An invitation to n-angulated categories

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Outline

Triangulated categories

n-angulated categories

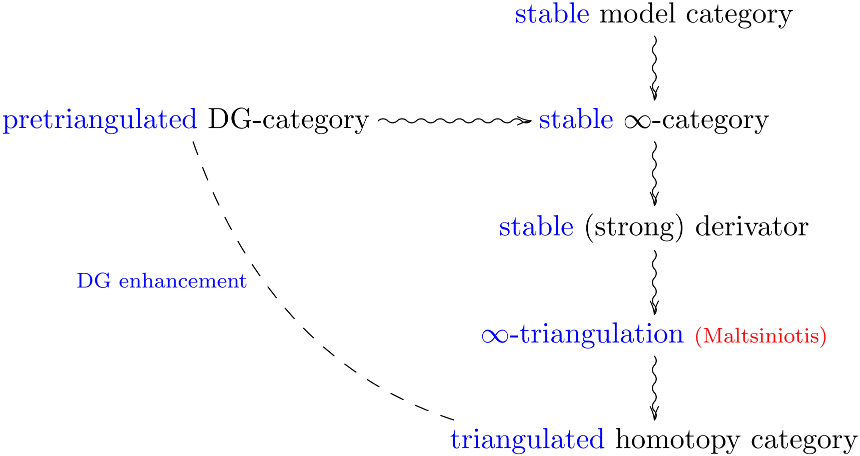
Examples

Toda brackets

Higher categorical structures

$$\begin{tikzcd}
 & \text{model category} \arrow[d, rightsquigarrow, "\text{has an underlying}"] & \mathcal{C} \\
\text{DG-category} \arrow[r, rightsquigarrow, "\text{DG-nerve}"] & \infty\text{-category} \arrow[d, rightsquigarrow] & \begin{array}{c} \mathrm{Ho}_\infty(\mathcal{C})=\mathcal{C}[W^{-1}] \\ (\infty\text{-categorical}) \end{array} \\
 & \text{derivator} \arrow[d, rightsquigarrow] & I \mapsto \mathrm{Ho}(\mathcal{C}^I) \\
 & \text{homotopy category} & \begin{array}{c} \mathrm{Ho}(\mathcal{C})=\mathcal{C}[W^{-1}] \\ (1\text{-categorical}) \end{array}
\end{tikzcd}$$

Stable homotopy



stable model category

pretriangulated DG-category

stable ∞-category

stable (strong) derivator

DG enhancement

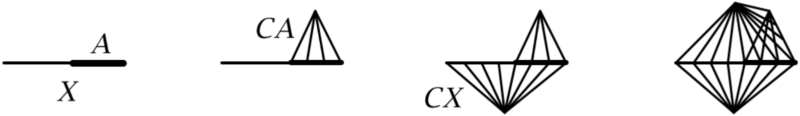
∞-triangulation (Maltsiniotis)

triangulated homotopy category

Triangulated categories

Idea: Triangles encode the structure of cofiber sequences

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X.$$



The mapping cone of $A \hookrightarrow X$. Image credit: Allen Hatcher.

Octahedral axiom, slogan version: Given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, the cofibers of f, g, and gf sit in a cofiber sequence

$$Y/X \to Z/X \to Z/Y.$$

Examples

Example. Some triangulated categories.

1. The stable homotopy category (spectra).
2. For a ring R, the naive homotopy category of chain complexes:

   $$\mathbf{K}(R) = \mathrm{Ch}(R)/\text{chain homotopy}$$

   and the derived category

   $$\mathrm{D}(R) = \mathrm{Ch}(R)[\text{quasi-iso}^{-1}].$$

   Here the "suspension" of C is the shift $C[1]$.
3. For G a finite p-group and $\Bbbk$ a field of characteristic p, the stable module category

   $$\mathrm{StMod}(\Bbbk G) = \Bbbk G\text{-Mod}/\text{stably null maps}.$$

Why n in the title?

Today: Not higher but **longer**!

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Pre-n-angulated categories

Idea: n-angulated ≈ "triangulated but with longer triangles".

Introduced by Geiss, Keller, and Oppermann (2013). Motivated by examples in quiver representation theory.

Let $\mathcal{C}$ be an additive category, $\Sigma \colon \mathcal{C} \xrightarrow{\cong} \mathcal{C}$ an automorphism, and $n \geq 3$.

Definition. An ***n*-Σ-sequence** is a diagram in $\mathcal{C}$ of the form

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1.$$

Definition. A **pre-*n*-angulation** of $\mathcal{C}$ is a collection $\mathscr{N}$ of n-Σ-sequences in $\mathcal{C}$, called ***n*-angles**, satisfying the following axioms.

The axioms

(N1)

- (a) $\mathscr{N}$ is closed under direct sums, direct summands and isomorphisms of n-Σ-sequences.
- (b) For all $X \in \mathcal{C}$, the trivial n-Σ-sequence

  $$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X_1$$

  belongs to $\mathscr{N}$.
- (c) For each morphism $f\colon X_1 \to X_2$ in $\mathcal{C}$, there exists an n-Σ-sequence in $\mathscr{N}$ whose first morphism is f.

(N2) An n-Σ-sequence belongs to $\mathscr{N}$ if and only if its rotation

$$X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2$$

belongs to $\mathscr{N}$.

The axioms (cont'd)

(N3) Given the solid part of the commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & X_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & X_3 \arrow[r, "f_3"] \arrow[d, dashed, "\phi_3"] & \cdots \arrow[r, "f_{n-1}"] & X_n \arrow[r, "f_n"] \arrow[d, dashed, "\phi_n"] & \Sigma X_1 \arrow[d, "\Sigma\phi_1"] \\
Y_1 \arrow[r, "g_1"] & Y_2 \arrow[r, "g_2"] & Y_3 \arrow[r, "g_3"] & \cdots \arrow[r, "g_{n-1}"] & Y_n \arrow[r, "g_n"] & \Sigma Y_1
\end{tikzcd}$$

with rows in $\mathscr{N}$, the dotted morphisms exist and give a morphism of n-Σ-sequences.

Remark. pre-3-angulated = pretriangulated.

Careful! Different meaning of the word "pretriangulated".

Getting rid of “pre”

Recall: triangulated = pretriangulated + octahedral axiom.

Definition. A pre-n-angulated category $\mathcal{C}$ is an ***n*-angulated category** if it also satisfies the “mapping cone axiom”, i.e., every fill-in problem admits a good fill-in.

Reformulated as a “higher octahedral axiom” by Bergh and Thaule (2013).

Non-unique "cofibers"

Main difference: When $n \geq 4$, an n-angle extension of $f \colon X_1 \to X_2$ is not unique up to isomorphism.

Example. In a 4-angulated category $\mathcal{C}$, given a 4-angle

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \Sigma X_1$$

and an object Y, we can add a trivial summand to obtain another 4-angle:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{\left[\begin{smallmatrix} f_2 \\ 0 \end{smallmatrix}\right]} X_3 \oplus Y \xrightarrow{\left[\begin{smallmatrix} f_3 & 0 \\ 0 & 1 \end{smallmatrix}\right]} X_4 \oplus Y \xrightarrow{\left[\begin{smallmatrix} f_4 & 0 \end{smallmatrix}\right]} \Sigma X_1.$$

Source of examples

Theorem (GKO 2013)**.** Let $\mathcal{T}$ be a triangulated category with an $(n-2)$-cluster tilting subcategory $\mathcal{C}$ closed under Σ^{n-2} where Σ denotes the suspension in $\mathcal{T}$. Then $(\mathcal{C}, \Sigma^{n-2}, \mathscr{N})$ is an n-angulated category where $\mathscr{N}$ is the class of all n-Σ^{n-2}-sequences

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma^{n-2} X_1$$

in $\mathcal{C}$ such that there exists a diagram in $\mathcal{T}$

$$\begin{tikzcd}
& X_2 \arrow[rr, "f_2"] \arrow[dr] & & X_3 \arrow[rr, "f_3"] \arrow[dr] & & \cdots \arrow[rr, "f_{n-2}"] & & X_{n-1} \arrow[dr, "f_{n-1}"] & \\
X_1 \arrow[ur, "f_1"] & & X_{2.5} \arrow[ll, "|" marking] \arrow[ur] & & X_{3.5} \arrow[ll, "|" marking] & \cdots \arrow[l, "|" marking] & X_{n-1.5} \arrow[l, "|" marking] \arrow[ur] & & X_n \arrow[ll, "|" marking]
\end{tikzcd}$$

with $X_i \in \mathcal{T}$ for all $i \notin \mathbb{Z}$ such that all triangles with base a degree-shifting morphism are triangles in $\mathcal{T}$, and f_n is the composition of the bottom row.

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Exotic example

Consider the category $\mathcal{C} = \mathrm{mod}^{\mathrm{ff}}\mathbb{Z}/p^2$ of finitely generated free modules over $R = \mathbb{Z}/p^2$, with $p = 2$ if n is odd (and p is any prime number if n is even). Take the identity automorphism $\Sigma = \mathrm{Id}$.

Muro, Schwede, and Strickland (2007) constructed an exotic triangulated structure on $\mathrm{mod}^{\mathrm{ff}}\mathbb{Z}/4$.

Bergh, Jasso, and Thaule (2016) constructed an exotic n-angulated structure on $\mathcal{C}$, for any $n \geq 3$.

Example. Taking $n = 4$, the diagram

$$\begin{tikzcd} R \arrow[r, "p"] & R \arrow[r, "p"] & R \arrow[r, "p"] & R \arrow[r, "p"] & R \end{tikzcd}$$

is a 4-angle in $\mathcal{C}$.

Quiver representation theory

Example. Consider the quiver Q with relations J^3 depicted here:

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[rrr, bend right, dashed] & 2 \arrow[r, "b"] & 3 \arrow[r, "c"] & 4
\end{tikzcd}$$

$J =$ arrow ideal, generated by paths of length 1.

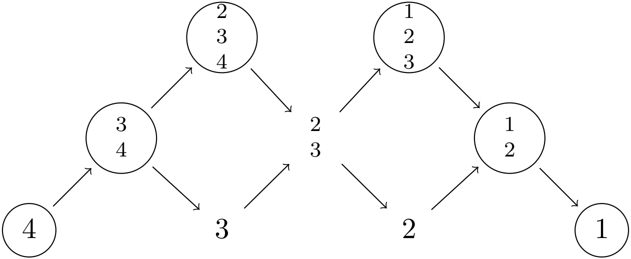
$J^3 =$ ideal generated by all paths of length 3, namely cba.

Path algebra $\Gamma = \Bbbk Q/J^3$.

Take $\mathcal{A} \coloneqq \text{mod-}\Gamma$ the category of finitely generated right Γ-modules.

Quiver example (cont'd)

The category $\mathcal{A}$ can be visualized as:



where we denote the quiver representation

$$\begin{smallmatrix}1\\2\end{smallmatrix} = \left(\mathbb{k} \xrightarrow{1} \mathbb{k} \longrightarrow 0 \longrightarrow 0\right)$$

and so on.

Take $M :=$ direct sum of the encircled modules.

Quiver example (cont'd)

In the category $\mathcal{A} = \text{mod-}\Gamma$, M is a 2-cluster tilting module in the sense of Iyama:

$$\begin{aligned} \operatorname{add} M &= \{X \in \mathcal{A} \mid \operatorname{Ext}^1_\Gamma(X, M) = 0\} \\ &= \{X \in \mathcal{A} \mid \operatorname{Ext}^1_\Gamma(M, X) = 0\}. \end{aligned}$$

Indecomposable projectives in $\mathcal{A}$:

$$P_1 = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \qquad P_2 = \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix} \qquad P_3 = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \qquad P_4 = 4.$$

Indecomposable injectives in $\mathcal{A}$:

$$I_1 = 1 \qquad I_2 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \qquad I_3 = P_1 \qquad I_4 = P_2.$$

Γ is a finite-dimensional $\Bbbk$-algebra of global dimension $\operatorname{gldim}\Gamma = 2$.

Quiver example (cont'd)

$\rightsquigarrow$ Get a 4-angulated structure on the subcategory

$$\mathcal{U} = \operatorname{add}\{M[2i] \mid i \in \mathbb{Z}\} \subseteq \mathrm{D}^b(\mathcal{A}).$$

Some 4-angles in $\mathcal{U}$:

$$P_4 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_3} I_1 \xrightarrow{f_4} \Sigma^2 P_4$$

$$P_4 \xrightarrow{g_1} P_3 \xrightarrow{g_2} P_1 \xrightarrow{g_3} I_2 \xrightarrow{g_4} \Sigma^2 P_4.$$

Here f_4 spans $\operatorname{Ext}^2_\Gamma(I_1, P_4) \cong \Bbbk$.

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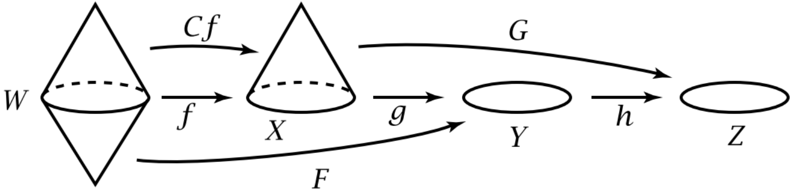
Toda brackets for spaces

Idea: A Toda bracket is constructed by picking nullhomotopies $f_{i+1}f_i \sim 0$ that witness $f_{i+1}f_i = 0$ in the homotopy category.



The Toda bracket $\langle h, g, f\rangle \subseteq [\Sigma W, Z]$. Image credit: Allen Hatcher.

Introduced by Toda (1962) to compute homotopy groups of spheres.

Still important for the computation of stable homotopy groups of spheres. See work of Isaksen, Wang, Xu, etc.

Triangulated case

Definition. Let $\mathcal{T}$ be a triangulated category and let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4$$

be a diagram in $\mathcal{T}$. The **Toda bracket** $\langle f_3, f_2, f_1 \rangle \subseteq \mathcal{T}(\Sigma X_1, X_4)$ consists of all composites $\Sigma(\beta^2 \beta^1) \colon \Sigma X_1 \to X_4$, where β^1 and β^2 appear in a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "\beta^1"] & X_2 \arrow[d, equal] & & \Sigma X_1 \arrow[d, "\Sigma\beta^1"] \\
Z_1 \arrow[r, "z_1"] & X_2 \arrow[r, "f_2"] & X_3 \arrow[r, "z_3"] \arrow[d, equal] & \Sigma Z_1 \arrow[d, "\Sigma\beta^2"] \\
& & X_3 \arrow[r, "f_3"] & X_4.
\end{tikzcd}$$

where the middle row is distinguished.

Remark. The bracket is non-empty if and only if $f_{i+1} f_i = 0$.

Indeterminacy

The Toda bracket is a coset of the **indeterminacy subgroup**

$$(f_3)_*\mathcal{T}(\Sigma X_1, X_3) + (\Sigma f_1)^*\mathcal{T}(\Sigma X_2, X_4) \subseteq \mathcal{T}(\Sigma X_1, X_4).$$

For higher n

What about Toda brackets in n-angulated categories?

Also available! Constructions, properties, examples.

Reference: M. Frankland, S. Martensen, and M. Thaule. Toda brackets in n-angulated categories. Available on arXiv.

Thank you!