Pure cocycle perturbations of E_0-semigroups

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Dedicated to the memory of Bill Arveson

Abstract. We establish a parameterization of the class of pure E_0-semigroups that are cocycle conjugate to a given E_0-semigroup $\rho = \{\rho_t\}_{t\geq 0}$ of $\mathscr{B}(H)$ in terms of the class of pure essential mixing states of the spectral C^*-algebra of the concrete product system of the semigroup $\rho = \{\rho_t\}_{t\geq 0}$.

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1. Introduction

Purification of E_0-semigroups is a central theme of the theory of noncommutative dynamics [1], a theme which has revolved around two main problems: (i) to decide whether or not every E_0-semigroup is cocycle conjugate to a pure E_0-semigroup [1][4], and; (ii) to describe the class of pure E_0-semigroups that are cocycle conjugate to a given E_0-semigroup [2][3]. A solution to the former problem was obtained in the affirmative by V. Liebscher, as a direct consequence of an intricate analysis of product systems of W^*-algebras [4, Theorem 7.23].

It is our main purpose, in paper, to initiate a study of the latter problem from the perspective of Arveson's theory of spectral C^*-algebras. More precisely, we show that Arveson's characterization of cocycle perturbations of an E_0-semigroup in terms of essential states of the spectral C^*-algebra pertains to the study of pure E_0-semigroups, and determines a surjective correspondence from the set of all pure essential mixing states of the spectral C^*-algebra $C^*(E_\rho)$ of the concrete product system E_ρ of an arbitrary E_0-semigroup $\rho = \{\rho_t\}_{t\geq 0}$ onto the set of all pure E_0-semigroups that are cocycle conjugate to the semigroup $\rho = \{\rho_t\}_{t\geq 0}$.

This self-contained paper is organized as follows. In Section 2, we collect several definitions and facts regarding E_0-semigroups. For a more detailed

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treatment of various aspects of this subject, we refer the reader to [1]. In Section 3, we show that the regular representation of the concrete product system E_ρ of an E_0-semigroup $\rho = \{\rho_t\}_{t\geq 0}$ induces a strongly continuous one-parameter contractive semigroup $\alpha^* = \{\alpha^*_t\}_{t\geq 0}$ on the dual space of the spectral C^*-algebra $C^*(E_\rho)$. We then obtain a complete characterization of the class of essential bounded linear functionals of $C^*(E_\rho)$ in terms of this semigroup. In Section 4, we introduce the new concept of mixing bounded linear functionals of the spectral C^*-algebra, which is defined with respect to the semigroup $\alpha^* = \{\alpha^*_t\}_{t\geq 0}$. We show that every mixing essential state completely defines the pureness of the natural E_0-semigroup acting on the von Neumann algebra generated by the spectral C^*-algebra under the GNS representation of the state. This result produces the aforementioned surjective correspondence between pure essential mixing states and pure E_0-semigroups.

2. Background on the theory of E_0-semigroups, product systems, and spectral C^*-algebras

Introduced by R. Powers in [5], E_0-semigroups are pointwise σ-weak continuous one parameter semigroups $\rho = \{\rho_t\}_{t\geq 0}$ of unit-preserving normal *-endomorphisms acting on a von Neumann algebra M, usually a type I_∞ factor $\mathscr{B}(H)$, where H is a separable Hilbert space. Non-unital E_0-semigroups are conveniently called E-semigroups. An E_0-semigroup $\rho = \{\rho_t\}_{t\geq 0}$ is said to be pure if its tail algebra is trivial, i.e.,

$$\bigcap_{t\geq 0} \rho_t(M) = \mathbb{C}\cdot \mathbb{1}.$$

The main focus of this theory is on the classification of E_0-semigroups up to conjugacy and cocycle conjugacy. Two E_0-semigroups $\rho = \{\rho_t\}_{t\geq 0}$ and $\sigma = \{\sigma_t\}_{t\geq 0}$ acting on $\mathscr{B}(H)$, respectively $\mathscr{B}(K)$, are said to be conjugate if there is a *-isomorphism $\theta : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\theta \circ \rho_t = \sigma_t \circ \theta$, $t \geq 0$. They are said to be cocycle conjugate if σ is conjugate to a cocycle perturbation of ρ, i.e., to an E_0-semigroup of the form $\{\mathrm{Ad}(u_t)\circ\rho_t\}_{t\geq 0}$, where $\{u_t\}_{t\geq 0} \subset \mathscr{B}(H)$ is a unitary ρ-cocycle, i.e., a strongly continuous family of unitaries u_t satisfying the cocycle relation $u_{s+t} = u_s\rho_s(u_t)$, for all s, $t \geq 0$.

As shown by Arveson, the classification problem of E_0-semigroups up to cocycle conjugacy is equivalent to the classification problem of product systems up to an isomorphism. More precisely, one can associate to each E_0-semigroup (or to an E-semigroup) $\rho = \{\rho_t\}_{\geq 0}$ acting on $\mathscr{B}(H)$, the Borel bundle $E_\rho = \{E_\rho(t)\}_{t>0} \subset \mathscr{B}(H)$ of Hilbert spaces of intertwining operators

$$E_\rho(t) = \{v \in \mathscr{B}(H) \,|\, \rho_t(x)v = vx, \text{ for all } x \in \mathscr{B}(H)\},$$

endowed with the inner product $\langle u, v\rangle_{E_\rho(t)} \cdot \mathbb{1} = v^*u$, u, $v \in E_\rho(t)$. The bundle $E_\rho = \{E_\rho(t)\}_{t>0}$, called the concrete product system of ρ, has the property that its isomorphism class is a complete cocycle conjugacy invariant, in the sense that two E_0-semigroups ρ and σ are cocycle conjugate iff E_ρ and E_σ are isomorphic as product systems. One should also note that there is a concept of abstract product system, which is not defined in terms of the intertwining spaces of an E_0-semigroup. However, any abstract product system can be transformed, via an isomorphism, into a concrete product system by a procedure similar to the one that follows.

One can describe all E_0-semigroups $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{\geq 0}$ that are cocycle conjugate to a given E_0-semigroup ρ acting on $\mathscr{B}(H)$ by making use of the representation theory of the concrete product system E_ρ. More precisely, if ϕ is a representation of E_ρ on a Hilbert space K, i.e., a measurable operator-valued function $\phi : E_\rho \to \mathscr{B}(K)$ satisfying the conditions (i) $\phi(v)^*\phi(u) = \langle u, v\rangle_{E(t)} \cdot \mathbb{1}$, u, $v \in E_\rho(t)$, $t > 0$; and (ii) $\phi(uv) = \phi(u)\phi(v)$, $u \in E_\rho(s)$, $y \in E_\rho(t)$, s, $t > 0$, then $\{\phi(E_\rho(t))\}_{t>0}$ is a Borel bundel of Hilbert spaces that is isomorphic to E_ρ, and $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$, where

$$\widetilde{\rho}_t(x) = \sum_{n=1}^{\infty} \phi(e_n(t))x\phi(e_n(t))^*,\ t > 0,\ x \in \mathscr{B}(K), \tag{2.1}$$

and $\rho_0 := \mathrm{Id}_{\mathscr{B}(K)}$, is a E-semigroup acting on $\mathscr{B}(K)$ with concrete product system $E_{\widetilde{\rho}} = \{\phi(E_\rho(t))\}_{t>0}$. Here $\{e_n(t)\}_{n\in\mathbb{N},\, t>0}$ is a Borel orthonormal basis for E_ρ, i.e., for every $n \in \mathbb{N}$, $(0, \infty) \ni t \mapsto e_n(t) \in E_\rho(t)$ is a Borel section of E_ρ in such a way that $\{e_n(t)\}_n$ is orthonormal basis for $E_\rho(t)$, for every $t > 0$. The semigroup $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$ is an E_0-semigroup if and only if

$$[\phi(E_\rho(t))K] = K,\ \ t > 0, \tag{2.2}$$

in which case we say that ϕ is an essential representation of the product system E_ρ. At the opposite end are the singular representations of E_ρ, i.e., representations ϕ satisfying $\bigcap_{t>0}[\phi(E_\rho(t))K] = \{0\}$.

Essential representations of a concrete product system $E_\rho = \{E_\rho(t)\}_{t>0}$ or, equivalently, E_0-semigroups that are cocycle conjugate to ρ, can be constructed by making use of the representation theory of the spectral C^*-algebra $C^*(E_\rho)$ of E_ρ, a C^*-algebra that plays the role of the “spectrum” of the E_0-semigroup ρ. To define this C^*-algebra, we consider, for $p \in \{1, 2\}$, the spaces $L^p(E_\rho)$ of all measurable sections $(0, \infty) \ni\mapsto f(t) \in E_\rho(t)$ that are p-integrable, i.e., $\int_{(0,\infty)} \|f(t)\|^p dt < \infty$, endowed with the usual norm $\|f\|_p = (\int_{(0,\infty)} \|f(t)\|^p dt)^{1/p}$. Every representation ϕ of E_ρ on a Hilbert space K extends to a contractive representation of the Banach algebra $L^1(E_\rho)$ on K by the formula

$$\phi(f) := \int_{(0,\infty)} \phi(f(t))dt,\ \ f \in L^1(E_\rho). \tag{2.3}$$

The spectral C^*-algebra $C^*(E_\rho) \subseteq \mathscr{B}(L^2(E_\rho))$ is then defined as the norm closure of the linear span of all products of the form $\ell_f\ell_g^*$ with f, $g \in L^1(E_\rho)$, where $\ell : E_\rho \to \mathscr{B}(L^2(E_\rho))$ is the regular representation of E_ρ,

$$(\ell_x f)(s) \;=\; \begin{cases} xf(s-t), & s > t, \\ 0, & 0 < s \leq t, \end{cases}$$

for $x \in E_\rho(t)$, $t > 0$, and $f \in L^2(E_\rho)$, and ℓ_f, ℓ_g are as in (2.3) for $f, g \in L^1(E_\rho)$. Note that the regular representation ℓ is singular.

The spectral C^*-algebra $C^*(E_\rho)$ is non-unital, separable, irreducible, nuclear, simple at least for spatial E_0-semigroups, and its representation theory echoes the representation theory of the product system E_ρ. More precisely, there is a 1-1 correspondence between nondegenerate representations $\pi : C^*(E_\rho) \to \mathscr{B}(K)$ of the C^*-algebra $C^*(E_\rho)$ and representations $\phi : E_\rho \to \mathscr{B}(K)$ of the product system E_ρ, given by

$$\pi(\ell_f \ell_g^*) = \phi(f)\phi(g)^*, \ \ f, g \in L^1(E_\rho). \tag{2.4}$$

A representation π of $C^*(E_\rho)$ is said to be essential iff the corresponding representation ϕ of E_ρ satisfying (2.4) is essential. In particular, a bounded linear functional $\omega \in C^*(E_\rho)^*$ is said to be essential iff the GNS representation of the positive linear functional $|\omega|$ obtained from the polar decomposition applied to ω is essential. As shown by Arveson, the set of all essential bounded linear functionals of $C^*(E_\rho)$ is a non-trivial, complementable norm-closed linear subspaces of $C^*(E_\rho)^*$.

3. A strongly continuous contractive semigroup on $C^*(E_\rho)^*$

Let $\rho = \{\rho_t\}_{t\geq 0}$ be an E_0-semigroup of $\mathscr{B}(H)$ with concrete product system $E_\rho = \{E_\rho(t)\}_{t>0}$. We infer from the definition of the spectral C^*-algebra $C^*(E_\rho)$ that the inclusion

$$\ell_u C^*(E_\rho)\ell_v^* \subseteq C^*(E_\rho), \tag{3.1}$$

holds true for all $u \in E_\rho(s)$, $v \in E_\rho(t)$, and $s, t > 0$. In particular, if $\{e_n(t)\}_{n\in\mathbb{N},,t>0}$ is a Borel orthogonal basis for E_ρ, then one can associate to each bounded linear functional $\omega \in C^*(E_\rho)^*$, the family of bounded linear functionals $\{\omega_{n,t}\}_{n\in\mathbb{N},\,t>0}$ on $C^*(E_\rho)$, defined as

$$\omega_{n,t}(x) = \omega(\ell_{e_n(t)} x \ell_{e_n(t)}^*), \ \ x \in C^*(E_\rho). \tag{3.2}$$

In the following lemma, we discuss the convergence of the series determined by these functionals.

Lemma 3.1. *Suppose $\rho = \{\rho_t\}_{t\geq 0}$ is an E_0-semigroup of $\mathscr{B}(H)$ with concrete product system $E_\rho = \{E_\rho(t)\}_{t>0}$. Let $\{e_n(t)\}_{n\in\mathbb{N},\,t>0}$ be a Borel orthogonal basis for E_ρ, and let $\omega \in C^*(E_\rho)^*$. Then for every $t > 0$, the series $\sum_n \omega_{n,t}$ converges uniformly in $C^*(E_\rho)^*$, and*

$$\sum_n \|\omega_{n,t}\| \leq \|\omega\|. \tag{3.3}$$

Proof. Applying the polar decomposition to ω, and the GNS construction to $|\omega|$, one can find a representation π of the C^*-algebra $C^*(E_\rho)$ on a Hilbert space K, and two cyclic vectors $\xi, \eta \in K$, $\|\xi\|^2 = \|\eta\|^2 = \|\omega\|$, such that

$$\omega(x) = \langle \pi(x)\xi, \eta \rangle, \ x \in C^*(E_\rho). \tag{3.4}$$

Let $\phi : E_\rho \to \mathscr{B}(K)$ be the representation of the product system E_ρ associated with π as in (2.4), and $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$ be the E-semigroup of $\mathscr{B}(K)$ constructed as in (2.1). Let $t > 0$ be fixed. For every positive integer n and every $x \in C^*(E_\rho)$, we have

$$\omega_{n,t}(x) = \omega(\ell_{e_n(t)} x \ell^*_{e_n(t)}) = \langle \pi_\omega(x)\xi_n(t), \eta_n(t)\rangle\,,$$

where $\xi_n(t) = \phi(e_n(t))^*\xi$ and $\eta_n(t) = \phi(e_n(t))^*\eta$. Since

$$\widetilde{\rho}_t(\mathbb{1}) = \sum_n \phi(e_n(t))\phi(e_n(t))^*$$

is an orthogonal projection, we deduce that

$$\sum_{n=1}^\infty \|\xi_n(t)\|^2 = \|\widetilde{\rho}_t(\mathbb{1})\xi\|^2 \leq \|\xi\|^2 = \|\omega\|,$$

and similarly $\sum_{n=1}^\infty \|\eta_n(t)\|^2 \leq \|\omega\|$. Using now the Schwarz inequality, we obtain

$$\begin{aligned}\sum_{n=1}^\infty \|\omega_{n,t}\| &= \sum_{n=1}^\infty \sup_{\|x\|\leq 1} |\langle \pi(x)\xi_n(t), \eta_n(t)\rangle| \\ &\leq \sum_{n=1}^\infty \|\xi_n(t)\|\|\eta_n(t)\| \\ &\leq \|\widetilde{\rho}_t(\mathbb{1})\xi\|\|\widetilde{\rho}_t(\mathbb{1})\eta\| \leq \|\omega\|.\end{aligned}$$

This concludes the proof of the lemma. □

Definition 3.2. Let $\rho = \{\rho_t\}_{t\geq 0}$ be an E_0-semigroup of $\mathscr{B}(H)$ with concrete product system $E_\rho = \{E_\rho(t)\}_{t>0}$, and $\{e_n(t)\}_{n\in\mathbb{N},\, t>0}$ be a Borel orthogonal basis for E_ρ. For every $t > 0$, consider the mapping $\alpha^*_t : C^*(E_\rho)^* \to C^*(E_\rho)^*$, defined as

$$\alpha^*_t(\omega)(x) = \sum_n \omega_{n,t}(x) = \sum_n \omega(\ell_{e_n(t)} x \ell^*_{e_n(t)}),$$

for $\omega \in C^*(E_\rho)^*$ and $x \in C^*(E_\rho)$. For $t = 0$, we set α^*_0 to be the identity mapping on $C^*(E_\rho)^*$.

It is readily seen that the above definition does not depend on the particular choice of the basis $\{e_n(t)\}_{n\in\mathbb{N}}$, for every $t > 0$.

Proposition 3.3. *$\alpha^* = \{\alpha^*_t\}_{t\geq 0}$ is a strongly continuous semigroup of contractions on the Banach space $C^*(E_\rho)^*$.*

Proof. We firstly show that $\{\alpha^*_t\}_{t\geq 0}$ is a one-parameter semigroup of contractions. Let $s, t > 0$ be fixed. Since $\{e_n(t)e_m(s)\}_{m,\, n\in\mathbb{N}}$ is an orthogonal basis for $E_\rho(s+t)$, we have

$$\begin{aligned}\alpha^*_{s+t}(\omega)(x) &= \sum_{m,n} \omega\left(\ell_{e_n(t)e_m(s)} x \ell^*_{e_n(t)e_m(s)}\right) = \sum_m \alpha^*_t(\omega)\left(\ell_{e_m(s)} x \ell^*_{e_m(s)}\right) \\ &= \alpha^*_s \circ \alpha^*_t(\omega)(x),\end{aligned}$$

for every $\omega \in C^*(E_\rho)^*$ and $x \in C^*(E_\rho)$. Therefore $\{\alpha_t^*\}_{t\geq 0}$ is a one-parameter semigroup, which is contractive by (3.3).

To show that the semigroup is strongly continuous, fix a bounded linear functional $\omega \in C^*(E_\rho)^*$, and consider the representation π of $C^*(E_\rho)$ associated with ω as in (3.4), as well as the corresponding E-semigroup $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$ of $\mathscr{B}(K)$, constructed as in (2.1). A direct calculation produces the following relation between $\widetilde{\rho}_t$ and $\alpha_t^*(\omega)$:

$$\alpha_t^*(\omega)(x) = \langle \widetilde{\rho}_t(\pi(x))\xi, \eta\rangle\,,\ x \in C^*(E_\rho). \tag{3.5}$$

Indeed, it is enough to verify (3.5) on elements of the form $x = \ell_f \ell_g^*$ with $f,\, g \in L^1(E_\rho)$. For such an x, we obtain by using (2.4) that

$$\begin{aligned} \alpha_t^*(\omega)(x) &= \sum_n \omega(\ell_{e_n(t)\cdot f}\ell^*_{e_n(t)\cdot g}) = \sum_n \left\langle \pi(\ell_{e_n(t)\cdot f}\ell^*_{e_n(t)\cdot g})\xi, \eta\right\rangle \\ &= \sum_n \left\langle \phi(e_n(t))\pi(\ell_f\ell_g^*)\phi(e_n(t)^*)\xi, \eta\right\rangle = \langle \widetilde{\rho}_t(\pi(x))\xi, \eta\rangle\,, \end{aligned}$$

where $e_n(t)\cdot f$ is the $L^1(E_\rho)$-section $(e_n(t)\cdot f)(s) = \begin{cases} e_n(t)f(s-t), & s > t, \\ 0, & 0 < s \leq t, \end{cases}$.

Now for every $t > 0$, we have:

$$\begin{aligned} \|\alpha_t^*(\omega) - \omega\| &= \sup_{\|x\|\leq 1} |\langle \widetilde{\rho}_t(\pi(x))\xi, \eta\rangle - \langle \pi(x)\xi, \eta\rangle| \\ &\leq \|\omega_{\xi,\,\eta} \circ \widetilde{\rho}_t - \omega_{\xi,\eta}\|, \end{aligned}$$

where $\omega_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$, $x \in \mathscr{B}(K)$. Since $\widetilde{\rho}$ is an E-semigroup, the last term tends to zero as $t \to 0$ by [1, Prop. 2.3.1], and thus $\{\alpha_t^*\}_{t\geq 0}$ is strongly continuous. □

Next, we describe the space of all essential bounded functionals on $C^*(E_\rho)$ in terms of the semigroup $\alpha^* = \{\alpha_t^*\}_{t\geq 0}$.

Theorem 3.4. *Let $\rho = \{\rho_t\}_{t\geq 0}$ be an E_0-semigroup of $\mathscr{B}(H)$ with concrete product system $E_\rho = \{E_\rho(t)\}_{t>0}$. A bounded linear functional $\omega \in C^*(E_\rho)^*$ is essential if and only if $\|\alpha_t^*(\omega)\| = \|\omega\|$, for all $t \geq 0$.*

Proof. It follows immediately from the polar decomposition that, in order to prove this result, it is enough to consider positive linear functionals on $C^*(E_\rho)$. For such a functional $\omega \in C^*(E_\rho)$, $\omega \geq 0$, we consider its GNS representation $\pi : C^*(E_\rho) \to \mathscr{B}(K)$ with cyclic vector ξ, $\|\xi\|^2 = \|\omega\|$, and the E-semigroup $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$ of $\mathscr{B}(K)$ constructed with respect to the product system representation ϕ. As in (3.5), we have

$$\alpha_t^*(\omega)(x) = \langle \widetilde{\rho}_t(\pi(x))\xi, \xi\rangle\,,\ x \in C^*(E_\rho). \tag{3.6}$$

We claim that

$$\|\alpha_t^*(\omega)\| = \|\widetilde{\rho}_t(\mathbb{1})\xi\|^2,\ \ t > 0. \tag{3.7}$$

Indeed, if $e_1 \leq e_2 \leq \ldots$ is an approximate identity for $C^*(E_\rho)$, then $\{\pi(e_n)\}_n$ converges strongly to the identity operator, and thus

$$\|\alpha_t^*(\omega)\| = \lim_{n\to\infty} \langle \widetilde{\rho}_t(\pi(e_n))\xi, \xi\rangle = \langle \widetilde{\rho}_t(\mathbb{1})\xi, \xi\rangle = \|\widetilde{\rho}_t(\mathbb{1})\xi\|^2.$$

We are now ready to prove the theorem. The implication “ $\Rightarrow$ ” follows immediately from (3.7). For the converse implication, let $\omega \in C^*(E_\rho)$, $\omega \geq 0$ satisfying $\|\alpha_t^*(\omega)\| = \|\omega\|$, for every $t > 0$, and let $\widetilde{K} = \bigcap_{t>0} K_t$, be the intersection of the decreasing family of Hilbert spaces

$$K_t = [\phi(E_\rho(t))K] = \widetilde{\rho}_t(\mathbb{1})K,\ t > 0.$$

Since $\|\xi\|^2 = \|\omega\| = \|\alpha_t^*(\omega)\| = \|\widetilde{\rho}_t(\mathbb{1})\xi\|$ for every $t > 0$, we have $\xi \in \widetilde{K}$. We claim that the Hilbert space $\widetilde{K}$ is invariant under $\phi(E_\rho) \bigcup \phi(E_\rho)^*$. It is obvious that

$$\phi(E_\rho(t))K_s \subseteq K_{t+s},$$

for all s, $t > 0$, so $\widetilde{K}$ is invariant under $\phi(E_\rho)$. Moreover, since $\{\phi(E_\rho(t)\}_{t>0}$ is the concrete product system of the E-semigroup $\widetilde{\rho}$, the Hilbert space $\phi(E_\rho(s))$ is the closed linear span of $\phi(E_\rho(t))\phi(E_\rho(s-t))$ for $s > t$, and we have

$$\begin{aligned}
\phi(E_\rho(t))^* K_s &\subseteq [\phi(E_\rho(t))^*\phi(E_\rho(s))K] \\
&= [\phi(E_\rho(t))^*\phi(E_\rho(t))\phi(E_\rho(s-t))K] \\
&= [\phi(E_\rho(s-t))K] = K_{s-t},
\end{aligned}$$

for $0 < t < s$. Therefore

$$\phi(E_\rho(t))^*\widetilde{K} \subseteq \bigcap_{s>t} [\phi(E_\rho(t))^* K_s] \subseteq \bigcap_{s>t} K_{s-t} = \widetilde{K},$$

for every $t > 0$, so $\widetilde{K}$ is also invariant under $\phi(E_\rho)^*$ as well, which proves our claim. It then follows that $\widetilde{K}$ is invariant under the von Neumann algebra generated by $\phi(E_\rho)$, which coincides with the von Neumann algebra $\pi(C^*(E_\rho))''$. But ξ is a cyclic vector for the representation π, and $\xi \in \widetilde{K}$, thus $\widetilde{K} = K$, i.e., ω is an essential functional. □

Observation 3.5. One can easily deduce from (3.7) that a representation ϕ of the product system E_ρ associated with $\omega \in C^*(E_\rho)^*$ is singular if and only if the orbit of ω under the semigroup $\{\alpha_t^*\}_{t\geq 0}$ is stable, i.e., $\lim_{t\to\infty} \|\alpha_t^*(\omega)\| = 0$.

4. Mixing states of $C^*(E_\rho)$ and purification

Let $\rho = \{\rho_t\}_{t\geq 0}$ be an E_0-semigroup of $\mathscr{B}(H)$ with concrete product system $E_\rho = \{E_\rho(t)\}_{t>0}$. We use the semigroup $\alpha^* = \{\alpha_t^*\}_{t\geq 0}$ from the previous section to introduce a property of bounded linear functionals on $C^*(E_\rho)$ that resembles the notion of mixing in ergodic theory.

Definition 4.1. A bounded linear functional $\omega \in C^*(E_\rho)^*$ is said to be mixing, if

$$\lim_{t\to\infty} \|\alpha_t^*(a\omega b) - \omega(ab)\alpha_t^*(\omega)\| = 0, \tag{4.1}$$

for every a, $b \in C^*(E_\rho)$. Here $a\omega b$ is the bounded linear functional $a\omega b(x) = \omega(axb)$, $x \in C^*(E_\rho)$.

The class of mixing bounded linear functionals is quite large, as shown by the following proposition.

Proposition 4.2. *Every vector state of $C^*(E_\rho)$ is mixing.*

Proof. Let $\xi \in L^2(E_\rho)$ be a unit vector, and ω_ξ be the corresponding vector state. Since the C^*-algebra $C^*(E_\rho)$ is irreducible, the GNS representation associated with ω_ξ can be taken to be the identity representation, having ξ as corresponding cyclic vector.

Let now $\alpha = \{\alpha_t\}_{t\geq 0}$ be the E-semigroup of $\mathscr{B}(L^2(E_\rho))$ associated with the regular representation ℓ of E_ρ. Since ℓ is a singular representation, we have $\alpha_t(\mathbb{1}) \downarrow 0$ as $t \to \infty$. Moreover, using (3.6), we obtain

$$\alpha_t^*(a\omega_\xi b)(x) = \langle \alpha_t(x)b\xi, a^*\xi\rangle = \langle \alpha_t(x)\alpha_t(\mathbb{1})b\xi, \alpha_t(\mathbb{1})a^*\xi\rangle\,,$$

for all a, b, $x \in C^*(E_\rho)$. Therefore

$$\sup_{\|x\|\leq 1} |\alpha_t^*(a\omega_\xi b)(x)| \leq \|\alpha_t(\mathbb{1})a^*\xi\|\|\alpha_t(\mathbb{1})b\xi\| \to 0$$

as $t \to \infty$. Using (3.7), we also have

$$\sup_{\|x\|\leq 1} |\omega_\xi(ab)\alpha_t^*(\omega_\xi)(x)| \leq |\omega_\xi(ab)| \cdot \|\alpha_t(\mathbb{1})\xi\|^2 \to 0,$$

as $t \to \infty$. In particular, $\|\alpha_t^*(a\omega_\xi b) - \omega_\xi(ab)\alpha_t^*(\omega_\xi)\| \to 0$ as $t \to \infty$. □

Observation 4.3. Although the vector states of $C^*(E_\rho)$ are mixing states, they fail to be essential states, as one can easily see from (3.7) and Theorem 3.4.

Next, we describe the relation between mixing essential states and pure E_0-semigroups. We need, for this purpose, the following proposition, which is a straightforward variation of Arveson's classical characterization of pure E_0-semigroups acting on a von Neumann algebra in terms of the asymptotic behavior of the action of the semigroup on the predual [2][1].

Proposition 4.4. *Suppose $\rho = \{\rho_t\}_{t\geq 0}$ is an E_0-semigroup of a von Neumann algebra $M \subseteq \mathscr{B}(K)$. The following conditions are equivalent:*

1. *ρ is pure;*
2. *for any two normal states ω, ψ of $\mathscr{B}(K)$, $\|\omega \restriction_{\rho_t(M)} - \psi \restriction_{\rho_t(M)}\|$ converges to 0 as $t \to \infty$;*
3. *for any two unit vectors $\xi, \eta \in K$, $\|\omega_\xi \circ \rho_t - \omega_\eta \circ \rho_t\|$ converges to 0 as $t \to \infty$.*

Proof. (1) ⇔ (2) is similar to the proof of [1, Proposition 2.9.2].
(2) ⇒ (3) follows immediately from the fact that $\|\omega \circ \rho_t\| = \|\omega \restriction_{\rho_t(M)}\|$, for every normal linear functional on $\mathscr{B}(K)$ and every $t \geq 0$.
(3) ⇒ (2) Let $\xi \in K$ be a fixed unit vector. Using the triangle inequality and the previous fact, it is enough to show that $\|\omega \circ \rho_t - \omega_\xi \circ \rho_t\|$ converges to 0 as $t \to \infty$, for every normal state ω of $\mathscr{B}(K)$. To prove this, realize the state ω

as $\omega = \sum_n \lambda_n \omega_{\xi_n}$, where $\{\xi_n\}_n \subset K$ is a sequence of unit vectors and $\{\lambda_n\}_n$ is a sequence of positive real numbers summing up to 1. We then have

$$\|\omega \circ \rho_t - \omega_\xi \circ \rho_t\| \leq \sum_n \lambda_n \|\omega_{\xi_n} \circ \rho_t - \omega_\xi \circ \rho_t\| \to 0,$$

as $t \to \infty$. The proof is complete. □

Theorem 4.5. *Let $\rho = \{\rho_t\}_{t\geq 0}$ be an E_0-semigroup of $\mathscr{B}(H)$ with concrete product system $E_\rho = \{E_\rho(t)\}_{t>0}$, and ω be an essential state of $C^*(E_\rho)$. Let $\pi : C^*(E_\rho) \to \mathscr{B}(K)$ be GNS representation of ω, and $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$ be the corresponding E_0-semigroup of $\mathscr{B}(K)$. The following statements are equivalent:*

1. *The restriction of $\widetilde{\rho}$ to the von Neumann algebra $M = \pi(C^*(E_\rho))''$ is a a pure E_0-semigroup of M.*
2. *ω is a mixing state.*

Proof. It is clear from the definition of the E_0-semigroup $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$ that $\widetilde{\rho}_t(M) \subseteq M$, so that $\widetilde{\rho}$ restricts to an E_0-semigroup of the von Neumann algebra M. Let $\xi \in K$ be the cyclic unit vector in the GNS construction associated with ω.
(1) $\Rightarrow$ (2): Consider an element $a \in C^*(E_\rho)$. If $\omega(a^*a) \neq 0$, then let $\eta \in K$ be the unit vector

$$\eta = \|\pi(a)\xi\|^{-1}\pi(a)\xi.$$

Since $\widetilde{\rho}$ is pure, we have

$$\lim_{t\to\infty} \|\omega_\xi \circ \rho_t - \omega_\eta \circ \rho_t\| = 0 \tag{4.2}$$

by the previous proposition. Substituting η into (4.2), and using (3.6), we obtain

$$\lim_{t\to\infty} \sup_{\|x\|\leq 1} \left| \alpha_t^*(\omega)(x) - \frac{\alpha_t^*(a^*\omega a)(x)}{\omega(a^*a)} \right| = 0 \tag{4.3}$$

so that one has

$$\lim_{t\to\infty} \|\omega(a^*a)\alpha_t^*(\omega) - \alpha_t^*(a^*\omega a)\| = 0. \tag{4.4}$$

Clearly (4.4) holds true if $\omega(a^*a) = 0$, and hence for all $a \in C^*(E_\rho)$. A direct polarization argument shows that (4.4) is equivalent to (4.1), and the conclusion follows.
(2) $\Rightarrow$ (1) is similar: if ω is mixing, then property (4.4) holds true for all $a \in C^*(E_\rho)$. This implies (4.3) when $\omega(a^*a) \neq 0$. Using the fact that the most general unit vector in $\pi(C^*(E_\rho))\xi$ is of the form $\|\pi(a)\xi\|^{-1}\pi(a)\xi$ where $\omega(a^*a) \neq 0$, (4.3) implies that (4.2) holds true for all unit vectors η in the dense linear manifold $\pi(C^*(E_\rho)\xi$ of K. In particular, since ξ is cyclic, (4.2) holds for all unit vectors $\eta \in H$. Now using the triangle inequality and Proposition 4.4, we obtain that $\widetilde{\rho}$ is a pure E_0-semigroup of M. This ends the proof of the theorem. □

Corollary 4.6. *Every essential mixing state of $C^*(E_\rho)$ is a factor state.*

Proof. It is easily seen that the center of the von Neumann algebra $M = \pi(C^*(E_\rho))''$ coincides with the fixed point algebra $\{x \in M \mid \widetilde{\rho}_t(x) = x, \forall t \geq 0\}$ of the restriction of the E_0-semigroup $\widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$ to M. Since the tail algebra of $\widetilde{\rho}$ contains the fixed point algebra, the conclusion follows. □

The main result of our paper is a straightforward consequence of the previous theorem.

Theorem 4.7. *Let $\rho = \{\rho_t\}_{t\geq 0}$ be an E_0-semigroup of $\mathscr{B}(H)$. The mapping*

$$\omega \longmapsto \widetilde{\rho} = \{\widetilde{\rho}_t\}_{t\geq 0}$$

that appears in Theorem 4.5 *restricts to a surjective mapping from the set of all pure essential mixing states of the spectral C^*-algebra $C^*(E_\rho)$ onto the set of all pure E_0-semigroups that are cocycle conjugate to ρ.*

Using the purification theorem of V. Liebscher that was mentioned in the introduction, we obtain the following:

Corollary 4.8. *For every E_0-semigroup $\rho = \{\rho_t\}_{t\geq 0}$ of $\mathscr{B}(H)$, there exists pure essential mixing states of the spectral C^*-algebra $C^*(E_\rho)$.*

The structure of the set of all pure essential mixing states will be discussed in a subsequent publication.

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