Enriched model categories and the Dold-Kan correspondence

Martin Frankland
University of Regina

Joint with Arnaud Ngopnang Ngompé

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Outline

Dold–Kan correspondence

Monoidal Quillen pairs

Change of enrichment

Normalization

Let R be a commutative ring. Given a simplicial R-module A, we can form its normalized chain complex $N(A)$.

Theorem (Dold–Kan correspondence)**.** The normalization functor

$$N\colon s\mathrm{Mod}_R \to \mathrm{Ch}_{\geq 0}(R)$$

is an equivalence of categories, with inverse equivalence the denormalization

$$\Gamma\colon \mathrm{Ch}_{\geq 0}(R) \to s\mathrm{Mod}_R.$$

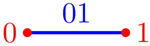
Examples

Example. For a constant simplicial R-modules $c(A)$, the normalization is

$$N(c(A)) = A[0] = (\cdots \to 0 \to 0 \to A).$$

Example. For the free abelian group $\mathbb{Z}\Delta^1$, the unnormalized chain complex is

$$\begin{array}{llllllll} C(\mathbb{Z}\Delta^1) = & \cdots & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2. \\ \text{Bases} & & & 000, 001, 011, 111 & & 00, 01, 11 & & 0, 1 \end{array}$$



The normalization is

$$N(\mathbb{Z}\Delta^1) \cong C_*^{\Delta}(\Delta^1_{\mathrm{sc}}) = (\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}^2)$$

the simplicial chain complex of the simplicial complex $\Delta^1_{\mathrm{sc}} = \mathcal{P}([1])$.

Example. More generally: $N(\mathbb{Z}\Delta^n) \cong C_*^{\Delta}(\Delta^n_{\mathrm{sc}})$.

Homotopical properties

Proposition.

1. Two maps of simplicial R-modules $f, g\colon A \to B$ are homotopic if and only if their normalizations $Nf, Ng\colon N(A) \to N(B)$ are chain homotopic.
2. The homotopy groups of a simplicial R-module A correspond to the homology groups of its normalization: $\pi_n(A) \cong H_n(NA)$.

Proposition. Both adjunctions $N \dashv \Gamma$ and $\Gamma \dashv N$ are Quillen equivalences.

Monoidal properties

Proposition. 1. Normalization is lax monoidal via the Eilenberg–Zilber map (also called *shuffle map*)

$$\mathrm{EZ}\colon N(A) \otimes N(B) \to N(A \otimes B).$$

2. Normalization is oplax monoidal via the Alexander–Whitney map

$$\mathrm{AW}\colon N(A \otimes B) \to N(A) \otimes N(B).$$

3. The composite

$$N(A) \otimes N(B) \xrightarrow{\mathrm{EZ}} N(A \otimes B) \xrightarrow{\mathrm{AW}} N(A) \otimes N(B)$$

is the identity, and the composite $\mathrm{EZ} \circ \mathrm{AW}$ is naturally chain homotopic to the identity.

$\Longrightarrow$ The chain complex $N(A) \otimes N(B)$ is a deformation retract of $N(A \otimes B)$.

Eilenberg–Zilber theorem

Example. For topological spaces X and Y, the Eilenberg–Zilber map between singular chain complexes

$$\mathrm{EZ}\colon C_*(X) \otimes C_*(Y) \xrightarrow{\simeq} C_*(X \times Y)$$

is a chain homotopy equivalence.

Remark. Moreover EZ satisfies associativity.

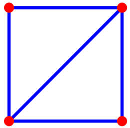
Not strong monoidal

Warning! Neither N nor Γ is strong monoidal.

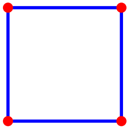
Example. $N(\mathbb{Z}\Delta^1 \otimes \mathbb{Z}\Delta^1) \ncong N(\mathbb{Z}\Delta^1) \otimes N(\mathbb{Z}\Delta^1)$.

$$N(\mathbb{Z}\Delta^1 \otimes \mathbb{Z}\Delta^1) = \qquad \cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^5 \longrightarrow \mathbb{Z}^4$$

$$N(\mathbb{Z}\Delta^1) \otimes N(\mathbb{Z}\Delta^1) = \qquad \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^4 \longrightarrow \mathbb{Z}^4.$$



versus



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Strong monoidal

Definition. For $\mathcal{V}$ and $\mathcal{W}$ two monoidal model categories, a **strong monoidal Quillen adjunction** is a Quillen adjunction

$$F\colon \mathcal{W} \rightleftarrows \mathcal{V}\colon G$$

such that the left adjoint F is strong monoidal and the map

$$F(q)\colon F(Q\mathbb{1}_{\mathcal{W}}) \xrightarrow{\sim} F(\mathbb{1}_{\mathcal{W}})$$

is a weak equivalence, where $q\colon Q\mathbb{1}_{\mathcal{W}} \xrightarrow{\sim} \mathbb{1}_{\mathcal{W}}$ is a cofibrant replacement of the tensor unit.

Note: The right adjoint G is lax monoidal.

Example

Example. The geometric realization / singular set adjunction

$$|\cdot|\colon s\mathrm{Set} \rightleftarrows \mathrm{Top}\colon \mathrm{Sing}$$

is a strong monoidal Quillen equivalence.

$$|X \times Y| \cong |X| \times |Y|$$

Note: Every simplicial set is cofibrant, in particular the tensor unit $\mathbb{1}_{s\mathrm{Set}} = * = \Delta^0$.

Non-Example. In Dold–Kan, neither N nor Γ is strong monoidal, but they are "strong monoidal up to homotopy"...

Weak monoidal

Definition. For $\mathcal{V}$ and $\mathcal{W}$ two monoidal model categories, a **weak monoidal Quillen adjunction** is a Quillen adjunction

$$F\colon \mathcal{W} \rightleftarrows \mathcal{V}\colon G$$

such that the right adjoint G is lax monoidal and the induced oplax monoidal structure on the left adjoint F satisfies:

(i) For all cofibrant objects $x, y \in \mathcal{W}$ the map

$$\delta_{x,y}\colon F(x \otimes y) \xrightarrow{\sim} F(x) \otimes F(y)$$

is a weak equivalence.

(ii) The composition $F(Q\mathbb{1}_{\mathcal{W}}) \xrightarrow{F(q)} F(\mathbb{1}_{\mathcal{W}}) \xrightarrow{\varepsilon} \mathbb{1}_{\mathcal{V}}$ is a weak equivalence.

Note: F is strong monoidal if the maps $\delta_{x,y}$ and ε are isomorphisms.

Example

Example. For R a commutative ring, both adjunctions

$$N\colon s\mathrm{Mod}_R \rightleftarrows \mathrm{Ch}_{\geq 0}(R)\colon \Gamma$$

and

$$\Gamma\colon \mathrm{Ch}_{\geq 0}(R) \rightleftarrows s\mathrm{Mod}_R\colon N$$

are weak monoidal Quillen equivalences.

Homotopy theory of monoids

Theorem (Schwede–Shipley 2003)**.** For nice monoidal model categories $\mathcal{V}$ and $\mathcal{W}$, a weak monoidal Quillen equivalence $F\colon \mathcal{W} \rightleftarrows \mathcal{V}\colon G$ induces a Quillen equivalence between categories of monoids

$$F^{\mathrm{mon}}\colon \mathrm{Mon}(\mathcal{W}) \rightleftarrows \mathrm{Mon}(\mathcal{V})\colon G.$$

Warning! The right adjoint is the functor induced by $G\colon \mathcal{V} \to \mathcal{W}$ on monoids. However, the left adjoint F^{mon} is **not** given by applying F in general.

Corollary. For R a commutative ring, Dold–Kan induces a Quillen equivalence

$$s\mathrm{Alg}_R \underset{\sim}{\rightleftarrows} \mathrm{DGA}_{R,\geq 0}.$$

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Change of enrichment

Let $G\colon \mathcal{V} \to \mathcal{W}$ be a lax monoidal functor between monoidal categories. Given any $\mathcal{V}$-enriched category $\mathcal{C}$, there is an associated $\mathcal{W}$-enriched category $G_*\mathcal{C}$ with the same objects, and hom-objects given by

$$\underline{G_*\mathcal{C}}(x, y) := G(\underline{\mathcal{C}}(x, y)).$$

This construction is called the **change of base** (or **change of enrichment**) along G, which forms a 2-functor

$$G_*\colon \mathcal{V}\text{-Cat} \to \mathcal{W}\text{-Cat}.$$

Underlying category

Lemma. For any monoidal category $\mathcal{V}$, the underlying set functor $U \colon \mathcal{V} \to \mathrm{Set}$

$$U(v) = \mathrm{Hom}_{\mathcal{V}}(\mathbb{1}, v)$$

is lax monoidal.

Definition. Given a $\mathcal{V}$-enriched category $\mathcal{C}$, the category $U\mathcal{C} \coloneqq U_*\mathcal{C}$ is called the **underlying category** of $\mathcal{C}$.

Preserving underlying categories

Lemma. Let $F\colon \mathcal{W} \rightleftarrows \mathcal{V}\colon G$ be an adjunction where the right adjoint G is lax monoidal. The following are equivalent.

1. The map $\varepsilon\colon F(\mathbb{1}_{\mathcal{W}}) \xrightarrow{\cong} \mathbb{1}_{\mathcal{V}}$ is an isomorphism.
2. G commutes with the underlying set functors:

$$\begin{tikzcd} \mathcal{V} \arrow[r, "G"] \arrow[dr, "\mathcal{V}(\mathbb{1}_{\mathcal{V}},-)=U"'] & \mathcal{W} \arrow[d, "\mathcal{W}(\mathbb{1}_{\mathcal{W}},-)=U"] \\ & \mathrm{Set}. \end{tikzcd}$$

3. Change of enrichment along $G\colon \mathcal{V} \to \mathcal{W}$ preserves the underlying category for every $\mathcal{V}$-enriched category $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{V}\text{-Cat} \arrow[r, "G_*"] \arrow[dr, "U_*"'] & \mathcal{W}\text{-Cat} \arrow[d, "U_*"] \\ & \mathrm{Cat}. \end{tikzcd}$$

Examples

Example. In Dold–Kan, both N and Γ preserve underlying sets:

$$U(N(A)) = Z_0(N(A)) = A_0 = U(A) \quad \text{for all } A \in s\mathrm{Mod}_R$$
$$U(\Gamma(C)) = \Gamma(C)_0 = C_0 = U(C) \quad \text{for all } C \in \mathrm{Ch}_{\geq 0}(R).$$

Example. Geometric realization $|\cdot|\colon s\mathrm{Set} \to \mathrm{Top}$ does **not** preserve underlying sets. For instance:

$$U(\Delta^1) = \Delta^1_0 = \{0, 1\}$$

versus

$$U(|\Delta^1|) = U(\Delta^1_{\mathrm{top}}) \cong [0, 1].$$

Enriched model categories

Definition. Let $\mathcal{V}$ be a symmetric monoidal model category. A **$\mathcal{V}$-model category** is a $\mathcal{V}$-enriched category $\mathcal{C}$ which is tensored and cotensored over $\mathcal{V}$, with a model structure on the underlying category $U\mathcal{C}$ satisfying:

(i) **SM7**: For a cofibration $i\colon a \to b$ and a fibration $p\colon x \to y$ in $\mathcal{C}$, their pullback-power

$$(i^*, p_*)\colon \underline{\mathcal{C}}(b, x) \to \underline{\mathcal{C}}(a, x) \times_{\underline{\mathcal{C}}(a,y)} \underline{\mathcal{C}}(b, y)$$

is a fibration in $\mathcal{V}$, which moreover is acyclic if i or p is.

(ii) **External unit axiom:** For every cofibrant object x in $\mathcal{C}$ the morphism $x \otimes Q\mathbb{1} \xrightarrow{x \otimes q} x \otimes \mathbb{1} \xrightarrow{\cong} x$ is a weak equivalence.

Example. A sSet-model category is a simplicial model category as introduced by Quillen (1967).

Unit axiom

Proposition. Let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category satisfying SM7 and tensored over $\mathcal{V}$. The following conditions are equivalent.

1. π_0 **of mapping space**: For any cofibrant $x \in \mathcal{C}$ and fibrant $y \in \mathcal{C}$, the canonical map $[x, y] \to [\mathbb{1}, \underline{\mathcal{C}}(x, y)]$ is a bijection.
2. **Detection property**: If a map $f\colon x \to y$ between cofibrant objects in $\mathcal{C}$ is such that the restriction map $f^*\colon \underline{\mathcal{C}}(y, z) \xrightarrow{\sim} \underline{\mathcal{C}}(x, z)$ is a weak equivalence in $\mathcal{V}$ for all fibrant object $z \in \mathcal{C}$, then f is a weak equivalence in $\mathcal{C}$.
3. **External unit axiom**: For any cofibrant object $x \in \mathcal{C}$, the composite $x \otimes Q\mathbb{1} \xrightarrow{x \otimes q} x \otimes \mathbb{1} \xrightarrow{\rho_x} x$ is a weak equivalence in $\mathcal{C}$.

When $\mathcal{C}$ is *weakly* tensored over $\mathcal{V}$, the implications (1) $\implies$ (2) $\implies$ (3) still hold.

Change of base theorem

Change of base theorem for enriched model categories:

Theorem. Let $F\colon \mathcal{W} \rightleftarrows \mathcal{V} \colon G$ be a strong monoidal Quillen adjunction between symmetric monoidal model categories. If $\mathcal{C}$ is a $\mathcal{V}$-model category, then $G_*\mathcal{C}$ is a $\mathcal{W}$-model category.

See Dugger (2006), Riehl (2014), Guillou–May (2020).

What about weak monoidal?

Proposition. Let $\mathcal{V}$ and $\mathcal{W}$ be closed symmetric monoidal categories and $F\colon \mathcal{W} \rightleftarrows \mathcal{V}\colon G$ an adjunction such that the right adjoint G is lax monoidal.

1. If F is strong monoidal, then for a $\mathcal{V}$-category $\mathcal{C}$, the $\mathcal{W}$-category $G_*\mathcal{C}$ admits a tensoring or a cotensoring over $\mathcal{W}$ given respectively by:

   $$x \otimes w := x \otimes Fw \quad \text{and} \quad x^w := x^{Fw}, \quad \text{for any } x \in \mathcal{C} \text{ and } w \in \mathcal{W}.$$

2. If $\varepsilon\colon F(\mathbb{1}_{\mathcal{W}}) \xrightarrow{\cong} \mathbb{1}_{\mathcal{V}}$ is an isomorphism and the $\mathcal{W}$-category $G_*\mathcal{V}$ admits a tensoring or a cotensoring over $\mathcal{W}$, then F is strong monoidal.

Upshot: Changing the enrichment along Dold–Kan definitely loses the tensoring and cotensoring! But everything else is fine...

The ingredients

Changing the enrichment of a $\mathcal{V}$-model category $\mathcal{C}$ along a weak monoidal Quillen adjunction $F\colon \mathcal{W} \rightleftarrows \mathcal{V}\colon G$ where $F(\mathbb{1}_{\mathcal{W}}) \cong \mathbb{1}_{\mathcal{V}}$.

In $\mathcal{C}$	In $G_*\mathcal{C}$
Enrichment	✓ G lax monoidal
Model structure	✓ $U(G_*\mathcal{C}) = U\mathcal{C}$
SM7	✓
Tensoring and cotensoring	Weakened!
Unit axiom π_0 of mapping space	✓

Weak $\mathcal{V}$-model category

Definition. Let $\mathcal{V}$ be a closed symmetric monoidal model category. A **weak $\mathcal{V}$-model category** is a $\mathcal{V}$-enriched model category $\mathcal{C}$ weakly tensored and cotensored over $\mathcal{V}$, satisfying SM7 and the π_0 of mapping space axiom.

Proposition. Let $\mathcal{C}$ be a weak $\mathcal{V}$-model category. Then the homotopy category $\mathrm{Ho}(\mathcal{C})$ is canonically enriched, tensored, and cotensored over $\mathrm{Ho}(\mathcal{V})$.

Theorem (F.–Ngopnang)**.** Let $F\colon \mathcal{W} \rightleftarrows \mathcal{V} \colon G$ be a weak monoidal Quillen pair such that the map $\varepsilon\colon F(\mathbb{1}_{\mathcal{W}}) \xrightarrow{\cong} \mathbb{1}_{\mathcal{V}}$ is an isomorphism. If $\mathcal{C}$ is a weak $\mathcal{V}$-model category, then $G_*\mathcal{C}$ is a weak $\mathcal{W}$-model category.

Thank you!