Categorical aspects of graphs

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Literature

Literature: Some overlap with the PhD thesis of Demitri Plessas (U Montana, 2011). Hat tip to Laura Scull.

Some pieces scattered in various papers.

Outline

Categories of graphs

Universal algebraic descriptions

Adjoints galore

Slice categories

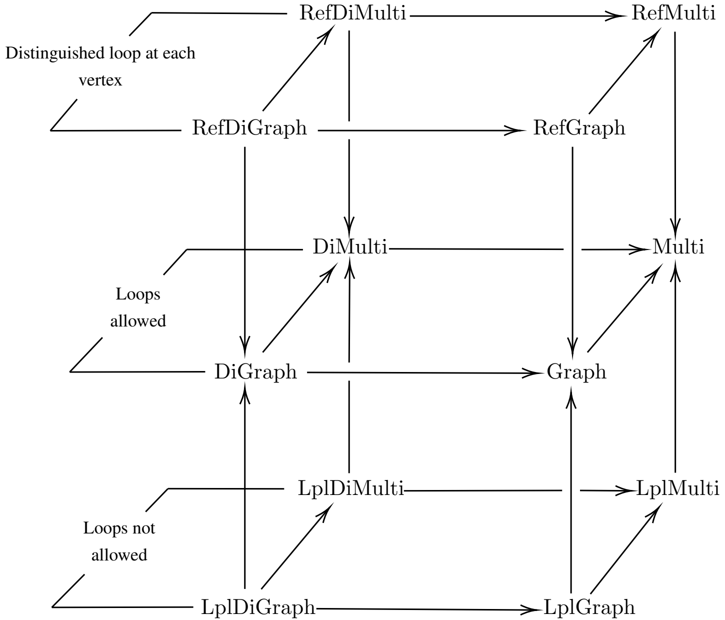
Graphs

Definition. A **graph** $G = (V, E)$ consists of a set of vertices V and a set of edges E connecting certain vertices.

Wait... What do you mean?

- Are the edges directed? $\rightsquigarrow$ **directed**
- Are parallel edges allowed? $\rightsquigarrow$ **multigraph**
- Are loops allowed?
  $\rightsquigarrow$ **loopless**: loops not allowed
  $\rightsquigarrow$ **reflexive**: distinguished loop at each vertex

Categories of graphs



Middle floor: Loops allowed

$$\begin{tikzcd}
 & \text{directed} & \text{undirected} & \\
\text{parallel edges allowed} & \mathrm{DiMulti} \arrow[r, "U_d"] \arrow[d, bend right, "Fl"'] \arrow[d, dashed, red, bend left=40, "\times"] \arrow[r, dashed, red, bend right=50, "\times"'] & \mathrm{Multi} \arrow[l, bend left=20, "D"] \arrow[l, dashed, red, bend right=40, "\times"'] \arrow[r, "\cong"] \arrow[d, bend right, "Fl"'] \arrow[d, dashed, red, bend left=40, "\times"] & \mathrm{DiMultiRev}_{\mathrm{self}} \\
\text{no parallel edges} & \mathrm{DiGraph} \arrow[u, "\iota"] \arrow[r, "U_d"] \arrow[r, bend right=50, "S"'] & \mathrm{Graph} \arrow[u, "\iota"] \arrow[l, bend left=20, "D"] \arrow[l, dashed, red, bend right=40, "\times"'] \arrow[r, "\cong"] & \mathrm{SymDiGraph}
\end{tikzcd}$$

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Directed multigraphs as presheaves

Directed multigraphs are diagrams of sets of shape

$$E \underset{t}{\overset{s}{\rightrightarrows}} V.$$

That is, presheaves on

$$[0] \underset{d^0}{\overset{d^1}{\rightrightarrows}} [1] \qquad \text{also known as } \Delta_{\text{inj},\leq 1}$$

where $(d^1)^* = d_1 = s$ and $(d^0)^* = d_0 = t$.

$$\begin{aligned} \text{DiMulti} &\cong \text{Fun}(\Delta^{\text{op}}_{\text{inj},\leq 1}, \text{Set}) \\ &= \text{1-truncated semisimplicial sets.} \end{aligned}$$

Allow parallel edges?

Having no parallel edges means satisfying the implication

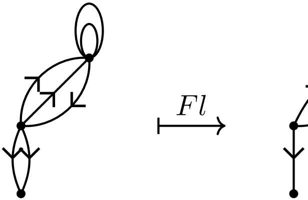
$$(s(e_1) = s(e_2)) \wedge (t(e_1) = t(e_2)) \implies e_1 = e_2$$

for all edges $e_1, e_2 \in E$.

Definition. The **flattening** of a directed multigraph $G = (V, E)$ is the directed graph $Fl(G)$ with the same vertex set V and set of arcs

$$E(Fl(G)) = \mathrm{im}(E \xrightarrow{(s,t)} V \times V).$$

Concretely, $Fl(G)$ has an arc from x to y if there was some arc e from x to y in G.



Allow parallel edges? (cont'd)

Proposition.
1. The flattening functor $Fl\colon \text{DiMulti} \to \text{DiGraph}$ is left adjoint to the inclusion functor $\iota\colon \text{DiGraph} \to \text{DiMulti}$.
2. The inclusion functor $\iota\colon \text{DiGraph} \to \text{DiMulti}$ admits no right adjoint.

Undirected analogue:

Proposition.
1. The flattening functor $Fl\colon \text{Multi} \to \text{Graph}$ is left adjoint to the inclusion functor $\iota\colon \text{Graph} \to \text{Multi}$.
2. The inclusion functor $\iota\colon \text{Graph} \to \text{Multi}$ admits no right adjoint.

$$
\begin{tikzcd}
\text{DiMulti} \arrow[d, bend right, "Fl"'] \arrow[d, dashed, bend left, "\times"] \\
\text{DiGraph} \arrow[u, "\iota"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\text{Multi} \arrow[d, bend right, "Fl"'] \arrow[d, dashed, bend left, "\times"] \\
\text{Graph} \arrow[u, "\iota"]
\end{tikzcd}
$$

Graphs as symmetric directed graphs

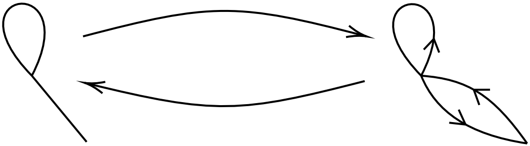
Definition. A directed graph $G = (V, E)$ is **symmetric** if its adjacency relation $E \subseteq V \times V$ is symmetric, that is:

$$(x, y) \in E \implies (y, x) \in E.$$

Proposition. The category of graphs is isomorphic to the category of symmetric directed graphs:

$$\mathrm{Graph} \cong \mathrm{SymDiGraph}.$$

Idea. Replace each undirected edge with a pair of arcs in both directions. Loops don't need to be duplicated. □



Reversing edges

Definition. For a directed multigraph $G = (V, E)$ a **reversal function** is a function $r\colon E \to E$ that "reverses edges", i.e.:

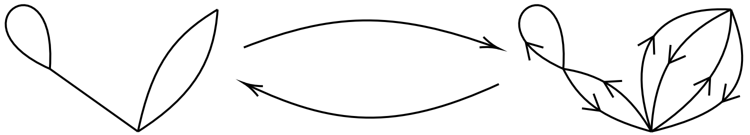
$$\begin{cases} r(r(e)) = e \\ s(r(e)) = t(e) \\ t(r(e)) = s(e). \end{cases}$$

Multigraphs as certain directed multigraphs

Proposition. The category of multigraphs is equivalent to the category of directed multigraphs with reversal and self-reverse loops, i.e., satisfying $r(\ell) = \ell$ for every loop $\ell \in E$:

$$\mathrm{Multi} \cong \mathrm{DiMultiRev}_{\mathrm{self}}.$$

Idea. Replace each undirected edge with a pair of arcs in both directions, keeping track of the pairing with the reversal $r\colon E \to E$. $\square$



Reflexive graphs

Reflexive directed multigraphs are diagrams of sets of shape

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V \arrow[l, "r"' description] \end{tikzcd}.$$

That is, presheaves on

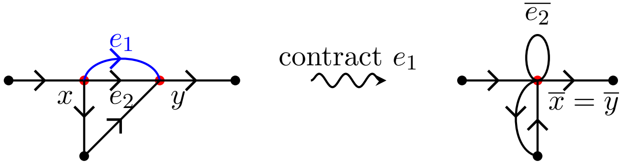
$$\begin{tikzcd} {[0]} \arrow[r, bend left, "d^1"] \arrow[r, bend right, "d^0"'] & {[1]} \arrow[l, "s^0" description] \end{tikzcd}$$ also known as $\Delta_{\leq 1}$

where $(d^1)^* = d_1 = s$, $(d^0)^* = d_0 = t$, and $(s^0)^* = s_0 = r$.

$$\begin{aligned} \mathrm{RefDiMulti} &\cong \mathrm{Fun}(\Delta_{\leq 1}^{\mathrm{op}}, \mathrm{Set}) \\ &= \text{1-truncated simplicial sets.} \end{aligned}$$

Edge contraction

Definition. Given a graph $G = (V, E)$, the **edge contraction** of $e \in E$ sends the endpoints of e to a single vertex and deletes e.



Proposition. The category of reflexive graphs is equivalent to the category of graphs with edge contractions allowed as morphisms.

Idea. Use the distinguished loops as dumpsters. □

Quasi-algebraic categories

The categories RefDiMulti and DiMulti are presheaf categories, in particular algebraic.

Remark. Presheaf categories are the algebraic categories that can be described using only unary operations.

Proposition. Multi, DiGraph, and Graph are quasi-algebraic categories.

Proof. We saw that they are implicational classes. □

Proposition. Multi, DiGraph and Graph are **not** algebraic categories. Equivalently, they are not exact in the sense of Barr.

Loopless graphs

Proposition. The four categories of loopless graphs (LplDiMulti, LplDiGraph, LplMulti and LplGraph) have no terminal object.

Corollary. The inclusion functor

$$\iota\colon \mathrm{LplDiMulti} \hookrightarrow \mathrm{DiMulti}$$

(and the other three) admits neither left adjoint nor right adjoint.

Proposition. The four categories LplDiMulti, LplDiGraph, LplMulti and LplGraph have all non-empty limits.

Idea. Check that they have non-empty products and equalizers. $\square$

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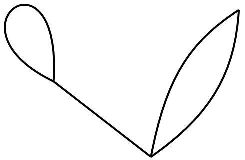
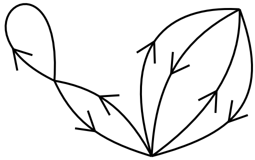
Directed versus undirected

Definition. The **directization** functor $D\colon \mathrm{Multi} \to \mathrm{DiMulti}$ is the composite

$$\mathrm{Multi} \xrightarrow{\cong} \mathrm{DiMultiRev}_{\mathrm{self}} \xrightarrow{\text{forget reversal}} \mathrm{DiMulti}.$$



$\overset{D}{\longmapsto}$



Proposition.

1. The directization functor $D\colon \mathrm{Multi} \to \mathrm{DiMulti}$ is right adjoint to the forgetful functor $U_d\colon \mathrm{DiMulti} \to \mathrm{Multi}$ that forgets the direction of the arcs.
2. The forgetful functor $U_d\colon \mathrm{DiMulti} \to \mathrm{Multi}$ admits no left adjoint.
3. The directization functor $D\colon \mathrm{Multi} \to \mathrm{DiMulti}$ admits no right adjoint.

Directed versus undirected, not multi

The situation without parallel edges is slightly different.

Definition. Given a directed graph $G = (V, E)$, the **symmetric part** of G is the subgraph SG with the same vertices V and keeping only the bidirected arcs of G:

$$(x, y) \in E(SG) \iff (x, y) \in E \text{ and } (y, x) \in E.$$

Via the isomorphism SymDiGraph $\cong$ Graph, we obtain the *symmetric part* functor

$$S\colon \text{DiGraph} \to \text{Graph}.$$

Directed versus undirected, not multi (cont'd)

Proposition. 1. The directization functor $D\colon \mathrm{Graph} \to \mathrm{DiGraph}$ is right adjoint to the forgetful functor $U_d\colon \mathrm{DiGraph} \to \mathrm{Graph}$ that forgets the direction of the arcs.

2. The forgetful functor $U_d\colon \mathrm{DiGraph} \to \mathrm{Graph}$ admits no left adjoint.

3. (Different!) The symmetric part functor $S\colon \mathrm{DiGraph} \to \mathrm{Graph}$ is right adjoint to the directization functor $D\colon \mathrm{Graph} \to \mathrm{DiGraph}$.

Reflexive or not

The forgetful functor

$$U_r \colon \mathrm{RefDiMulti} \to \mathrm{DiMulti}$$

that forgets the distinguished loops is the restriction along the inclusion

$$i \colon \Delta_{\mathrm{inj}, \leq 1} \hookrightarrow \Delta_{\leq 1}.$$

$\implies$ U_r has both adjoints, given by Kan extensions.

The left adjoint $L \colon \mathrm{DiMulti} \to \mathrm{RefDiMulti}$ adds a distinguished loop at each vertex.

Exercise. Compute the right adjoint $R \colon \mathrm{DiMulti} \to \mathrm{RefDiMulti}$.

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Induced adjunctions

Recall: Any adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

induces adjunctions on slice categories

$$\mathcal{C}/c \underset{\eta_c^* \circ G}{\overset{F}{\rightleftarrows}} \mathcal{D}/Fc \quad \text{and} \quad \mathcal{C}/Gd \underset{G}{\overset{(\epsilon_d)_! \circ F}{\rightleftarrows}} \mathcal{D}/d$$

$$\begin{tikzcd} \eta_c^* E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ c \arrow[r, "\eta_c"] & GFc \end{tikzcd} \qquad \begin{tikzcd} E \arrow[d] \arrow[dr] & \\ FGd \arrow[r, "\epsilon_d"] & d. \end{tikzcd}$$

Goal: Convince graph theorists to care about slice categories (and of course adjoint functors).

Graph colorings

Example. A graph homomorphism $G \to K_n$ is the same as an n-coloring of G, where

$$K_n = \text{complete (loopless) graph on } n \text{ vertices.}$$

$$\mathrm{LplGraph}/K_n = \{\text{graphs equipped with an } n\text{-coloring}\}.$$

Chromatic number of G:

$$\chi(G) = \min\{n \mid G \text{ admits an } n\text{-coloring}\}.$$

Cycles in graphs

A graph homomorphism $C_n \to G$ is an n-cycle in G, where

$$C_n = \text{cycle graph with } n \text{ vertices.}$$

$$C_n/\mathrm{LplGraph} = \{\text{graphs equipped with an } n\text{-cycle}\}.$$

Girth of G = length of a shortest cycle in G.

Thank you!