any quasi-algebraic category has nice simplicial objects.

Proposition. If $\mathcal{C}$ has nice simplicial objects and $s\mathcal{C}$ is cofibrantly generated, then $T\mathcal{C}$ has nice simplicial objects.

Homotopy theory of simplicial modules

Proposition. The category of Beck modules over a simplicial object $X_\bullet$ in $s\mathcal{C}$

$$\mathrm{Mod}(X_\bullet) = (s\mathcal{C}/X_\bullet)_{\mathrm{ab}}$$

admits the model structure right-induced along the forgetful functor

$$U_{X_\bullet} \colon (s\mathcal{C}/X_\bullet)_{\mathrm{ab}} \to s\mathcal{C}/X_\bullet.$$

Recovers the model structure on simplicial modules over a simplicial commutative ring $R_\bullet$ (Quillen 1967, Schwede 1997).

Lemma. There is an equivalence of categories $sT\mathcal{C} \cong T(s\mathcal{C})$ exhibiting $sT\mathcal{C}$ as the tangent category of $\mathcal{C}$:

$$\begin{array}{ccc} sT\mathcal{C} & \overset{\cong}{\longrightarrow} & T(s\mathcal{C}) \\ {\scriptstyle s\pi_{\mathcal{C}}}\downarrow & \swarrow {\scriptstyle \pi_{s\mathcal{C}}} & \\ s\mathcal{C}. & & \end{array}$$

Simplicial modules (cont'd)

More explicitly: A module over $X_\bullet$ is the same as a module M_n over X_n for each $n \geq 0$ together with face maps $d_i \colon M_n \to M_{n-1}$ that are maps of modules over the face maps $d_i \colon X_n \to X_{n-1}$, and likewise for degeneracies.

Lemma. The standard model structure on $sT\mathcal{C}$ restricts to each fiber $(s\mathcal{C}/X_\bullet)_{\mathrm{ab}}$ to the model structure induced from that of $s\mathcal{C}/X_\bullet$.

Proposition. Under the identification $sT\mathcal{C} \cong T(s\mathcal{C})$, the standard model structure on $sT\mathcal{C}$ corresponds to the integral model structure on $T(s\mathcal{C})$ in the sense of Harpaz–Prasma (2015).

Future steps

Develop tools to compute $\mathrm{HQ}_*(X_\bullet; M_\bullet)$ and $\mathrm{HQ}^*(X_\bullet; M_\bullet)$ analogous to Quillen's work:

- Transitivity sequence
- Flat base change
- Universal coefficient spectral sequences.

Thank you!