Modules over bialgebroids and Beck modules

Martin Frankland
University of Regina

Joint with Raveen Tehara

Descent Methods in Algebra, Geometry, and Topology
CMS Summer Meeting
Memorial University, St. John's, NL
June 4, 2022

Outline

Beck modules

Representing ringoids

Tensor product of Beck modules

Bialgebras with many objects

Coefficients for cohomology

Cohomology theories for algebraic structures:

- Group cohomology
- Lie algebra cohomology
- Hochschild cohomology of associative algebras
- Barr–Beck triple cohomology
- André–Quillen cohomology of commutative rings
- etc.

Need a good notion of *coefficient module* M over X:

$$H^*(X; M).$$

Beck modules

Definition (Beck 1967)**.** For an object X in a category $\mathcal{C}$, a **Beck module** over X is an abelian group object in the slice category $\mathcal{C}/X$.

Denote the category of Beck modules over X

$$\mathrm{Mod}(X) \coloneqq (\mathcal{C}/X)_{\mathrm{ab}}.$$

Pullback and pushforward

Definition. The pullback functor $f^*\colon \mathcal{C}/Y \to \mathcal{C}/X$ induces a functor

$$f^*\colon \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$$

also called the **pullback**.

Its left adjoint

$$f_!\colon \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$$

is called the **pushforward** along f.

$$\text{pullback} = \text{“restriction of scalars”}$$

$$\text{pushforward} = \text{“extension of scalars”}$$

Fibered category

The assignment

$$\mathrm{Mod}(-)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{AbCat}$$

sending an object X to its category of Beck modules $\mathrm{Mod}(X)$ and a map $f\colon X \to Y$ to the pullback functor $f^*\colon \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$ is a pseudo-functor: $(gf)^* \cong f^* g^*$.

Definition. The Grothendieck construction of the pseudo-functor $\mathrm{Mod}(-)$ yields a fibered category

$$\pi\colon T\mathcal{C} \to \mathcal{C}$$

called the **tangent category** of $\mathcal{C}$.

An object of $T\mathcal{C}$ is (X, M), where M is a module over X.

Rings

$\mathcal{C} = \mathrm{Alg}_{\Bbbk}$, the category of (unital) $\Bbbk$-algebras.

For a $\Bbbk$-algebra A:

$$\mathrm{Mod}(A) \cong {}_A\mathrm{Bimod}_A.$$

A Beck module over A is a split extension of A with square zero kernel:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & A \oplus M \arrow[r, shift left, "p"] & A \arrow[l, shift left, "s"] \arrow[r] & 0. \end{tikzcd}$$

The two actions on M are given by

$$(a, m)(a', m') = (aa', a \cdot m' + m \cdot a')$$

and they coincide for scalars in $\Bbbk$.

Rings, cont'd

For a map of $\Bbbk$-algebras $f\colon A \to B$, the pullback functor

$$f^*\colon \mathrm{Mod}(B) \to \mathrm{Mod}(A)$$

is the usual restriction of scalars:

$$a \cdot m \cdot a' \coloneqq f(a) \cdot m \cdot f(a').$$

The pushforward functor is

$$\begin{aligned} f_!\colon \mathrm{Mod}(A) &\to \mathrm{Mod}(B) \\ f_!(M) &= B \otimes_A M \otimes_A B. \end{aligned}$$

Commutative rings

$\mathcal{C} = \mathrm{Com}_{\Bbbk}$, the category of commutative $\Bbbk$-algebras.

For a commutative $\Bbbk$-algebra A:

$$\mathrm{Mod}(A) \cong \mathrm{Mod}_A \quad \text{in the usual sense.}$$

Same correspondence as for algebras, except that $A \oplus M$ must be commutative. This forces the two actions to coincide:

$$a \cdot m = m \cdot a.$$

For a map of commutative k-algebras $f\colon A \to B$, the pushforward functor is extension of scalars:

$$f_! \colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$$
$$f_!(M) = B \otimes_A M.$$

Groups

$\mathcal{C} = \mathrm{Gp}$, the category of groups.

For a group G:

$$\begin{aligned}\mathrm{Mod}(G) &\cong G\text{-modules in the usual sense}\\ &\cong \mathbb{Z}G\text{-Mod}.\end{aligned}$$

A Beck module over G is a split extension of G with abelian kernel:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] & G \ltimes K \arrow[r, shift left, "p"] & G \arrow[l, shift left, "e"] \arrow[r] & 1. \end{tikzcd}$$

The G-action on K is given by $e(g)k = (g, g \cdot k)$. In other words:

$$(g,k)(g',k') = (gg', k + g \cdot k').$$

Groups, cont'd

For a map of groups $f\colon G \to H$, the pushforward functor is

$$\begin{aligned} f_!\colon \mathrm{Mod}(G) &\to \mathrm{Mod}(H) \\ f_!(M) &= \mathbb{Z}H \otimes_{\mathbb{Z}G} M. \end{aligned}$$

Abelian groups

$\mathcal{C} = \mathrm{Ab}$, the category of abelian groups.

For an abelian group A:

$$\mathrm{Mod}(A) \cong \mathrm{Ab}.$$

Same correspondence as for groups, except that $A \ltimes K$ must be abelian. This forces the A-action on K to be trivial:

$$a \cdot k = k.$$

More generally:

Example (Beck 1967)**.** In an additive category $\mathcal{A}$ with finite limits, Beck modules over any object X are:

$$\begin{aligned} \mathrm{Mod}(X) &\cong \mathcal{A} \\ (p \colon E \twoheadrightarrow X) &\mapsto \ker(p). \end{aligned}$$

Lie algebras

$\Bbbk$ = a field of characteristic not 2.

$\mathcal{C} = \mathrm{Lie}_{\Bbbk}$, the category of Lie algebras over $\Bbbk$.

For a Lie algebra L:

$$\begin{aligned} \mathrm{Mod}(L) &\cong L\text{-modules in the usual sense} \\ &\cong U(L)\text{-Mod} \end{aligned}$$

where $U(L)$ is the universal enveloping algebra of L.

A Beck module over L is a split extension of L with abelian kernel:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & L \oplus M \arrow[r, shift left, "p"] & L \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

with $[m, m'] = 0$ for all $m, m' \in M$.

Lie algebras (cont'd)

The action of L on M is given by

$$[(\ell, 0), (0, m)] = (0, \ell \cdot m),$$

which satisfies

$$[\ell, \ell'] \cdot m = \ell \cdot (\ell' \cdot m) - \ell' \cdot (\ell \cdot m).$$

Commutative monoids

$\mathcal{C} = \mathrm{CMon}$, the category of commutative monoids.

A Beck module M over a commutative monoid A consists of a family of abelian groups $\{M_a\}_{a \in A}$ and for each $a, b \in A$, a map of abelian groups

$$b \cdot (-) \colon M_a \to M_{ba}$$

subject to $1 \cdot m = m$ and $(ab) \cdot m = a \cdot (b \cdot m)$.

Equivalently, a graded module M over the A-graded monoid ring $\mathbb{Z}A$.

Equivalenty, a functor $M \colon H(A) \to \mathrm{Ab}$ from the *Leech category* of A (a.k.a. action category), whose objects are $a \in A$ and morphisms are pairs (b, a):

$$a \xrightarrow{(b,a)} ba.$$

Outline

Beck modules

Representing ringoids

Tensor product of Beck modules

Bialgebras with many objects

Ringoids

Reference: Mitchell, Rings with several objects, 1972.

Definition.
1. A **ringoid** is a small pre-additive category, i.e., a small category enriched in abelian groups.
2. A **morphism of ringoids** is an Ab-enriched functor $F\colon \mathcal{R} \to \mathcal{S}$, a.k.a. an *additive functor*.
3. For a ringoid $\mathcal{R}$, a **left $\mathcal{R}$-module** is a covariant additive functor from $\mathcal{R}$ to abelian groups
$$M\colon \mathcal{R} \to \mathrm{Ab}.$$
4. A **right $\mathcal{R}$-module** is a contravariant additive functor from $\mathcal{R}$ to abelian groups
$$M\colon \mathcal{R}^{\mathrm{op}} \to \mathrm{Ab}.$$

Example. A one-object ringoid is a ring. The notions of modules recover the usual ones.

Modules over ringoids

Modules over a ringoid $\mathcal{R}$ form an abelian category. Which abelian categories are of that form?

Theorem (Freyd 1964)**.**

1. An abelian category $\mathcal{A}$ is equivalent to a category of modules over a ring if and only if $\mathcal{A}$ is cocomplete and has a small projective generator P.
   In that case $\mathcal{A} \cong \text{Mod-}\operatorname{End}(P)$.
2. $\mathcal{A}$ is equivalent to a category of modules over a ringoid if and only if $\mathcal{A}$ is cocomplete and has a set S of small projective generators.
   Given such a set S, take as ringoid the full subcategory on S.

The case of Beck modules

Proposition. Let $\mathcal{C}$ be an algebraic category (a.k.a. "equational class"). For every object X in $\mathcal{C}$, the category of Beck modules $\mathrm{Mod}(X)$ is equivalent to modules over some ringoid.

Representing ringoid

Some examples of representing ringoids:

1. For a ring A, the enveloping ring $A \otimes A^{\mathrm{op}}$.
2. For a commutative ring R, the ring R itself.
3. For a group G, the group ring $\mathbb{Z}G$.
4. For an abelian group A, the ring $\mathbb{Z}$.
5. For a Lie algebra L, the universal enveloping algebra $U(L)$.
6. For a commutative monoid A, the $\mathbb{Z}$-linearization of the Leech category $H(A)$.

Outline

Beck modules

Representing ringoids

Tensor product of Beck modules

Bialgebras with many objects

Common tensor products

We know how to tensor modules over a commutative ring R:

$$M \otimes_R N.$$

We also know how to tensor bimodules over a ring A:

$$M \otimes_A N.$$

Goal

Find a broad enough framework where Beck modules have a natural tensor product, recovering many known examples.

Idea: Look for comultiplicative structure in the representing ringoids.

Bialgebras

Example. The comultiplication of the group ring $\mathbb{Z}G$

$$\Delta\colon \mathbb{Z}G \to \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G$$
$$\Delta(g) = g \otimes g$$

induces a G-action on the tensor product of G-modules $M \otimes_{\mathbb{Z}} N$:

$$g \cdot (m \otimes n) = (g \cdot m) \otimes (g \cdot n).$$

Example. Likewise for the universal enveloping algebra of a Lie algebra L:

$$\Delta\colon U(L) \to U(L) \otimes_{\Bbbk} U(L)$$
$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

induces an action of L on the tensor product of L-modules $M \otimes_{\Bbbk} N$:

$$\ell \cdot (m \otimes n) = (\ell \cdot m) \otimes n + m \otimes (\ell \cdot n).$$

Bialgebras (cont'd)

More generally, the comultiplication on a bialgebra B induces a B-module structure on the tensor product of B-modules $M \otimes_{\Bbbk} N$:

$$\begin{tikzcd}
B \otimes_{\Bbbk} M \otimes_{\Bbbk} N \arrow[r, "\Delta\otimes\mathrm{id}"] \arrow[drr, bend right, "\lambda_{M\otimes N}"'] & B \otimes_{\Bbbk} B \otimes_{\Bbbk} M \otimes_{\Bbbk} N \arrow[r, "\cong"] & B \otimes_{\Bbbk} M \otimes_{\Bbbk} B \otimes_{\Bbbk} N \arrow[d, "\lambda_M\otimes\lambda_N"] \\
 & & M \otimes_{\Bbbk} N.
\end{tikzcd}$$

Pointwise tensor product

For a group G:

$$G\text{-Mod} \cong \mathrm{Fun}(BG, \mathrm{Ab})$$

where BG denotes G viewed as a one-object groupoid.

The tensor product of G-modules agrees with the pointwise tensor product in $\mathrm{Fun}(BG, \mathrm{Ab})$.

Definition. A category $\mathcal{C}$ has **representable Beck modules** if for all object X in $\mathcal{C}$:

$$\mathrm{Mod}(X) \cong \mathrm{Fun}(J_X, \mathrm{Ab})$$

for some small category J_X, pseudo-functorial in X.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathrm{Mod}(-)"] \arrow[dr, dashed, "J_{(-)}"'] & & \mathrm{AbCat} \\
& \mathrm{Cat} \arrow[ur, "\mathrm{Fun}(-{,}\mathrm{Ab})"'] &
\end{tikzcd}$$

Pointwise tensor product (cont'd)

Example. The following categories have representable Beck modules.

1. Sets: $J_X = X$ viewed as a discrete category.
2. Groups: $J_G = BG$, the one-object groupoid.
3. Abelian groups: $J_A = *$, the trivial category.
4. Commutative monoids: $J_A = H(A)$, the Leech category of A.
5. Monoids.

Example. The following categories do not have representable Beck modules.

1. Rings.
2. Commutative rings.

$\mathbb{Z}$-linearization

Definition. Given a small category $\mathcal{C}$, the $\mathbb{Z}$-linearization of $\mathcal{C}$ is the $\mathbb{Z}$-linear (i.e. Ab-enriched) category $\mathbb{Z}\mathcal{C}$ obtained by applying the free abelian group functor to the hom-sets in $\mathcal{C}$:

$$(\mathbb{Z}\mathcal{C})(X, Y) := \mathbb{Z}\left(\mathcal{C}(X, Y)\right).$$

Composition in $\mathbb{Z}\mathcal{C}$ is defined as the bilinear extension of composition in $\mathcal{C}$.

Lemma.
1. $\mathbb{Z}$-linearization is a (strict) 2-functor Cat $\to$ $\mathbb{Z}$-Cat.
2. Universal property:
$$\mathrm{Fun}_{\mathbb{Z}}(\mathbb{Z}\mathcal{C}, \mathcal{A}) \cong \mathrm{Fun}(\mathcal{C}, \mathcal{A}).$$
3. $\mathbb{Z}$-linearization sends the Cartesian product of categories to the tensor product of ringoids:
$$\mathbb{Z}(\mathcal{C} \times \mathcal{D}) \cong \mathbb{Z}\mathcal{C} \otimes \mathbb{Z}\mathcal{D}.$$

Comultiplicative structure

For any small category $\mathcal{C}$, the diagonal functor and the constant functor

$$\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$
$$\epsilon\colon \mathcal{C} \to *$$

make $\mathcal{C}$ into a comonoid in Cat.

Applying the $\mathbb{Z}$-linearization produces ringoid maps

$$\Delta\colon \mathbb{Z}(\mathcal{C}) \to \mathbb{Z}(\mathcal{C} \times \mathcal{C}) \cong \mathbb{Z}\mathcal{C} \otimes \mathbb{Z}\mathcal{C}$$
$$\epsilon\colon \mathbb{Z}(\mathcal{C}) \to \mathbb{Z}(*) \cong \mathbb{Z}$$

making $\mathbb{Z}\mathcal{C}$ into a comonoid in ringoids.

Outline

Beck modules

Representing ringoids

Tensor product of Beck modules

Bialgebras with many objects

Comonoids in ringoids

Not all Beck modules are represented by a bialgebra. Look for a "many objects" generalization, but in which direction?

$$\text{bialgebra} = \text{monoid in coalgebras} = \text{comonoid in algebras}$$

Generalizing rings to ringoids yields one notion of "bialgebra with many objects": **comonoids in ringoids**.

The $\mathbb{Z}$-linearization $\mathbb{Z}\mathcal{C}$ is an example.

This notion was studied in the literature (Day, Street 1997).

Commutative bialgebroids

In stable homotopy theory, we often stumble upon:

commutative Hopf algebroid = internal cogroupoid in commutative algebras.

Dropping the antipode yields:

commutative bialgebroid = internal cocategory in commutative algebras.

Bialgebroids

Introduced by Takeuchi (1977). Comprehensive reference: Böhm (2009).

A bialgebroid consists of:

- algebras A and H
- algebra maps $A \to H$ and $A^{\mathrm{op}} \to H$
- a bimodule map $H \to H \otimes_A H$
- a character $H \to A$
- + more structure and properties.

Salient feature: Both modules and comodules over a bialgebroid have a tensor product.

Example: enveloping ring

Example. Given a $\Bbbk$-algebra A, its enveloping algebra $A \otimes_{\Bbbk} A^{\mathrm{op}}$ admits a canonical bialgebroid structure.

The induced tensor product of $A \otimes_{\Bbbk} A^{\mathrm{op}}$-modules recovers the usual tensor product of A-bimodules.

Question. Are there many instances where Beck modules are represented by a bialgebroid?

Thank you!