Modules over bialgebroids and Beck modules

Martin Frankland University of Regina

Joint with Raveen Tehara

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Outline

Beck modules

Representing ringoids

Tensor product of Beck modules

Bialgebras with many objects

Coefficients for cohomology

Cohomology theories for algebraic structures:

- Group cohomology
- Lie algebra cohomology
- Hochschild cohomology of associative algebras
- Barr–Beck triple cohomology
- André–Quillen cohomology of commutative rings

• etc.

Need a good notion of *coefficient module* M over X:

 $H^*(X;M).$

Beck modules

Definition (Beck 1967). For an object X in a category C, a **Beck** module over X is an abelian group object in the slice category C/X.

Denote the category of Beck modules over X

 $\operatorname{Mod}(X) \coloneqq (\mathcal{C}/X)_{\operatorname{ab}}.$

Pullback and pushforward

Definition. The pullback functor $f^* \colon \mathcal{C}/Y \to \mathcal{C}/X$ induces a functor

 $f^* \colon \operatorname{Mod}(Y) \to \operatorname{Mod}(X)$

also called the **pullback**.

Its left adjoint

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f_! \colon \operatorname{Mod}(X) \to \operatorname{Mod}(Y)
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is called the **pushforward** along f.

pullback = "restriction of scalars"
pushforward = "extension of scalars"

Fibered category

The assignment

$$\operatorname{Mod}(-) \colon \mathcal{C}^{\operatorname{op}} \to \operatorname{AbCat}$$

sending an object X to its category of Beck modules Mod(X) and a map $f: X \to Y$ to the pullback functor $f^*: Mod(Y) \to Mod(X)$ is a pseudo-functor: $(gf)^* \cong f^*g^*$.

Definition. The Grothendieck construction of the pseudo-functor Mod(-) yields a fibered category

$$\pi\colon T\mathcal{C}\to \mathcal{C}$$

called the **tangent category** of C.

An object of TC is (X, M), where M is a module over X.

Rings

 $\mathcal{C} = \mathrm{Alg}_{\Bbbk},$ the category of (unital) $\Bbbk\text{-algebras}.$

For a k-algebra A:

 $Mod(A) \cong {}_ABimod_A.$

A Beck module over A is a split extension of A with square zero kernel:

$$0 \longrightarrow M \longrightarrow A \oplus M \xrightarrow[]{p} A \longrightarrow 0.$$

The two actions on M are given by

$$(a,m)(a',m')=(aa',a\cdot m'+m\cdot a')$$

and they coincide for scalars in \Bbbk .

Rings, cont'd

For a map of k-algebras $f \colon A \to B$, the pullback functor

 $f^*\colon \operatorname{Mod}(B) \to \operatorname{Mod}(A)$

is the usual restriction of scalars:

$$a \cdot m \cdot a' \coloneqq f(a) \cdot m \cdot f(a').$$

The pushforward functor is

$$f_! \colon \operatorname{Mod}(A) \to \operatorname{Mod}(B)$$
$$f_!(M) = B \otimes_A M \otimes_A B.$$

Commutative rings

 $\mathcal{C} = \operatorname{Com}_{\Bbbk}$, the category of commutative \Bbbk -algebras.

For a commutative k-algebra A:

 $Mod(A) \cong Mod_A$ in the usual sense.

Same correspondence as for algebras, except that $A \oplus M$ must be commutative. This forces the two actions to coincide:

 $a \cdot m = m \cdot a.$

For a map of commutative k-algebras $f: A \to B$, the pushforward functor is extension of scalars:

 $f_! \colon \operatorname{Mod}(A) \to \operatorname{Mod}(B)$ $f_!(M) = B \otimes_A M.$

Groups

 $\mathcal{C} = \mathrm{Gp}$, the category of groups.

For a group G:

$$\operatorname{Mod}(G) \cong G$$
-modules in the usual sense
 $\cong \mathbb{Z}G$ -Mod.

A Beck module over G is a split extension of G with abelian kernel:

$$1 \longrightarrow K \longrightarrow G \ltimes K \xrightarrow[e]{p} G \longrightarrow 1.$$

The G-action on K is given by $e(g)k = (g, g \cdot k)$. In other words:

$$(g,k)(g',k') = (gg',k+g\cdot k').$$

Groups, cont'd

For a map of groups $f: G \to H$, the pushforward functor is

 $f_! \colon \operatorname{Mod}(G) \to \operatorname{Mod}(H)$ $f_!(M) = \mathbb{Z}H \otimes_{\mathbb{Z}G} M.$

Abelian groups

 $\mathcal{C} = Ab$, the category of abelian groups.

For an abelian group A:

 $Mod(A) \cong Ab.$

Same correspondence as for groups, except that $A \ltimes K$ must be abelian. This forces the A-action on K to be trivial:

$$a \cdot k = k.$$

More generally:

Example (Beck 1967). In an additive category \mathcal{A} with finite limits, Beck modules over any object X are:

$$\operatorname{Mod}(X) \cong \mathcal{A}$$

 $(p \colon E \twoheadrightarrow X) \mapsto \ker(p).$

Lie algebras

 $\mathbb{k} = a$ field of characteristic not 2.

 $\mathcal{C} = \operatorname{Lie}_{\Bbbk}$, the category of Lie algebras over \Bbbk .

For a Lie algebra L:

 $\operatorname{Mod}(L) \cong L$ -modules in the usual sense $\cong U(L)$ -Mod

where U(L) is the universal enveloping algebra of L.

A Beck module over L is a split extension of L with abelian kernel:

$$0 \longrightarrow M \longrightarrow L \oplus M \xrightarrow[s]{p} L \longrightarrow 0$$

with [m, m'] = 0 for all $m, m' \in M$.

Lie algebras (cont'd)

The action of L on M is given by

$$[(\ell, 0), (0, m)] = (0, \ell \cdot m),$$

which satisfies

$$[\ell,\ell']\cdot m = \ell \cdot (\ell' \cdot m) - \ell' \cdot (\ell \cdot m).$$

Commutative monoids

 $\mathcal{C} = CMon$, the category of commutative monoids.

A Beck module M over a commutative monoid A consists of a family of abelian groups $\{M_a\}_{a \in A}$ and for each $a, b \in A$, a map of abelian groups

$$b \cdot (-) \colon M_a \to M_{ba}$$

subject to $1 \cdot m = m$ and $(ab) \cdot m = a \cdot (b \cdot m)$.

Equivalently, a graded module M over the A-graded monoid ring $\mathbb{Z}A.$

Equivalenty, a functor $M: H(A) \to Ab$ from the *Leech category* of A (a.k.a. action category), whose objects are $a \in A$ and morphisms are pairs (b, a):

$$a \xrightarrow{(b,a)} ba.$$

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Ringoids

Reference: Mitchell, Rings with several objects, 1972.

- **Definition.** 1. A **ringoid** is a small pre-additive category, i.e., a small category enriched in abelian groups.
 - 2. A morphism of ringoids is an Ab-enriched functor $F: \mathcal{R} \to \mathcal{S}$, a.k.a. an *additive functor*.
 - 3. For a ringoid \mathcal{R} , a left \mathcal{R} -module is a covariant additive functor from \mathcal{R} to abelian groups

$$M \colon \mathcal{R} \to \mathrm{Ab}.$$

4. A right \mathcal{R} -module is a contravariant additive functor from \mathcal{R} to abelian groups

$$M: \mathcal{R}^{\mathrm{op}} \to \mathrm{Ab}.$$

Example. A one-object ringoid is a ring. The notions of modules recover the usual ones.

Modules over ringoids

Modules over a ringoid \mathcal{R} form an abelian category. Which abelian categories are of that form?

- **Theorem** (Freyd 1964). 1. An abelian category \mathcal{A} is equivalent to a category of modules over a ring if and only if \mathcal{A} is cocomplete and has a small projective generator P. In that case $\mathcal{A} \cong \text{Mod-End}(P)$.
 - 2. \mathcal{A} is equivalent to a category of modules over a ringoid if and only if \mathcal{A} is cocomplete and has a set S of small projective generators.

Given such a set S, take as ringoid the full subcategory on S.

The case of Beck modules

Proposition. Let C be an algebraic category (a.k.a. "equational class"). For every object X in C, the category of Beck modules Mod(X) is equivalent to modules over some ringoid.

Representing ringoid

Some examples of representing ringoids:

- 1. For a ring A, the enveloping ring $A \otimes A^{\text{op}}$.
- 2. For a commutative ring R, the ring R itself.
- 3. For a group G, the group ring $\mathbb{Z}G$.
- 4. For an abelian group A, the ring \mathbb{Z} .
- 5. For a Lie algebra L, the universal enveloping algebra U(L).
- 6. For a commutative monoid A, the \mathbb{Z} -linearization of the Leech category H(A).

Beck modules

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Bialgebras with many objects

Common tensor products

We know how to tensor modules over a commutative ring R:

 $M \otimes_R N.$

We also know how to tensor bimodules over a ring A:

 $M \otimes_A N.$

Goal

Find a broad enough framework where Beck modules have a natural tensor product, recovering many known examples.

Idea: Look for comultiplicative structure in the representing ringoids.

Bialgebras

Example. The comultiplication of the group ring $\mathbb{Z}G$

 $\Delta \colon \mathbb{Z}G \to \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G$ $\Delta(g) = g \otimes g$

induces a G-action on the tensor product of G-modules $M \otimes_{\mathbb{Z}} N$:

$$g \cdot (m \otimes n) = (g \cdot m) \otimes (g \cdot n).$$

Example. Likewise for the universal enveloping algebra of a Lie algebra *L*:

$$\Delta \colon U(L) \to U(L) \otimes_{\Bbbk} U(L)$$
$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

induces an action of L on the tensor product of L-modules $M \otimes_{\Bbbk} N$:

$$\ell \cdot (m \otimes n) = (\ell \cdot m) \otimes n + m \otimes (\ell \cdot n).$$

Bialgebras (cont'd)

More generally, the comultiplication on a bialgebra B induces a B-module structure on the tensor product of B-modules $M \otimes_{\Bbbk} N$:

$$B \otimes_{\Bbbk} M \otimes_{\Bbbk} N \xrightarrow{\Delta \otimes \operatorname{id}} B \otimes_{\Bbbk} B \otimes_{\Bbbk} M \otimes_{\Bbbk} N \xrightarrow{\cong} B \otimes_{\Bbbk} M \otimes_{\Bbbk} B \otimes_{\Bbbk} N \xrightarrow{} \lambda_{M \otimes N} \lambda_{M} \otimes_{\lambda_{N}} M \otimes_{\Bbbk} N.$$

Pointwise tensor product

For a group G:

G-Mod \cong Fun(BG, Ab)

where BG denotes G viewed as a one-object groupoid.

The tensor product of G-modules agrees with the pointwise tensor product in Fun(BG, Ab).

Definition. A category C has representable Beck modules (over \mathbb{Z}) if for all object X in C:

 $Mod(X) \cong Fun(J_X, Ab)$

for some small category J_X , pseudo-functorial in X.



Pointwise tensor product (cont'd)

Example. The following categories have representable Beck modules.

- 1. Sets: $J_X = X$ viewed as a discrete category.
- 2. Groups: $J_G = BG$, the one-object groupoid.
- 3. Abelian groups: $J_A = *$, the trivial category.
- 4. Commutative monoids: $J_A = H(A)$, the Leech category of A.
- 5. Monoids: J_M = twisted arrow category of M.

Example. The following categories do not have representable Beck modules.

- 1. Rings.
- 2. Commutative rings.
- 3. Lie algebras over k.

\mathbb{Z} -linearization

Definition. Given a small category C, the \mathbb{Z} -linearization of C is the \mathbb{Z} -linear (i.e. Ab-enriched) category $\mathbb{Z}C$ obtained by applying the free abelian group functor to the hom-sets in C:

$$(\mathbb{ZC})(X,Y) \coloneqq \mathbb{Z}(\mathcal{C}(X,Y)).$$

Composition in $\mathbb{Z}\mathcal{C}$ is defined as the bilinear extension of composition in \mathcal{C} .

Lemma. 1. \mathbb{Z} -linearization is a (strict) 2-functor Cat $\rightarrow \mathbb{Z}$ -Cat. 2. Universal property:

$$\operatorname{Fun}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C},\mathcal{A})\cong\operatorname{Fun}(\mathcal{C},\mathcal{A}).$$

3. Z-linearization sends the Cartesian product of categories to the tensor product of ringoids:

$$\mathbb{Z}(\mathcal{C} \times \mathcal{D}) \cong \mathbb{Z}\mathcal{C} \otimes \mathbb{Z}\mathcal{D}.$$

Comultiplicative structure

For any small category \mathcal{C} , the diagonal functor and the constant functor

$$\Delta \colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$

$$\epsilon \colon \mathcal{C} \to *$$

make ${\mathcal C}$ into a comonoid in Cat.

Applying the \mathbb{Z} -linearization produces ringoid maps

$$\Delta \colon \mathbb{Z}(\mathcal{C}) \to \mathbb{Z}(\mathcal{C} \times \mathcal{C}) \cong \mathbb{Z}\mathcal{C} \otimes \mathbb{Z}\mathcal{C}$$

$$\epsilon \colon \mathbb{Z}(\mathcal{C}) \to \mathbb{Z}(*) \cong \mathbb{Z}$$

making $\mathbb{Z}\mathcal{C}$ into a comonoid in ringoids.

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Comonoids in ringoids

Not all Beck modules are represented by a bialgebra. Look for a "many objects" generalization, but in which direction?

bialgebra = monoid in coalgebras = comonoid in algebras

Generalizing rings to ringoids yields one notion of "bialgebra with many objects": **comonoids in ringoids**.

The \mathbb{Z} -linearization $\mathbb{Z}\mathcal{C}$ is an example.

This notion was studied in the literature (Day, Street 1997).

Commutative bialgebroids

In stable homotopy theory, we often stumble upon:

Dropping the antipode yields:

Bialgebroids

Introduced by Takeuchi (1977). Comprehensive reference: Böhm (2009).

- A bialgebroid consists of:
 - \bullet algebras A and H
 - $\bullet\,$ algebra maps $A \to H$ and $A^{\mathrm{op}} \to H$
 - a bimodule map $H \to H \otimes_A H$
 - \bullet a character $H \to A$
 - + more structure and properties.

Salient feature: Both modules and comodules over a bialgebroid have a tensor product.

Example: enveloping ring

Example. Given a k-algebra A, its enveloping algebra $A \otimes_{k} A^{\text{op}}$ admits a canonical bialgebroid structure.

The induced tensor product of $A \otimes_{\Bbbk} A^{\text{op}}$ -modules recovers the usual tensor product of A-bimodules.

Question. Are there many instances where Beck modules are represented by a bialgebroid?

Thank you!