Multiparameter persistence modules in the large scale

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Outline

Persistence modules

Localized persistence modules

Classification of indecomposables

Which subcategories are we quotienting out?

Rank invariant

Topological data analysis pipeline

Example. Let X be a finite metric space — "data set". The **Vietoris–Rips complex** $VR(X)_{\epsilon}$ is the simplicial complex on the vertex set X with

 $\{x_0, \ldots, x_n\}$ is an *n*-simplex $\iff d(x_i, x_j) \le \epsilon$ for all i, j.

Filtered simplicial complex



Image source: Robert Ghrist, Barcodes: The persistent topology of data.

As ϵ varies, $VR(X)_{\epsilon}$ forms a filtered simplicial complex with one parameter $\epsilon \geq 0$, i.e., a functor

$$VR(X): \mathbb{R}_+ \to \operatorname{SimpCpx}.$$

If instead we let ϵ increase by a fixed small step, we obtain one discrete parameter $\mathbb{N} \to \text{SimpCpx.}$

Filtered space

Example. Let X be a smooth manifold and $f: X \to \mathbb{R}$ a Morse function. Filtration by sublevel sets:

$$X_s = \{x \in X \mid f(x) \le s\} = f^{-1}((-\infty, s]).$$

As s varies, X_s forms a filtered space with one parameter $s \in \mathbb{R},$ i.e., a functor

$$X_{\bullet} \colon \mathbb{R} \to \text{Top.}$$

Given another Morse function $g: X \to \mathbb{R}$, consider the joint sublevel sets:

$$X_{s,t} = \{x \in X \mid f(x) \le s, g(x) \le t\}.$$

Get a filtered space with two parameters $s, t \in \mathbb{R}$, i.e., a functor $X_{\bullet,\bullet} : \mathbb{R}^2 \to \text{Top.}$

In applications, often need *multiple* parameters.

In this project, we focus on *discrete* parameters.

Persistence modules

Fix a ground field $\Bbbk.$

Definition. For $m \ge 1$, an *m*-parameter persistence module is a diagram

$$\mathbb{N}^m \to \operatorname{Vect}_{\mathbb{k}}$$
.

 \cong graded module over the graded polynomial algebra

$$R \coloneqq \Bbbk[t_1, \ldots, t_m],$$

which is \mathbb{N}^m -graded with multigrading

$$|t_i| = \vec{e_i} = (0, \dots, \overbrace{1}^{i}, \dots, 0).$$

Write $M(\vec{d})$ for the k-vector space in multidegree $\vec{d} \in \mathbb{N}^m$.

Goal / Dream

Work with *finitely generated R*-modules:

R-mod := R-Mod^{fin.gen.} $\subset R$ -Mod.

Goal Classify the indecomposable objects in R-mod.

One-parameter case

For m = 1, finitely generated $\Bbbk[t]$ -modules decompose into interval modules

$$\begin{aligned} [a,b) &\coloneqq t^a \mathbb{k}[t]/t^b \mathbb{k}[t] \\ &= \operatorname{coker} \left(t^a \mathbb{k}[t] \xrightarrow{t^{b-a}} t^a \mathbb{k}[t] \right). \end{aligned}$$

Example.

$$M = t^4 \mathbb{k}[t] \oplus t^2 \mathbb{k}[t] / t^7 \mathbb{k}[t]$$
$$= [4, \infty) \oplus [2, 7).$$

 \rightsquigarrow **Barcode**: multiset of intervals. List of intervals appearing in the decomposition (with multiplicity).

Multiparameter case

Not available for $m \ge 2$, because $\mathbb{k}[t_1, t_2]$ has wild representation type.

How to deal with that?

One approach: Extract invariants that are both computable and significant. Rank invariant and various refinements. Many authors...

Another approach: Focus on certain families of modules admitting a nice decomposition, such as rectangle-decomposable modules.

Approach: localize

Our approach: We localize R-mod until the resulting category admits a classification of indecomposables, or at least a partial classification.

Related work: [Harrington, Otter, Schenck, Tillmann] and [Bauer, Botnan, Oppermann, Steen].

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Inverting some variables

Fact. The homogeneous prime ideals of R are those of the form $(t_{i_1}, \dots, t_{i_k})$.

The various localizations of a module M fit together.

Example. For M a module over $R = \Bbbk[t_1, t_2]$:

$$\begin{aligned} & \mathbb{k}[t_1, t_2^{\pm}] \otimes_R M \xrightarrow{\text{invert } t_1} \mathbb{k}[t_1^{\pm}, t_2^{\pm}] \otimes_R M \\ & \text{invert } t_2 \\ & M \xrightarrow{\text{invert } t_1} \mathbb{k}[t_1^{\pm}, t_2] \otimes_R M. \end{aligned}$$

Inverting some variables (cont'd)

Notation. 1. $[m] \coloneqq \{1, 2, \dots, m\}$

2. For a subset $\sigma \subseteq [m]$, denote the localization of rings

$$R_{\sigma} \coloneqq R[t_i^{-1} \mid i \in \sigma],$$

which is $\sigma^{-1}\mathbb{N}^m$ -graded.

3. $\varphi_i \coloneqq$ the localization map "invert t_i ".

Example. With this notation, the previous square becomes:

$$R_{\{2\}} \otimes_R M \xrightarrow{\varphi_1} R_{\{1,2\}} \otimes_R M$$

$$\varphi_2 \uparrow \qquad \uparrow \varphi_2$$

$$M \xrightarrow{\varphi_1} R_{\{1\}} \otimes_R M.$$

K-localized persistence modules

Idea: Forget the module M but keep some of its localizations.

If we keep a localization $R_{\sigma} \otimes_R M$, we should keep all further localizations $R_{\tau} \otimes_R M$ for $\sigma \subseteq \tau$.

Definition. Let K be a simplicial complex on the vertex set [m]. A **K-localized persistence module** M consists of:

- 1. For each missing face $\sigma \notin K$, a finitely generated R_{σ} -module M_{σ} .
- 2. For each $\sigma \subseteq \tau$ with $\sigma \notin K$ (and hence $\tau \notin K$), a map of R_{σ} -modules $\varphi_{\sigma,\tau} \colon M_{\sigma} \to M_{\tau}$ such that the induced map of R_{τ} -modules

$$R_{\tau} \otimes_{R_{\sigma}} M_{\sigma} \xrightarrow{\cong} M_{\tau}$$

is an isomorphism.

Let $\mathcal{D}(K)$ denote the category of K-localized persistence modules.

The role of K

Small $K \rightsquigarrow$ Localize a little.

Big $K \rightsquigarrow$ Localize a lot.

Example. Extreme cases:

1.
$$K = \{\} = \operatorname{sk}_{-2} \Delta^{m-1} \rightsquigarrow \text{Don't localize:}$$

 $\mathcal{D}(K) \cong R\text{-mod.}$

2.
$$K = \partial \Delta^{m-1} = \operatorname{sk}_{m-2} \Delta^{m-1} \rightsquigarrow \text{ Invert all the } t_i:$$

$$\mathcal{D}(K) = R_{[m]} \operatorname{-mod} \cong \operatorname{vect}_{\Bbbk}.$$

3. Even more extreme! $K = \Delta^{m-1} \rightsquigarrow$ Localize everything into oblivion:

$$\mathcal{D}(K) = 0.$$

Example: m = 2

Take m = 2 and $K = \{\emptyset\} = \operatorname{sk}_{-1} \Delta^1$.

A K-localized persistence module M consists of modules

where φ_i inverts t_i .

Example: m = 3

Take m = 3 and $K = \operatorname{sk}_0 \Delta^2 = \{\emptyset, \{1\}, \{2\}, \{3\}\}.$

A K-localized persistence module M consists of modules



where $\sigma_i := [m] \setminus \{i\} \quad \rightsquigarrow$ "all but t_i have been inverted" $\varphi_i := \varphi_{[m] \setminus \{i\}, [m]} \colon M_{[m] \setminus \{i\}} \to M_{[m]} \quad \rightsquigarrow$ "invert t_i ".

A Serre quotient

Consider the canonical functor

 $L_K \colon R\operatorname{-mod} \to \mathcal{D}(K)$

that keeps the relevant localizations of M:

$$L_K(M)_{\sigma} = R_{\sigma} \otimes_R M.$$

Lemma. L_K is exact.

Proposition. L_K is a Serre quotient functor:

$$\begin{array}{c|c} R\operatorname{-mod} & \xrightarrow{L_K} \mathcal{D}(K). \\ & q \\ & & \swarrow \\ R\operatorname{-mod}/ \ker(L_K) \end{array}$$

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A hopeless dream?

Denote $K_m := \operatorname{sk}_{m-3} \Delta^{m-1} \rightsquigarrow$ Allow at most one non-inverted t_i .

Proposition. For any smaller simplicial complex $K \subset K_m$, $\mathcal{D}(K)$ has wild representation type.

Proof. $\mathcal{D}(K)$ contains a copy of $\Bbbk[s, t]$ -mod as a retract.

In this section, focus on $\mathcal{D}(K_m)$.

Some indecomposables



Some indecomposable objects in $\mathcal{D}(K_2)$.

Theorem (F.–Stanley). Every object in $\mathcal{D}(K_2)$ decomposes (in a unique way) as a direct sum of:

- "vertical strips" $[a, b)_1$
- "horizontal strips" $[a, b)_2$
- "quadrants" $[\vec{a}, \infty)$.

A torsion pair

Consider two full subcategories of $\mathcal{D}(K_m)$:

$$\mathcal{T} = \{ M \mid M_{\sigma_i} \text{ is a torsion } R_{\sigma_i} \text{-module for all } i \in [m] \}$$
$$= \{ M \mid M_{[m]} = 0 \}$$

 $\mathcal{F} = \{ M \mid M_{\sigma_i} \text{ is a torsion-free } R_{\sigma_i} \text{-module for all } i \in [m] \}.$

Proposition. 1. $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\mathcal{D}(K_m)$. 2. For all M in $\mathcal{D}(K_m)$, the natural short exact sequence

$$0 \to T(M) \to M \to F(M) \to 0$$

splits.

 $\implies M \cong T(M) \oplus F(M)$

Torsion objects

For $a < b < \infty$, consider the "interval [a, b) in the *i*th direction":

$$[a,b)_i = L_{K_m} \left(t_i^a R / t_i^b R \right).$$

Lemma. $[a,b)_i$ is indecomposable in $\mathcal{D}(K_m)$.

Proposition. Each torsion object $M \in \mathcal{T}$ decomposes as a direct sum of objects of the form $[a, b)_i$.

Dimension count arguments

After inverting all variables, we are left with a vector space:

$$\begin{aligned} R_{[m]}\operatorname{-mod} &\xrightarrow{\cong} \operatorname{vect}_{\Bbbk} \\ M &\mapsto M(\vec{0}). \end{aligned}$$

Remark. The grading is crucial here. An *ungraded* $\Bbbk[t^{\pm}]$ -module corresponds to a \Bbbk -vector space V equipped with an automorphism $\mu_t \colon V \xrightarrow{\cong} V.$

Definition. 1. For $\sigma \subseteq [m]$ and an R_{σ} -module M, the rank of M is

$$\operatorname{rank} M \coloneqq \dim_{\mathbb{K}} \left(R_{[m]} \otimes_{R_{\sigma}} M \right).$$

2. The **rank** of an object M in $\mathcal{D}(K)$ is rank $M \coloneqq \operatorname{rank} M_{[m]}$ = rank M_{σ} for any missing face $\sigma \notin K$.

Torsion-free objects

Notation. For a multidegree $\vec{a} \in \mathbb{N}^m$, consider the object of $\mathcal{D}(K_m)$

$$(\vec{a},\infty) = L_{K_m}(t^{\vec{a}}R),$$

where $t^{\vec{a}} \coloneqq t_1^{a_1} \cdots t_m^{a_m}$.

"quadrant module starting at \vec{a} "

Lemma. 1. [*a*, ∞) is torsion-free, of rank 1, and indecomposable.
2. Any torsion-free object of rank 1 is of the form [*a*, ∞).

Proposition. Every torsion-free object M in $\mathcal{D}(K_2)$ decomposes as a direct sum of modules of the form $[\vec{a}, \infty)$.

Proof sketch...

Scanning process



Scan for an element x of lowest degree $\vec{d} \in \mathbb{N}^2$ in lexicographic order.

The quotient $M/\left\langle x\right\rangle$ is still torsion-free.

Battleship game



Scan, mod out, repeat r - 1 times, where $r = \operatorname{rank} M$.

The composite epimorphism $M \twoheadrightarrow [\vec{a}, \infty)$ admits a section, splitting off a rank 1 summand from M.

In higher dimension $m \ge 3$

Warning: $[\vec{a}, \infty)$ is **not** projective in $\mathcal{D}(K_2)$.

Example. Consider the map in $\mathcal{D}(K_2)$

$$[(1,0),\infty) \oplus [(0,1),\infty) \xrightarrow{f=[\operatorname{inc} \operatorname{inc}]} [(0,0),\infty).$$

The map f is an epimorphism but does not admit a section.

Proposition. For any $m \geq 3$, there exists a torsion-free object in $\mathcal{D}(K_m)$ that is of rank 2 and indecomposable.

How complicated can the torsion-free objects in $\mathcal{D}(K_m)$ get? Answer: Pretty complicated!

Classification in higher dimension

Steffen Oppermann kindly provided the following argument.

Theorem (F.–Oppermann–Stanley). The category $\mathcal{D}(K_3)$ has wild representation type.

Proof idea. Reduce to the known fact that this quiver has wild representation type:



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Tensor ideals

Recall: Serre quotient

 $\mathcal{D}(K) \cong R\text{-mod}/\ker(L_K).$

The subcategory $\ker(L_K) \subseteq R$ -mod is a "tensor ideal": a Serre subcategory closed under tensoring with a \mathbb{Z}^m -graded R-module as long as the result is still \mathbb{N}^m -graded.

 \rightsquigarrow Allow shifting the degrees *down*, but not below 0.

Classification of tensor ideals

Recall: The **support** of an *R*-module *M* is the set of homogeneous prime ideals $P \subset R$ for which the localization $M_P \neq 0$.

For $P = (t_{i_1}, \dots, t_{i_k})$, we will record the *complement* $[m] \setminus \{i_1, \dots, i_k\}$, the variables that *may* be inverted.

Proposition. [F.–(Don)Stanley] There is a bijection



Morever $\ker(L_k)$ = the "tensor ideal" generated by $\Bbbk[K]$.

Stanley–Reisner ring

Definition. The **Stanley–Reisner ring** or *face ring* of a simplicial complex K is the polynomial ring modulo the monomials corresponding to missing faces:

$$\Bbbk[K] \coloneqq R/(t_{\sigma} \colon \sigma \notin K).$$

Example. $K = \operatorname{sk}_{-1} \Delta^{m-1} = \{\emptyset\}$

$$\Longrightarrow \mathbb{k}[K] = R/(t_1, \dots, t_m) = \mathbb{k}$$
$$\Longrightarrow \operatorname{Supp} \mathbb{k}[K] = \{(t_1, \dots, t_m)\}$$
$$\Longrightarrow \operatorname{complemented} \operatorname{Supp} \mathbb{k}[K] = \{\emptyset\} = K.$$

Simple objects

What about $\ker(L_{K_m})$?

Proposition. $\mathcal{D}(K_m)$ is obtained from *R*-mod by quotienting out the Serre subcategory generated by the simple objects m-1 times successively.

Corollary. $\mathcal{D}(K_2)$ is the category of 2-parameter persistence modules up to finite diagrams:

 $\mathcal{D}(K_2) \cong \mathbb{k}[t_1, t_2] \text{-mod}/\{\text{finite modules}\}.$

 \rightsquigarrow Large-scale behavior of the persistence module.

A link with toric geometry

Coherent sheaves on the projective line \mathbb{P}^1_{\Bbbk} :

 $\operatorname{Coh}(\mathbb{P}^1) \cong \mathbb{Z}$ -graded $\Bbbk[s, t]$ -mod/{finite modules}.

The \mathbb{Z}^2 -graded variant of our category $\mathcal{D}(K_2)$ is the bigraded analogue of the right-hand side:

$$\mathcal{D}(K_2)_{\mathbb{Z}^2} \cong \mathbb{Z}^2$$
-graded $\Bbbk[s, t]$ -mod/{finite modules}.

Colin Ingalls pointed out that this category is equivalent to *torus-equivariant* coherent sheaves on \mathbb{P}^1 .

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Definition. Let M be an R-module. The rank invariant of M is the function assigning to each pair of multidegrees $\vec{a}, \vec{b} \in \mathbb{N}^m$ with $\vec{a} \leq \vec{b}$ the integer

$$\operatorname{rk}_M(\vec{a}, \vec{b}) = \operatorname{rank}\left(M(\vec{a}) \xrightarrow{t^{\vec{b}-\vec{a}}} M(\vec{b})\right).$$

Introduced by Carlsson and Zomorodian (2009). Widely studied invariant of multiparameter persistence modules.

Rank invariant is not enough

The rank invariant of an *R*-module does not determine $L_K(M)$.

Example. In the case m = 2:

$$M = (t_1, t_2) \oplus t_1 t_2 R$$
$$N = t_1 R \oplus t_2 R$$

have the same rank invariant.

However, they have different K_2 -localizations in $\mathcal{D}(K_2)$:

$$L_{K_2}(M) \cong [(0,0),\infty) \oplus [(1,1),\infty)$$
$$L_{K_2}(N) \cong [(1,0),\infty) \oplus [(0,1),\infty).$$

Same rank invariant



Rank invariant is sometimes enough

Proposition. For $K = K_m$, the rank invariant of an *R*-module *M* determines the R_{σ_i} -modules M_{σ_i} and the k-vector space $M_{[m]}$.

Proposition. If M lies in the image of the right adjoint (delocalization functor)

$$\rho_{K_2} \colon \mathcal{D}(K_2) \to \Bbbk[t_1, t_2] \text{-mod},$$

then the rank invariant of M determines the localization $L_{K_2}(M)$.

Question. Which refinement of the rank invariant determines the torsion-free part of $L_{K_2}(M)$ in $\mathcal{D}(K_2)$?

Thank you!