

Multiparameter persistence modules in the large scale

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Outline

Persistence modules

Localized persistence modules

Classification of indecomposables

Which subcategories are we quotienting out?

Persistence modules

Fix a ground field \mathbb{k} .

Definition. For $m \geq 1$, an **m -parameter persistence module** is a diagram

$$\mathbb{N}^m \rightarrow \text{Vect}_{\mathbb{k}}.$$

\cong graded module over the graded polynomial algebra

$$R := \mathbb{k}[t_1, \dots, t_m],$$

which is \mathbb{N}^m -graded with multigrading

$$|t_i| = \vec{e}_i = (0, \dots, \overbrace{1}^i, \dots, 0).$$

Write $M(\vec{d})$ for the \mathbb{k} -vector space in multidegree $\vec{d} \in \mathbb{N}^m$.

Goal / Dream

Work with *finitely generated* R -modules:

$$R\text{-mod} := R\text{-Mod}^{\text{fin.gen.}} \subset R\text{-Mod}.$$

Goal

Classify the indecomposable objects in $R\text{-mod}$.

One-parameter case

For $m = 1$, finitely generated $\mathbb{k}[t]$ -modules decompose into **interval modules**

$$\begin{aligned}[a, b) &:= t^a \mathbb{k}[t] / t^b \mathbb{k}[t] \\ &= \operatorname{coker} \left(t^a \mathbb{k}[t] \xrightarrow{t^{b-a}} t^a \mathbb{k}[t] \right).\end{aligned}$$

Example.

$$\begin{aligned}M &= t^4 \mathbb{k}[t] \oplus t^2 \mathbb{k}[t] / t^7 \mathbb{k}[t] \\ &= [4, \infty) \oplus [2, 7).\end{aligned}$$

\rightsquigarrow Barcodes.

Multiparameter case

Not available for $m \geq 2$, because $\mathbb{k}[t_1, t_2]$ has **wild** representation type.

How to deal with that?

One approach: Extract invariants that are both computable and significant. Rank invariant and various refinements. (Many authors, probably you.)

Another approach: Focus on certain families of modules admitting a nice decomposition, such as rectangle-decomposable modules. [Botnan, Lebovici, Oudot], [Asashiba, Buchet, Escolar, Nakashima, Yoshiwaki], etc.

Approach: localize

Our approach: We localize $R\text{-mod}$ until the resulting category admits a classification of indecomposables, or at least a partial classification.

Related work: [Harrington, Otter, Schenck, Tillmann] and [Bauer, Botnan, Oppermann, Steen].

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Inverting some variables

Fact. The homogeneous prime ideals of R are those of the form $(t_{i_1}, \dots, t_{i_k})$.

The various localizations of a module M fit together.

Example. For M a module over $R = \mathbb{k}[t_1, t_2]$:

$$\begin{array}{ccc} \mathbb{k}[t_1, t_2^\pm] \otimes_R M & \xrightarrow{\text{invert } t_1} & \mathbb{k}[t_1^\pm, t_2^\pm] \otimes_R M \\ \text{invert } t_2 \uparrow & & \uparrow \text{invert } t_2 \\ M & \xrightarrow{\text{invert } t_1} & \mathbb{k}[t_1^\pm, t_2] \otimes_R M. \end{array}$$

Inverting some variables (cont'd)

Notation. 1. $[m] := \{1, 2, \dots, m\}$

2. For a subset $\sigma \subseteq [m]$, denote the localization of rings

$$R_\sigma := R[t_i^{-1} \mid i \in \sigma],$$

which is $\sigma^{-1}\mathbb{N}^m$ -graded.

3. $\varphi_i :=$ the localization map “invert t_i ”.

Example. With this notation, the previous square becomes:

$$\begin{array}{ccc} R_{\{2\}} \otimes_R M & \xrightarrow{\varphi_1} & R_{\{1,2\}} \otimes_R M \\ \varphi_2 \uparrow & & \uparrow \varphi_2 \\ M & \xrightarrow{\varphi_1} & R_{\{1\}} \otimes_R M. \end{array}$$

K -localized persistence modules

Idea: Forget the module M but keep some of its localizations.

If we keep a localization $R_\sigma \otimes_R M$, we should keep all further localizations $R_\tau \otimes_R M$ for $\sigma \subseteq \tau$.

Definition. Let K be a simplicial complex on the vertex set $[m]$. A **K -localized persistence module** M consists of:

1. For each missing face $\sigma \notin K$, a finitely generated R_σ -module M_σ .
2. For each $\sigma \subseteq \tau$ with $\sigma \notin K$ (and hence $\tau \notin K$), a map of R_σ -modules $\varphi_{\sigma,\tau}: M_\sigma \rightarrow M_\tau$ such that the induced map of R_τ -modules

$$R_\tau \otimes_{R_\sigma} M_\sigma \xrightarrow{\cong} M_\tau$$

is an isomorphism.

Let $\mathcal{D}(K)$ denote the category of K -localized persistence modules.

The role of K

Small $K \rightsquigarrow$ Localize a little.

Big $K \rightsquigarrow$ Localize a lot.

Example. Extreme cases:

1. $K = \{\} = \mathrm{sk}_{-2} \Delta^{m-1} \rightsquigarrow$ Don't localize:

$$\mathcal{D}(K) \cong R\text{-mod}.$$

2. $K = \partial\Delta^{m-1} = \mathrm{sk}_{m-2} \Delta^{m-1} \rightsquigarrow$ Invert all the t_i :

$$\mathcal{D}(K) = R_{[m]}\text{-mod} \cong \mathrm{vect}_{\mathbb{k}}.$$

3. Even more extreme! $K = \Delta^{m-1} \rightsquigarrow$ Localize everything into oblivion:

$$\mathcal{D}(K) = 0.$$

Example: $m = 2$

Take $m = 2$ and $K = \{\emptyset\} = \text{sk}_{-1} \Delta^1$.

A K -localized persistence module M consists of modules

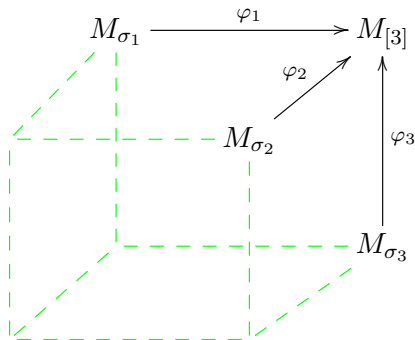
$$\begin{array}{ccc} M_{\{2\}} & \xrightarrow{\varphi_1} & M_{\{1,2\}} \\ \vdots & & \uparrow \varphi_2 \\ & \text{---} & M_{\{1\}} \end{array}$$

where φ_i inverts t_i .

Example: $m = 3$

Take $m = 3$ and $K = \text{sk}_0 \Delta^2 = \{\emptyset, \{1\}, \{2\}, \{3\}\}$.

A K -localized persistence module M consists of modules



where $\sigma_i := [m] \setminus \{i\} \rightsquigarrow$ “all but t_i have been inverted”

$\varphi_i := \varphi_{[m] \setminus \{i\}, [m]}: M_{[m] \setminus \{i\}} \rightarrow M_{[m]} \rightsquigarrow$ “invert t_i ”.

A Serre quotient

Consider the canonical functor

$$L_K: R\text{-mod} \rightarrow \mathcal{D}(K)$$

that keeps the relevant localizations of M :

$$L_K(M)_\sigma = R_\sigma \otimes_R M.$$

Lemma. L_K is exact.

Proposition. L_K is a Serre quotient functor:

$$\begin{array}{ccc} R\text{-mod} & \xrightarrow{L_K} & \mathcal{D}(K). \\ q \downarrow & \nearrow \cong & \\ R\text{-mod}/\ker(L_K) & & \end{array}$$

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A hopeless dream?

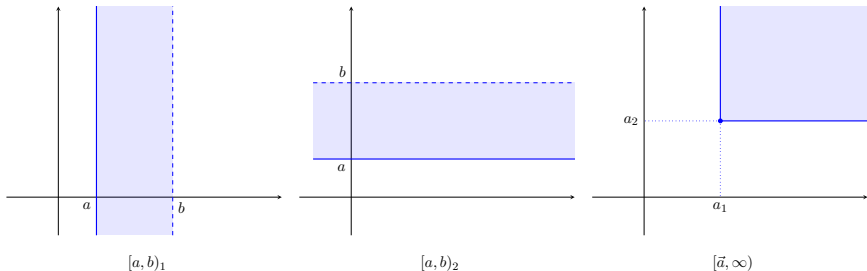
Denote $K_m := \text{sk}_{m-3} \Delta^{m-1} \rightsquigarrow$ Allow **at most one** non-inverted t_i .

Proposition. For any smaller simplicial complex $K \subset K_m$, $\mathcal{D}(K)$ has wild representation type.

Proof. $\mathcal{D}(K)$ contains a copy of $\mathbb{k}[x, y]$ -mod as a retract. □

In this section, focus on $\mathcal{D}(K_m)$.

Some indecomposables



Some indecomposable objects in $\mathcal{D}(K_2)$.

Theorem (F.–Stanley). Every object in $\mathcal{D}(K_2)$ decomposes in a unique way as a direct sum of:

- “vertical strips” $[a, b)_1$
- “horizontal strips” $[a, b)_2$
- “quadrants” $[\vec{a}, \infty)$.

A torsion pair

Consider two full subcategories of $\mathcal{D}(K_m)$:

$$\begin{aligned}\mathcal{T} &= \{M \mid M_{\sigma_i} \text{ is a torsion } R_{\sigma_i}\text{-module for all } i \in [m]\} \\ &= \{M \mid M_{[m]} = 0\}\end{aligned}$$

$$\mathcal{F} = \{M \mid M_{\sigma_i} \text{ is a torsion-free } R_{\sigma_i}\text{-module for all } i \in [m]\}.$$

Proposition. 1. $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\mathcal{D}(K_m)$.

2. For all M in $\mathcal{D}(K_m)$, the natural short exact sequence

$$0 \rightarrow T(M) \rightarrow M \rightarrow F(M) \rightarrow 0$$

splits.

$$\implies M \cong T(M) \oplus F(M)$$

Torsion objects

For $a < b < \infty$, consider the “interval $[a, b)$ in the i^{th} direction”:

$$[a, b)_i = L_{K_m} \left(t_i^a R / t_i^b R \right).$$

Lemma. $[a, b)_i$ is indecomposable in $\mathcal{D}(K_m)$.

Proposition. Each torsion object $M \in \mathcal{T}$ decomposes as a direct sum of objects of the form $[a, b)_i$.

Dimension count arguments

After inverting all variables, we are left with a vector space:

$$\begin{aligned} R_{[m]} \text{-mod} &\xrightarrow{\cong} \text{vect}_{\mathbb{k}} \\ M &\mapsto M(\vec{0}). \end{aligned}$$

Remark. The grading is crucial here. An *ungraded* $\mathbb{k}[t^{\pm}]$ -module corresponds to a \mathbb{k} -vector space V equipped with an automorphism $\mu_t: V \xrightarrow{\cong} V$.

Definition. 1. For $\sigma \subseteq [m]$ and an R_{σ} -module M , the **rank** of M is

$$\text{rank } M := \dim_{\mathbb{k}} (R_{[m]} \otimes_{R_{\sigma}} M).$$

2. The **rank** of an object M in $\mathcal{D}(K)$ is

$$\begin{aligned} \text{rank } M &:= \text{rank } M_{[m]} \\ &= \text{rank } M_{\sigma} \quad \text{for any missing face } \sigma \notin K. \end{aligned}$$

Torsion-free objects

Notation. For a multidegree $\vec{a} \in \mathbb{N}^m$, consider the object of $\mathcal{D}(K_m)$

$$[\vec{a}, \infty) = L_{K_m}(t^{\vec{a}}R),$$

where $t^{\vec{a}} := t_1^{a_1} \cdots t_m^{a_m}$.

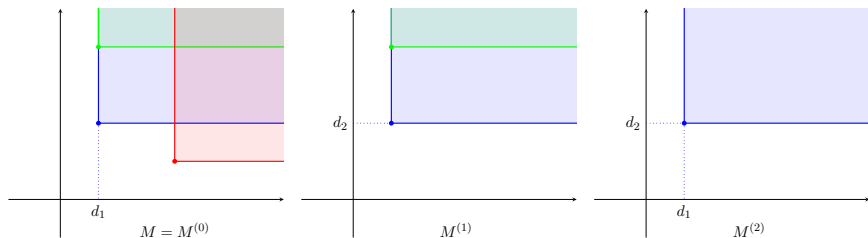
“quadrant module starting at \vec{a} ”

Lemma. 1. $[\vec{a}, \infty)$ is torsion-free, of rank 1, and indecomposable.
2. Any torsion-free object of rank 1 is of the form $[\vec{a}, \infty)$.

Proposition. Every torsion-free object M in $\mathcal{D}(K_2)$ decomposes as a direct sum of modules of the form $[\vec{a}, \infty)$.

Proof sketch...

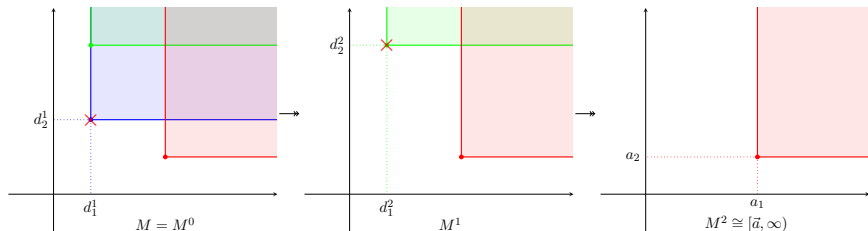
Scanning process



Scan for an element x of lowest degree $\vec{d} \in \mathbb{N}^2$ in lexicographic order.

The quotient $M / \langle x \rangle$ is still torsion-free.

Battleship game



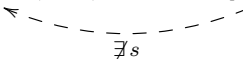
Scan, mod out, repeat $r - 1$ times, where $r = \text{rank } M$.

The composite epimorphism $M \twoheadrightarrow [\vec{a}, \infty)$ admits a section, splitting off a rank 1 summand from M .

In higher dimension $m \geq 3$

Warning: $[\vec{a}, \infty)$ is **not** projective in $\mathcal{D}(K_2)$.

Example. Consider the map in $\mathcal{D}(K_2)$

$$[(1, 0), \infty) \oplus [(0, 1), \infty) \xrightarrow{f=[\text{inc} \text{ inc}]} [(0, 0), \infty).$$


The map f is an epimorphism but does not admit a section.

Proposition. For any $m \geq 3$, there exists a torsion-free object in $\mathcal{D}(K_m)$ that is of rank 2 and indecomposable.

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Tensor ideals

Recall: Serre quotient

$$\mathcal{D}(K) \cong R\text{-mod} / \ker(L_K).$$

The subcategory $\ker(L_K) \subseteq R\text{-mod}$ is a “**tensor ideal**”: a Serre subcategory closed under tensoring with a \mathbb{Z}^m -graded R -module as long as the result is still \mathbb{N}^m -graded.

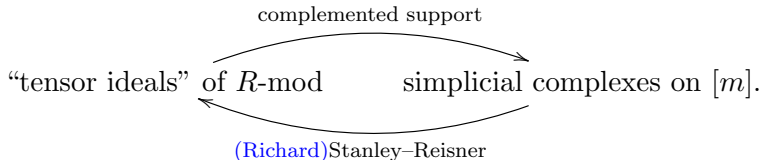
\rightsquigarrow Allow shifting the degrees *down*, but not below 0.

Classification of tensor ideals

Recall: The **support** of an R -module M is the set of homogeneous prime ideals $P \subset R$ for which the localization $M_P \neq 0$.

For $P = (t_{i_1}, \dots, t_{i_k})$, we will record the *complement* $[m] \setminus \{i_1, \dots, i_k\}$, the variables that *may* be inverted.

Proposition. [F.–(Don)Stanley] There is a bijection



Moreover $\ker(L_k) =$ the “tensor ideal” generated by $\mathbb{k}[K]$.

Stanley–Reisner ring

Definition. The **Stanley–Reisner ring** or *face ring* of a simplicial complex K is the polynomial ring modulo the monomials corresponding to missing faces:

$$\mathbb{k}[K] := R/(t_\sigma : \sigma \notin K).$$

Example. $K = \text{sk}_{-1} \Delta^{m-1} = \{\emptyset\}$

$$\implies \mathbb{k}[K] = R/(t_1, \dots, t_m) = \mathbb{k}$$

$$\implies \text{Supp } \mathbb{k}[K] = \{(t_1, \dots, t_m)\}$$

$$\implies \text{complemented } \text{Supp } \mathbb{k}[K] = \{\emptyset\} = K.$$

Simple objects

What about $\ker(L_{K_m})$?

Proposition. $\mathcal{D}(K_m)$ is obtained from $R\text{-mod}$ by quotienting out the Serre subcategory generated by the simple objects $m - 1$ times successively.

Corollary. $\mathcal{D}(K_2)$ is the category of 2-parameter persistence modules up to finite diagrams:

$$\mathcal{D}(K_2) \cong \mathbb{k}[t_1, t_2]\text{-mod}/\{\text{finite}\}.$$

\rightsquigarrow Large-scale behavior of the persistence module.

Some questions

Question. Which invariants of a $\mathbb{k}[t_1, t_2]$ -module M allow us to read off the decomposition of $L_{K_2}(M)$ in $\mathcal{D}(K_2)$?

Question. Does $\mathcal{D}(K_3)$ have wild representation type?

Thank you!