Multiparameter persistence modules in the large scale

Martin Frankland
University of Regina

Joint with Don Stanley

Session on Applied Topology
CMS Summer Meeting
University of Saskatchewan
June 1, 2024

Outline

Persistence modules

Localized persistence modules

Classification of indecomposables

Which subcategories are we quotienting out?

Rank invariant

Topological data analysis pipeline

$$\begin{array}{c}\text{filtered space / simplicial complex}\\ \Big\downarrow{\scriptstyle H_r(-;\mathbb{k})}\\ \text{filtered } \mathbb{k}\text{-vector space}\end{array}$$

Example. Let X be a finite metric space — "data set". The **Vietoris–Rips complex** $VR(X)_\epsilon$ is the simplicial complex on the vertex set X with

$$\{x_0, \ldots, x_n\} \text{ is an } n\text{-simplex} \iff d(x_i, x_j) \leq \epsilon \text{ for all } i, j.$$

Filtered simplicial complex

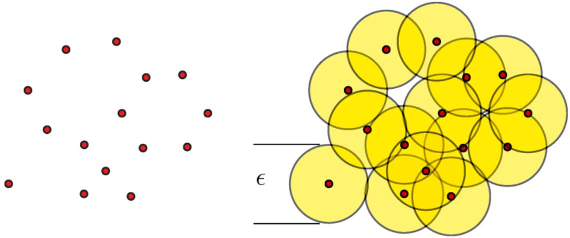


Image source: Robert Ghrist, *Barcodes: The persistent topology of data.*

As ϵ varies, $VR(X)_\epsilon$ forms a filtered simplicial complex with one parameter $\epsilon \geq 0$, i.e., a functor

$$VR(X) \colon \mathbb{R}_+ \to \text{SimpCpx}.$$

If instead we let ϵ increase by a fixed small step, we obtain one *discrete* parameter $\mathbb{N} \to \text{SimpCpx}$.

Filtered space

Example. Let X be a smooth manifold and $f\colon X \to \mathbb{R}$ a Morse function. Filtration by sublevel sets:

$$X_s = \{x \in X \mid f(x) \leq s\} = f^{-1}((-\infty, s]).$$

As s varies, X_s forms a filtered space with one parameter $s \in \mathbb{R}$, i.e., a functor

$$X_\bullet\colon \mathbb{R} \to \mathrm{Top}.$$

Given another Morse function $g\colon X \to \mathbb{R}$, consider the joint sublevel sets:

$$X_{s,t} = \{x \in X \mid f(x) \leq s,\ g(x) \leq t\}.$$

Get a filtered space with two parameters $s, t \in \mathbb{R}$, i.e., a functor $X_{\bullet,\bullet}\colon \mathbb{R}^2 \to \mathrm{Top}$.

Multiple parameters

In applications, often need *multiple* parameters.

Good survey: M.B. Botnan and M. Lesnick, *An introduction to multiparameter persistence* (2023).

In this project, we focus on *discrete* parameters.

Persistence modules

Fix a ground field $\Bbbk$.

Definition. For $m \geq 1$, an ***m*-parameter persistence module** is a diagram

$$\mathbb{N}^m \to \mathrm{Vect}_{\Bbbk}.$$

$\cong$ graded module over the graded polynomial algebra

$$R \coloneqq \Bbbk[t_1, \ldots, t_m],$$

which is $\mathbb{N}^m$-graded with multigrading

$$|t_i| = \vec{e}_i = (0, \ldots, \overbrace{1}^{i}, \ldots, 0).$$

Write $M(\vec{d})$ for the $\Bbbk$-vector space in multidegree $\vec{d} \in \mathbb{N}^m$.

Goal / Dream

Work with *finitely generated* R-modules:

$$R\text{-mod} \coloneqq R\text{-Mod}^{\text{fin.gen.}} \subset R\text{-Mod}.$$

Goal

Classify the indecomposable objects in R-mod.

One-parameter case

For $m = 1$, finitely generated $\Bbbk[t]$-modules decompose into **interval modules**

$$\begin{aligned}[a, b) &:= t^a \Bbbk[t]/t^b \Bbbk[t] \\ &= \operatorname{coker}\left(t^a \Bbbk[t] \xrightarrow{t^{b-a}} t^a \Bbbk[t]\right).\end{aligned}$$

Example.

$$\begin{aligned}M &= t^4 \Bbbk[t] \oplus t^2 \Bbbk[t]/t^7 \Bbbk[t] \\ &= [4, \infty) \oplus [2, 7).\end{aligned}$$

$\rightsquigarrow$ **Barcode**: multiset of intervals. List of intervals appearing in the decomposition (with multiplicity).

Multiparameter case

Not available for $m \geq 2$, because $\Bbbk[t_1, t_2]$ has wild representation type.

How to deal with that?

One approach: Extract invariants that are both computable and significant. Rank invariant and various refinements. Many authors...

Another approach: Focus on certain families of modules admitting a nice decomposition, such as rectangle-decomposable modules.

Approach: localize

Our approach: We localize R-mod until the resulting category admits a classification of indecomposables, or at least a partial classification.

Related work: [Harrington, Otter, Schenck, Tillmann] and [Bauer, Botnan, Oppermann, Steen].

Outline

Persistence modules

Localized persistence modules

Classification of indecomposables

Which subcategories are we quotienting out?

Rank invariant

Inverting some variables

Fact. The homogeneous prime ideals of R are those of the form $(t_{i_1}, \cdots, t_{i_k})$.

The various localizations of a module M fit together.

Example. For M a module over $R = \Bbbk[t_1, t_2]$:

$$\begin{array}{ccc} \Bbbk[t_1, t_2^{\pm}] \otimes_R M & \xrightarrow{\text{invert } t_1} & \Bbbk[t_1^{\pm}, t_2^{\pm}] \otimes_R M \\ {\scriptstyle \text{invert } t_2}\uparrow & & \uparrow{\scriptstyle \text{invert } t_2} \\ M & \xrightarrow{\text{invert } t_1} & \Bbbk[t_1^{\pm}, t_2] \otimes_R M. \end{array}$$

Inverting some variables (cont'd)

Notation. 1. $[m] \coloneqq \{1, 2, \ldots, m\}$

2. For a subset $\sigma \subseteq [m]$, denote the localization of rings

$$R_\sigma \coloneqq R[t_i^{-1} \mid i \in \sigma],$$

which is $\sigma^{-1}\mathbb{N}^m$-graded.

3. $\varphi_i \coloneqq$ the localization map "invert t_i".

Example. With this notation, the previous square becomes:

$$\begin{array}{ccc} R_{\{2\}} \otimes_R M & \xrightarrow{\varphi_1} & R_{\{1,2\}} \otimes_R M \\ \varphi_2 \uparrow & & \uparrow \varphi_2 \\ M & \xrightarrow{\varphi_1} & R_{\{1\}} \otimes_R M. \end{array}$$

K-localized persistence modules

Idea: Forget the module M but keep some of its localizations.

If we keep a localization $R_\sigma \otimes_R M$, we should keep all further localizations $R_\tau \otimes_R M$ for $\sigma \subseteq \tau$.

Definition. Let K be a simplicial complex on the vertex set $[m]$. A ***K*-localized persistence module** M consists of:

1. For each missing face $\sigma \notin K$, a finitely generated R_σ-module M_σ.
2. For each $\sigma \subseteq \tau$ with $\sigma \notin K$ (and hence $\tau \notin K$), a map of R_σ-modules $\varphi_{\sigma,\tau} \colon M_\sigma \to M_\tau$ such that the induced map of R_τ-modules
$$R_\tau \otimes_{R_\sigma} M_\sigma \xrightarrow{\cong} M_\tau$$
is an isomorphism.

Let $\mathcal{L}(K)$ denote the category of K-localized persistence modules.

The role of K

Small $K \rightsquigarrow$ Localize a little.

Big $K \rightsquigarrow$ Localize a lot.

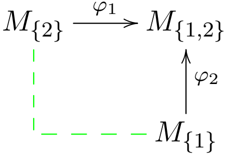
Example. Extreme cases:

1. $K = \{\} = \operatorname{sk}_{-2} \Delta^{m-1} \rightsquigarrow$ Don't localize:
$$\mathcal{L}(K) \cong R\text{-mod}.$$
2. $K = \partial \Delta^{m-1} = \operatorname{sk}_{m-2} \Delta^{m-1} \rightsquigarrow$ Invert all the t_i:
$$\mathcal{L}(K) = R_{[m]}\text{-mod} \cong \operatorname{vect}_{\mathbb{k}}.$$
3. Even more extreme! $K = \Delta^{m-1} \rightsquigarrow$ Localize everything into oblivion:
$$\mathcal{L}(K) = 0.$$

Example: $m = 2$

Take $m = 2$ and $K = \{\emptyset\} = \mathrm{sk}_{-1}\,\Delta^1$.

A K-localized persistence module M consists of modules

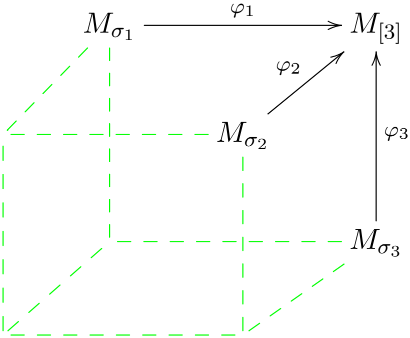


where φ_i inverts t_i.

Example: $m = 3$

Take $m = 3$ and $K = \mathrm{sk}_0 \Delta^2 = \{\emptyset, \{1\}, \{2\}, \{3\}\}$.

A K-localized persistence module M consists of modules



where $\sigma_i \coloneqq [m] \setminus \{i\}$ $\rightsquigarrow$ "all but t_i have been inverted"

$\varphi_i \coloneqq \varphi_{[m]\setminus\{i\},[m]} \colon M_{[m]\setminus\{i\}} \to M_{[m]}$ $\rightsquigarrow$ "invert t_i".

A Serre quotient

Consider the canonical functor

$$L_K \colon R\text{-mod} \to \mathcal{L}(K)$$

that keeps the relevant localizations of M:

$$L_K(M)_\sigma = R_\sigma \otimes_R M.$$

Lemma. L_K is exact.

Proposition. L_K is a Serre quotient functor:

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{\;L_K\;} & \mathcal{L}(K). \\ {\scriptstyle q}\downarrow & \overset{\cong}{\nearrow} & \\ R\text{-mod}/\ker(L_K) & & \end{array}$$

Outline

Persistence modules

Localized persistence modules

Classification of indecomposables

Which subcategories are we quotienting out?

Rank invariant

A hopeless dream?

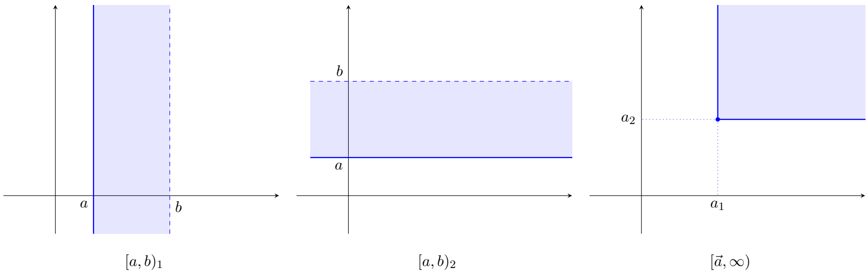
Denote $K_m \coloneqq \mathrm{sk}_{m-3}\,\Delta^{m-1} \rightsquigarrow$ Allow **at most one** non-inverted t_i.

Proposition. For any smaller simplicial complex $K \subset K_m$, $\mathcal{L}(K)$ has wild representation type.

Proof. $\mathcal{L}(K)$ contains a copy of $\Bbbk[s,t]$-mod as a retract. □

In this section, focus on $\mathcal{L}(K_m)$.

Some indecomposables



Some indecomposable objects in $\mathcal{L}(K_2)$.

Theorem (F.–Stanley)**.** Every object in $\mathcal{L}(K_2)$ decomposes (in a unique way) as a direct sum of:

- "vertical strips" $[a,b)_1$
- "horizontal strips" $[a,b)_2$
- "quadrants" $[\vec{a},\infty)$.

In higher dimension $m \geq 3$

The analogue in higher dimension is false.

Proposition. For any $m \geq 3$, there exists a torsion-free object in $\mathcal{L}(K_m)$ that is of rank 2 and indecomposable.

How complicated can the torsion-free objects in $\mathcal{L}(K_m)$ get?

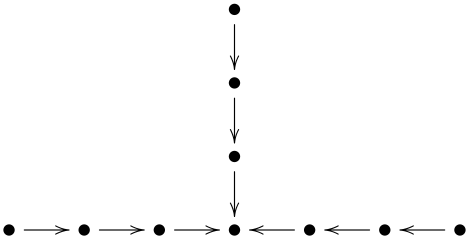
Answer: Pretty complicated!

Classification in higher dimension

Steffen Oppermann kindly provided the following argument.

Theorem (F.–Oppermann–Stanley)**.** The category $\mathcal{L}(K_3)$ has wild representation type.

Proof idea. Reduce to the known fact that this quiver has wild representation type:



□

Outline

Persistence modules

Localized persistence modules

Classification of indecomposables

Which subcategories are we quotienting out?

Rank invariant

Tensor ideals

Recall: Serre quotient

$$\mathcal{L}(K) \cong R\text{-mod}/\ker(L_K).$$

The subcategory $\ker(L_K) \subseteq R$-mod is a "**tensor ideal**": a Serre subcategory closed under tensoring with a $\mathbb{Z}^m$-graded R-module as long as the result is still $\mathbb{N}^m$-graded.

$\rightsquigarrow$ Allow shifting the degrees *down*, but not below 0.

Classification of tensor ideals

Proposition. [F.–Stanley] There is a bijection

$$\text{“tensor ideals” of } R\text{-mod} \rightleftarrows \text{simplicial complexes on } [m].$$

Via the bijection: $\ker(L_K) \leftrightsquigarrow K$.

Simple objects

What about $\ker(L_{K_m})$?

Proposition. $\mathcal{L}(K_m)$ is obtained from R-mod by quotienting out the Serre subcategory generated by the simple objects $m-1$ times successively.

Corollary. $\mathcal{L}(K_2)$ is the category of 2-parameter persistence modules up to finite diagrams:

$$\mathcal{L}(K_2) \cong \Bbbk[t_1, t_2]\text{-mod}/\{\text{finite modules}\}.$$

$\rightsquigarrow$ Large-scale behavior of the persistence module.

Outline

Persistence modules

Localized persistence modules

Classification of indecomposables

Which subcategories are we quotienting out?

Rank invariant

Rank invariant

Definition. Let M be an R-module. The **rank invariant** of M is the function assigning to each pair of multidegrees $\vec{a}, \vec{b} \in \mathbb{N}^m$ with $\vec{a} \leq \vec{b}$ the integer

$$\mathrm{rk}_M(\vec{a}, \vec{b}) = \mathrm{rank}\left(M(\vec{a}) \xrightarrow{t^{\vec{b}-\vec{a}}} M(\vec{b}) \right).$$

Introduced by Carlsson and Zomorodian (2009). Widely studied invariant of multiparameter persistence modules.

Rank invariant is not enough

The rank invariant of an R-module does not determine $L_K(M)$.

Example. In the case $m = 2$:

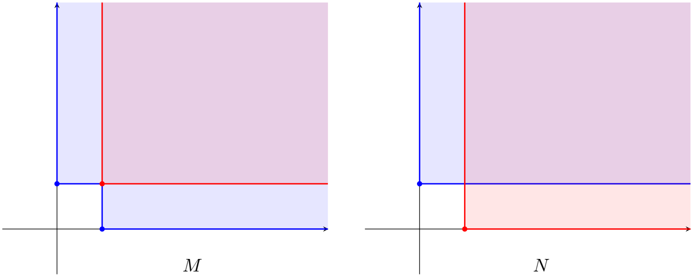
$$M = (t_1, t_2) \oplus t_1 t_2 R$$
$$N = t_1 R \oplus t_2 R$$

have the same rank invariant.

However, they have different K_2-localizations in $\mathcal{L}(K_2)$:

$$L_{K_2}(M) \cong [(0,0), \infty) \oplus [(1,1), \infty)$$
$$L_{K_2}(N) \cong [(1,0), \infty) \oplus [(0,1), \infty).$$

Same rank invariant



Rank invariant is sometimes enough

Proposition. For $K = K_m$, the rank invariant of an R-module M determines the R_{σ_i}-modules M_{σ_i} and the $\Bbbk$-vector space $M_{[m]}$.

Proposition. If M lies in the image of the right adjoint (delocalization functor)

$$\rho_{K_2} \colon \mathcal{L}(K_2) \to \Bbbk[t_1, t_2]\text{-mod},$$

then the rank invariant of M determines the localization $L_{K_2}(M)$.

Proposition. The following refinement of the rank invariant of M determines the localization $L_{K_2}(M)$. For all three bidegrees $\vec{a}, \vec{b}, \vec{c} \in \mathbb{N}^2$ satisfying $\vec{a} \leq \vec{c}$ and $\vec{b} \leq \vec{c}$, take the dimension of the intersection of images

$$\dim_{\Bbbk}\left(\operatorname{im}\left(M(\vec{a}) \xrightarrow{t^{\vec{c}-\vec{a}}} M(\vec{c})\right) \cap \operatorname{im}\left(M(\vec{b}) \xrightarrow{t^{\vec{c}-\vec{b}}} M(\vec{c})\right)\right).$$

Towards applications

Question. Given an R-module M, find efficient algorithms to compute the decomposition of $L_{K_2}(M)$ in $\mathcal{L}(K_2)$.

Question. Are there applications of persistent homology where the large-scale behavior is useful?

Thank you!