

# Quillen cohomology of divided power algebras over an operad

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# Outline

Introduction

Quillen cohomology

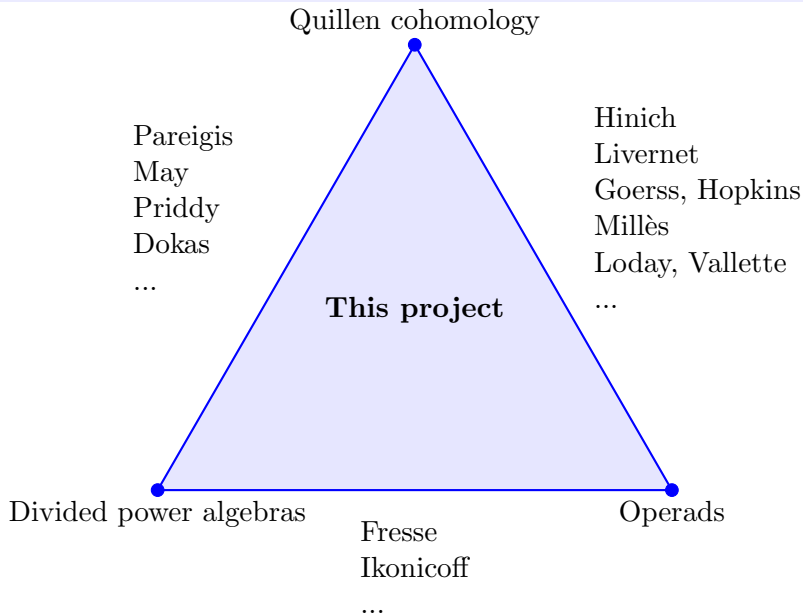
Divided power algebras (classical)

Restricted Lie algebras

Operads

Divided power algebras (operadic)

# Overview



# André–Quillen (co)homology

- Cohomology theory for commutative rings.
- Developed by André and Quillen in the 1960s.
- Non-additive derived functors constructed using simplicial methods.
- Used to solve problems in commutative algebra and algebraic geometry.
- Makes sense for any algebraic structure.

# Applications in topology

A sampler of applications in topology.

- Unstable Adams spectral sequence (Miller, Goerss).
- Obstruction theory for ring spectra (Goerss–Hopkins–Miller, Lurie)
- Realization and classification problems (Blanc, Blanc–Dwyer–Goerss, F., Biedermann–Raptis–Stelzer).
- Higher homotopy operations (Baues–Blanc, Blanc–Johnson–Turner).
- Knot theory: Quillen homology of racks and quandles (Szymik, Berest).

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# Cohomology in algebra

Cohomology theories for algebraic structures:

- Group cohomology
- Lie algebra cohomology
- Hochschild cohomology of associative algebras
- André–Quillen cohomology of commutative rings
- etc.

Unified approach: Barr–Beck triple cohomology, based on simplicial resolutions (1969).

Put into the framework of model categories by Quillen (1967).

**Idea:** cohomology  $\approx$  derived functors of derivations.

# Beck modules

**Setup:** An “algebraic” category  $\mathcal{C}$ , e.g., groups, abelian groups, rings, commutative rings,  $R$ -modules, Lie algebras, etc.

Need a good notion of *coefficient module*  $M$  over  $X$ :

$$H^*(X; M).$$

**Definition** (Beck 1967). For an object  $X$  in a category  $\mathcal{C}$ , a **Beck module** over  $X$  is an abelian group object in the slice category  $\mathcal{C}/X$ .

Denote the category of Beck modules over  $X$

$$\text{Mod}(X) := (\mathcal{C}/X)_{\text{ab}}.$$



# Pullback and pushforward

**Definition.** The pullback functor  $f^*: \mathcal{C}/Y \rightarrow \mathcal{C}/X$  induces a functor

$$f^*: \text{Mod}(Y) \rightarrow \text{Mod}(X)$$

also called the **pullback**.

Its left adjoint

$$f_!: \text{Mod}(X) \rightarrow \text{Mod}(Y)$$

is called the **pushforward** along  $f$ .

pullback = “restriction of scalars”

pushforward = “extension of scalars”

# Groups

$\mathcal{C} = \text{Gp}$ , the category of groups.

For a group  $G$ :

$$\begin{aligned}\text{Mod}(G) &\cong G\text{-modules in the usual sense} \\ &\cong \mathbb{Z}G\text{-Mod.}\end{aligned}$$

A Beck module over  $G$  is a split extension of  $G$  with abelian kernel:

$$1 \longrightarrow K \longrightarrow G \ltimes K \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{e} \end{array} G \longrightarrow 1.$$

The  $G$ -action on  $K$  is given by  $e(g)k = (g, g \cdot k)$ . In other words:

$$(g, k)(g', k') = (gg', k + g \cdot k').$$

## Groups, cont'd

For a map of groups  $f: G \rightarrow H$ , the pushforward functor is

$$f_! : \text{Mod}(G) \rightarrow \text{Mod}(H)$$
$$f_!(M) = \mathbb{Z}H \otimes_{\mathbb{Z}G} M.$$

# Rings

$\mathcal{C} = \text{Alg}_{\mathbb{k}}$ , the category of (unital) algebras over a commutative ring  $\mathbb{k}$ .

For a  $\mathbb{k}$ -algebra  $A$ :

$$\text{Mod}(A) \cong {}_A\text{Bimod}_A.$$

A Beck module over  $A$  is a split extension of  $A$  with square zero kernel:

$$0 \longrightarrow M \longrightarrow A \oplus M \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} A \longrightarrow 0.$$

The two actions on  $M$  are given by

$$(a, m)(a', m') = (aa', a \cdot m' + m \cdot a')$$

and they coincide for scalars in  $\mathbb{k}$ .

## Rings, cont'd

For a map of  $\mathbb{k}$ -algebras  $f: A \rightarrow B$ , the pullback functor

$$f^*: \text{Mod}(B) \rightarrow \text{Mod}(A)$$

is the usual restriction of scalars:

$$a \cdot m \cdot a' := f(a) \cdot m \cdot f(a').$$

The pushforward functor is

$$\begin{aligned} f_! : \text{Mod}(A) &\rightarrow \text{Mod}(B) \\ f_!(M) &= B \otimes_A M \otimes_A B. \end{aligned}$$

# Commutative rings

$\mathcal{C} = \text{Com}_{\mathbb{k}}$ , the category of commutative  $\mathbb{k}$ -algebras.

For a commutative  $\mathbb{k}$ -algebra  $A$ :

$$\text{Mod}(A) \cong \text{Mod}_A \quad \text{in the usual sense.}$$

Same correspondence as for algebras, except that  $A \oplus M$  must be commutative. This forces the two actions to coincide:

$$a \cdot m = m \cdot a.$$

For a map of commutative  $k$ -algebras  $f: A \rightarrow B$ , the pushforward functor is extension of scalars:

$$\begin{aligned} f_! : \text{Mod}(A) &\rightarrow \text{Mod}(B) \\ f_!(M) &= B \otimes_A M. \end{aligned}$$

**Remark.** The notion of Beck module depends on the ambient category! A Beck module over  $A$  in  $\text{Com}_{\mathbb{k}}$  is **not** the same as in  $\text{Alg}_{\mathbb{k}}$ .

# Lie algebras

$\mathbb{k}$  = a field.

$\mathcal{C} = \text{Lie}_{\mathbb{k}}$ , the category of Lie algebras over  $\mathbb{k}$ .

For a Lie algebra  $L$ :

$$\begin{aligned}\text{Mod}(L) &\cong L\text{-modules in the usual sense} \\ &\cong U(L)\text{-Mod}\end{aligned}$$

where  $U(L)$  is the universal enveloping algebra of  $L$ .

A Beck module over  $L$  is a split extension of  $L$  with abelian kernel:

$$0 \longrightarrow M \longrightarrow L \oplus M \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} L \longrightarrow 0$$

with  $[m, m'] = 0$  for all  $m, m' \in M$ .

## Lie algebras (cont'd)

The action of  $L$  on  $M$  is given by

$$[(\ell, 0), (0, m)] = (0, \ell \cdot m),$$

which satisfies

$$[\ell, \ell'] \cdot m = \ell \cdot (\ell' \cdot m) - \ell' \cdot (\ell \cdot m).$$



# Universal enveloping algebra

Usually, the category of Beck modules over  $X$  is equivalent to (left) modules over a ring  $\mathbb{U}_{\mathcal{C}}(X)$ , called the **universal enveloping algebra** of  $X$ :

$$\text{Mod}(X) \cong \mathbb{U}_{\mathcal{C}}(X)\text{-Mod.}$$

Examples of universal enveloping algebras:

1. For a group  $G$ , the group ring  $\mathbb{U}_{\text{Gp}}(G) = \mathbb{Z}G$ .
2. For a ring  $A$ , the enveloping ring  $\mathbb{U}_{\text{Alg}_{\mathbb{Z}}}(A) = A \otimes A^{\text{op}}$ .
3. For a commutative ring  $R$ , the ring itself  $\mathbb{U}_{\text{Com}_{\mathbb{Z}}}(R) = R$ .
4. For a Lie algebra  $L$  over  $\mathbb{k}$ , the (classical) universal enveloping algebra  $\mathbb{U}_{\text{Lie}_{\mathbb{k}}}(L) = U(L)$ .

# Derivations and differentials

**Definition.** A **derivation** of  $X$  with coefficients in a Beck module  $p: E \rightarrow X$  is a section of  $p$ :

$$\begin{array}{ccc} X & \xrightarrow{s} & E \\ & \searrow & \downarrow p \\ & & X. \end{array}$$

The abelian group of derivations is

$$\begin{aligned} \text{Der}(X, E) &:= \text{Hom}_{\mathcal{C}/X}(X \xrightarrow{\text{id}} X, E \xrightarrow{p} X) \\ &\cong \text{Hom}_{(\mathcal{C}/X)_{\text{ab}}}(Ab_X(X \xrightarrow{\text{id}} X), E \xrightarrow{p} X), \end{aligned}$$

where  $Ab_X: \mathcal{C}/X \rightarrow (\mathcal{C}/X)_{\text{ab}}$  is the **abelianization** functor, i.e., the left adjoint to the forgetful functor  $(\mathcal{C}/X)_{\text{ab}} \rightarrow \mathcal{C}/X$ .

The module of **Kähler differentials** is  $\Omega_{\mathcal{C}}(X) = Ab_X X$ , which represents derivations.

## Example: Commutative rings

Take  $\mathcal{C} = \text{Com}_{\mathbb{k}}$ . Given a commutative  $\mathbb{k}$ -algebra  $A$  and an  $A$ -module  $M$ , a  $\mathbb{k}$ -linear map

$$(\text{id}, f): A \rightarrow A \oplus M$$

is a (unital) ring homomorphism if and only if

$$\begin{cases} f(1_A) = 0 \\ f(ab) = a \cdot f(b) + f(a) \cdot b \quad \rightsquigarrow \text{Leibniz rule.} \end{cases}$$

$\text{Der}(A, M)$  = derivations in the usual sense.

The  $A$ -module  $Ab_A A$  is:

$$Ab_A A = I_A / I_A^2 = \Omega_{A/k},$$

where  $I_A = \ker(\mu: A \otimes_{\mathbb{k}} A \rightarrow A)$ . Classical module of Kähler differentials, which represents  $k$ -derivations:

$$\text{Hom}_A(\Omega_{A/k}, M) \cong \text{Der}_k(A, M).$$

# Geometric interpretation

Analogy:

Kähler differentials in algebraic geometry



differential 1-forms in differential geometry

**Example.** 1.  $A = \mathbb{k}[x, y]$

$$\begin{aligned}\Omega_{A/\mathbb{k}} &\cong \{p dx + q dy \mid p, q \in \mathbb{k}[x, y]\} \\ &\cong A \langle dx, dy \rangle \quad \text{free } A\text{-module.}\end{aligned}$$

2.  $A = \mathbb{k}[x, y]/(f)$

$$\Omega_{A/\mathbb{k}} \cong A \langle dx, dy \rangle / \left\langle \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right\rangle.$$

# Quillen (co)homology

Quillen homology of  $X \approx$  (simplicially) derived functors of Kähler differentials.

Quillen cohomology of  $X$  with coefficients in a module  $M \approx$  (simplicially) derived functors of derivations with coefficients in  $M$ .

**The “package”:**

1. Beck modules
2. Universal enveloping algebra
3. Derivations
4. Kähler differentials

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# Divided power algebras

Introduced by Cartan in the 1950s. Appear in positive characteristic.

Divided power operations  $\approx$  operations  $\gamma_n$  that look like  $\gamma_n(x) = \frac{x^n}{n!}$ . More precisely...

## Divided power algebras (cont'd)

**Definition.** Let  $A$  be a commutative  $\mathbb{k}$ -algebra. A **system of divided powers** on an ideal  $I \subseteq A$  is a collection of maps  $\gamma_i: I \rightarrow A$  for  $i \geq 0$  satisfying:

$$\gamma_0(a) = 1$$

$$\gamma_1(a) = a$$

$$\gamma_i(a) \in I, \quad i \geq 1$$

$$\gamma_i(a+b) = \sum_{k=0}^{i} \gamma_k(a)\gamma_{i-k}(b), \quad a, b \in I, \quad i \geq 0$$

$$\gamma_i(ab) = a^i \gamma_i(b), \quad a \in A, \quad i \geq 0$$

$$\gamma_i(a)\gamma_j(a) = \frac{(i+j)!}{i!j!} \gamma_{i+j}(a), \quad a \in I, \quad i, j \geq 0$$

$$\gamma_i(\gamma_j(a)) = \frac{(ij)!}{i!(j!)^i} \gamma_{ij}(a), \quad a \in I, \quad i \geq 0, \quad j \geq 1.$$

Usually  $A$  has an augmentation  $\epsilon: A \rightarrow \mathbb{k}$  and  $I = \ker(\epsilon)$ .



## Examples

**Example.** The free divided power algebra over  $\mathbb{Z}$  on one generator  $x$  is

$$\mathbb{Z}[x, \frac{x^2}{2}, \dots, \frac{x^n}{n!}, \dots] \subset \mathbb{Q}[x].$$

**Example.** Over a field  $\mathbb{k}$  of characteristic 0, divided powers must in fact be given by

$$\gamma_n(x) = \frac{x^n}{n!}.$$

$\rightsquigarrow$  No additional structure in that case.

Divided powers are a **positive characteristic** phenomenon.

## In positive characteristic

**Proposition** (Soublin 1987). Over a field  $\mathbb{k}$  of characteristic  $p > 0$ , a system of divided power operations is the same data as just the operation  $\pi = \gamma_p: I \rightarrow I$  satisfying:

$$a^p = 0, \quad a \in I$$

$$\pi(a + b) = \pi(a) + \pi(b) + \sum_{k=1}^{k=p-1} \frac{(-1)^k}{k} a^k b^{p-k}, \quad a, b \in I$$

$$\pi(ab) = 0, \quad a, b \in I$$

$$\pi(\lambda a) = \lambda^p \pi(a), \quad a \in I, \lambda \in \mathbb{k}.$$

Note: The operation  $\pi$  is non-additive!

**Theorem** (Dokas 2009). Worked out the “package” for divided power algebras over a field  $\mathbb{k}$  of characteristic  $p > 0$ .

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## Restricted Lie algebras

Over a field  $\mathbb{k}$  of characteristic  $p > 0$ , Lie algebras usually come with extra structure, that of a *restricted* Lie algebra.

Textbook account in Jacobson (1962).

Important in topology. See work of Milnor–Moore, May, Rector, Bousfield, Curtis, Priddy, etc. in the 1960s.

## Restricted Lie algebras (cont'd)

**Definition.** A **restricted Lie algebra**  $L$  over  $\mathbb{k}$  is a Lie algebra together with a map  $(-)^{[p]}: L \rightarrow L$  called the  $p$ -map satisfying:

$$(\alpha x)^{[p]} = \alpha^p x^{[p]}, \quad \alpha \in \mathbb{k}$$

$$[x, y^{[p]}] = [\underbrace{\cdots [x, y], y}_{p}, \cdots, y]$$

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$$

where  $s_i(x, y)$  is the coefficient of  $\lambda^{i-1}$  in  $\text{ad}_{\lambda x + y}^{p-1}(x)$ .

Here  $\text{ad}_x: L \rightarrow L$  denotes the adjoint representation

$$\text{ad}_x(y) := [y, x], \quad x, y \in L.$$

**Example.** For  $p = 2$ , the last equation becomes:

$$(x + y)^{[2]} = x^{[2]} + [x, y] + y^{[2]}.$$

## Examples

**Example.** For a  $\mathbb{k}$ -algebra  $A$ , the underlying restricted Lie algebra of  $A$  has the commutator bracket

$$[x, y] = xy - yx$$

and the  $p^{\text{th}}$  power map

$$x^{[p]} = x^p.$$

**Example.** The free restricted Lie algebra on a  $\mathbb{k}$ -vector space  $V$  is the subspace

$$L^r(V) \subseteq T(V)$$

obtained by taking  $V$  and closing it under commutators and  $p^{\text{th}}$  powers.

**Theorem** (Dokas 2004). Worked out the “package” for restricted Lie algebras over a field  $\mathbb{k}$  of characteristic  $p > 0$ .

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# Operads in sets

**Idea:** Encode a type of algebraic structure as

$$\mathcal{P}(n) = \{\text{the allowed } n\text{-ary operations}\}.$$

**Definition.** A (symmetric) **operad** in sets  $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$  is a sequence of sets  $\mathcal{P}(n)$  with a right action of the symmetric group  $\Sigma_n$  together with composition maps

$$\circ: \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \rightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

and a unit  $\eta \in \mathcal{P}(1)$  such that composition is associative, unital, and equivariant.



# Algebras over operads

**Example.** For any set  $X$ , the **endomorphism operad** of  $X$  is:

$$\left\{ \begin{array}{l} \text{End}_X(n) := \text{Hom}(X^n, X) \\ \circ = \text{usual composition} \\ \eta = \text{id}_X \in \text{Hom}(X, X) \\ \Sigma_n \text{ acts by permuting the factors of } X^n. \end{array} \right.$$

**Definition.** A  **$\mathcal{P}$ -algebra** is a set  $X$  equipped with action maps

$$\alpha_n : \mathcal{P}(n) \rightarrow \text{Hom}(X^n, X)$$

compatible with composition, unit, and equivariance. In other words, an operad map

$$\alpha : \mathcal{P} \rightarrow \text{End}_X .$$

Each  $n$ -ary operation symbol  $\mu \in \mathcal{P}(n)$  yields an actual  $n$ -ary operation on  $X$ .

## Examples

**Example.** The (unital) associative operad:

$$\text{As}^{\text{un}}(n) = \Sigma_n, \quad n \geq 0.$$

$\text{As}^{\text{un}}$ -algebra = monoid.

**Example.** The (non-unital) associative operad:

$$\text{As}(n) = \begin{cases} \Sigma_n, & n \geq 1 \\ \emptyset, & n = 0. \end{cases}$$

$\text{As}$ -algebra = semigroup.

## Examples (cont'd)

**Example.** The (unital) commutative operad:

$$\text{Com}^{\text{un}}(n) = *, \quad n \geq 0.$$

$\text{Com}^{\text{un}}$ -algebra = commutative monoid.

**Example.** The (non-unital) commutative operad:

$$\text{Com}(n) = \begin{cases} *, & n \geq 1 \\ \emptyset, & n = 0. \end{cases}$$

$\text{Com}$ -algebra = commutative semigroup.

# Operads in vector spaces

What if we want  $\mathbb{k}$ -vector spaces equipped with operations?

Instead of working in  $(\text{Set}, \times)$ , work in  $(\text{Vect}_{\mathbb{k}}, \otimes_{\mathbb{k}})$ .

$\rightsquigarrow$  Replace the Cartesian product  $\times$  with the tensor product  $\otimes_{\mathbb{k}}$  everywhere.

**Definition.** A (symmetric) **operad** in  $\mathbb{k}$ -vector spaces  $\mathcal{P} = \{\mathcal{P}(n)\}_{n \geq 0}$  is a sequence of  $\mathbb{k}$ -vector spaces  $\mathcal{P}(n)$  with a right action of the symmetric group  $\Sigma_n$  together with composition maps

$$\circ: \mathcal{P}(k) \otimes_{\mathbb{k}} \mathcal{P}(n_1) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} \mathcal{P}(n_k) \rightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

and a unit  $\eta \in \mathcal{P}(1)$  such that composition is associative, unital, and equivariant.

**Warning.** This forces operations  $\mu \in \mathcal{P}(n)$  to be  **$\mathbb{k}$ -multilinear**.

## Examples

Taking the  $\mathbb{k}$ -vector space spanned by an operad in sets yields an operad in  $\mathbb{k}$ -vector spaces.

**Example.** The (unital) associative operad:

$$\text{As}^{\text{un}}(n) = \mathbb{k}\Sigma_n, \quad n \geq 0.$$

$\text{As}^{\text{un}}$ -algebra = (unital)  $\mathbb{k}$ -algebra.

**Example.** The (non-unital) associative operad:

$$\text{As}(n) = \begin{cases} \mathbb{k}\Sigma_n, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

$\text{As}$ -algebra = (non-unital)  $\mathbb{k}$ -algebra.

## Examples (cont'd)

**Example.** The (unital) commutative operad:

$$\text{Com}^{\text{un}}(n) = \mathbb{k}, n \geq 0.$$

$\text{Com}^{\text{un}}$ -algebra = (unital) commutative  $\mathbb{k}$ -algebra.

**Example.** The (non-unital) commutative operad:

$$\text{Com}(n) = \begin{cases} \mathbb{k}, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

$\text{Com}$ -algebra = (non-unital) commutative  $\mathbb{k}$ -algebra.

## Examples (cont'd)

**Example.** The Lie operad  $\text{Lie}$  is generated by an operation

$$[-, -] \in \text{Lie}(2) \quad \text{Lie bracket } [x, y]$$

subject to relations

$$\left\{ \begin{array}{l} [-, -] = -[-, -] \cdot (12) \in \text{Lie}(2) \\ [x, y] = -[y, x] \quad \text{skew-symmetry} \\ [-, [-, -]] + [-, [-, -]] \cdot (132) + [-, [-, -]] \cdot (123) = 0 \in \text{Lie}(3) \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{Jacobi identity.} \end{array} \right.$$

Lie-algebra = Lie algebra if  $\text{char}(\mathbb{k}) \neq 2$ .

**Warning.** In the case  $\text{char}(\mathbb{k}) = 2$ , Lie-algebra = “quasi-Lie algebra”. Can’t encode the equation  $[x, x] = 0$  with an operad.

## Quillen cohomology of $\mathcal{P}$ -algebra

For the category  $\mathcal{C} = \mathcal{P}\text{-Alg}$  of  $\mathcal{P}$ -algebras, the “package” has been worked out. Nice account in Loday–Vallette (2012).



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## Free $\mathcal{P}$ -algebras

**Setup:**  $\mathbb{k}$  = field of characteristic  $p > 0$ .

$\mathcal{P}$  = operad in  $\mathbb{k}$ -vector spaces that is *reduced*, i.e.,  $\mathcal{P}(0) = 0$ .

Free  $\mathcal{P}$ -algebra functor ([forget-of-free monad](#))

$$S(\mathcal{P}): \text{Vect}_{\mathbb{k}} \rightarrow \text{Vect}_{\mathbb{k}}$$

$$S(\mathcal{P})(V) = \bigoplus_{n \geq 1} (\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n}$$

## Example: Tensor algebra

**Example.** Associative operad:

$$\begin{aligned} S(\text{As})(V) &= \bigoplus_{n \geq 1} (\text{As}(n) \otimes V^{\otimes n})_{\Sigma_n} \\ &= \bigoplus_{n \geq 1} (\mathbb{k}\Sigma_n \otimes V^{\otimes n})_{\Sigma_n} \\ &\cong \bigoplus_{n \geq 1} V^{\otimes n} \\ &= T(V) \end{aligned}$$

the (non-unital) tensor algebra.

## Example: Symmetric algebra

**Example.** Commutative operad:

$$\begin{aligned} S(\text{Com})(V) &= \bigoplus_{n \geq 1} (\text{Com}(n) \otimes V^{\otimes n})_{\Sigma_n} \\ &= \bigoplus_{n \geq 1} (\mathbb{k} \otimes V^{\otimes n})_{\Sigma_n} \\ &\cong \bigoplus_{n \geq 1} \text{Sym}^n(V) \\ &= \text{Sym}(V) \end{aligned}$$

the (non-unital) symmetric algebra.

# Divided power algebras

Instead of taking coinvariants, take *invariants*. Norm map:

$$\begin{array}{ccc} \bigoplus_{n \geq 1} (\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n} & \longrightarrow & \bigoplus_{n \geq 1} (\mathcal{P}(n) \otimes V^{\otimes n})^{\Sigma_n} \\ \parallel & & \parallel \text{def} \\ S(\mathcal{P})(V) & & \Gamma(\mathcal{P})(V) \end{array}$$

**Proposition.**  $\Gamma(\mathcal{P}): \text{Vect}_{\mathbb{k}} \rightarrow \text{Vect}_{\mathbb{k}}$  is a monad.

**Definition.** A **divided power  $\mathcal{P}$ -algebra** is an algebra for the monad  $\Gamma(\mathcal{P})$ .

$\Gamma(\mathcal{P})$ -algebra  $\approx$   $\mathcal{P}$ -algebra + extra operations, usually **non-additive**.

## Examples

**Example.** If  $\text{char}(\mathbb{k}) = 0$ , then  $S(\mathcal{P}) \xrightarrow{\cong} \Gamma(\mathcal{P})$ .

$\rightsquigarrow$  No extra operations in that case.

Divided power operations are a **positive characteristic** phenomenon.

**Theorem** (Fresse 2000). 1. Associative operad:  $S(\text{As}) \xrightarrow{\cong} \Gamma(\text{As})$ ,  
so a  $\Gamma(\text{As})$ -algebra is just an As-algebra, i.e., a  $\mathbb{k}$ -algebra.

2. Commutative operad:  $\Gamma(\text{Com})$ -algebra = classical divided power algebra.

3. Lie operad:  $\Gamma(\text{Lie})$ -algebra = restricted Lie algebra.

# Results

**Theorem** (Ikonicoff 2020). Equational presentation for  $\Gamma(\mathcal{P})$ -algebras.

$\rightsquigarrow$  List of operations satisfying a (big) list of equations.

**Theorem** (Dokas–F.–Ikonicoff). Worked out the “package” for  $\Gamma(\mathcal{P})$ -algebras.

Reality check: Recovered the examples of classical divided power algebras ( $\mathcal{P} = \text{Com}$ ) and restricted Lie algebras ( $\mathcal{P} = \text{Lie}$ ).

## Results (cont'd)

Comparison maps involving Quillen cohomology of  $\Gamma(\mathcal{P})$ -algebras.

A map of operads  $f: \mathcal{P} \rightarrow \mathcal{Q}$  induces adjunctions:

$$\begin{array}{ccc} \mathcal{P}\text{-Alg} & \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \end{array} & \mathcal{Q}\text{-Alg} \\ \begin{array}{c} \text{free} \downarrow \\ \uparrow \text{forget} \end{array} & & \begin{array}{c} \text{free} \downarrow \\ \uparrow \text{forget} \end{array} \\ \Gamma(\mathcal{P})\text{-Alg} & \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \end{array} & \Gamma(\mathcal{Q})\text{-Alg}. \end{array}$$

$\rightsquigarrow$  Comparison maps between Quillen cohomology in those different categories.

**Example.** Inclusion of operads  $\text{Lie} \rightarrow \text{As}$ .



**Danke!**