ABSENCE OF GAUGE-INVARIANT FUNCTIONALS ON SPECTRAL C^*-ALGEBRAS

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ABSTRACT. We show that Arveson's spectral C^*-algebra $C^*(E)$ of an arbitrary product system $E = \{E(t)\}_{t>0}$ admits no non-zero gauge-invariant positive functionals.

1. INTRODUCTION AND BACKGROUND

The study of gauge-invariant states has played a central role in the development of the representation theory of Cuntz algebras [5][7][1] or, equivalently, in the theory of *-endomorphisms of the algebra $\mathscr{B}(H)$ of all bounded linear operators on a Hilbert space H [4]. A continuous analogue of the Cuntz algebras, called the spectral C^*-algebra of a product system, was introduced by Arveson in [2], with the purpose of constructing and classifying E_0-semigroups of $\mathscr{B}(H)$, i.e., weak*-continuous one parameter semigroups $\rho = \{\rho_t\}_{t\geq 0}$ of unit-preserving normal *-endomorphisms of $\mathscr{B}(H)$. Ever since its introduction, the spectral C^*-algebra of a product system has emerged as a pivotal concept in the classification theory of E_0-semigroups [3], and has been the subject of numerous investigations, mostly based on methods and techniques specific to the representation theory of Cuntz algebras[3][8][6].

It is our main purpose, in this paper, to show in full generality that the analogy between the theory of Cuntz algebras and that of spectral C^*-algebras has certain limitations (see also [9] for similar results obtained in the particular case of spatial product systems). More precisely, we prove the following theorem:

Theorem. *Let $E = \{E(t)\}_{t>0}$ be a product system, and ω be a positive linear functional on the spectral C^*-algebra $C^*(E)$ of E. If ω is gauge-invariant in the sense that $\omega \circ \gamma_\lambda = \omega$, $\lambda \in \mathbb{R}$, where $\{\gamma_\lambda\}_{\lambda\in\mathbb{R}}$ is the gauge group of $C^*(E)$, then $\omega = 0$.*

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It is noteworthy that, despite the lack of non-trivial gauge-invariant positive functionals, the spectral C^*-algebra of a product system admits certain gauge-invariant weights that encounter similar behavior to some well-known functionals on Cuntz algebras, provided that the product system is spatial (see [9]). Whether or not such weights exist for arbitrary product systems is still unknown.

In the remaining of this section, we present some background on product systems and spectral C^*-algebras. For more detailed information, we refer the reader to [3]. A product system is a standard Borel bundel $E = \{E(t)\}_{t>0}$ of infinite dimensional separable Hilbert spaces $E(t)$, which is trivial in the sense that $E \cong E(t_0) \times (0, \infty)$ as a measurable family of Hilbert spaces, and which is endowed with a measurable family $\{U_{s,t}\}_{s,\,t>0}$ of unitary operators

$$E(s) \otimes E(t) \ni x \otimes y \longmapsto U_{s,t}(x \otimes y) =: x \cdot y \in E(t+s),\ \ s,\, t > 0,$$

that obey the associativity rule

$$U_{t_1,t_2+t_3}(\mathbb{1}_{E(t_1)} \otimes U_{t_2,t_3}) = U_{t_1+t_2,t_3}(U_{t_1,t_2} \otimes \mathbb{1}_{E(t_3)}),\ \ t_1,\, t_2,\, t_3 > 0.$$

For every $p \in \{1, 2\}$, we consider the spaces $L^p(E)$ of all measurable sections $(0, \infty) \ni\mapsto f(t) \in E(t)$ that are p-integrable, i.e., $\int_{(0,\infty)} \|f(t)\|^p dt < \infty$, endowed with the usual norm $\|f\|_p = (\int_{(0,\infty)} \|f(t)\|^p dt)^{1/p}$. Every representation ϕ of E on a Hilbert space H, i.e., a measurable operator-valued function $\phi : E \to \mathscr{B}(H)$ satisfying the conditions (i) $\phi(y)^*\phi(x) = \langle x, y\rangle \cdot \mathbb{1}$, $x,\, y \in E(t)$, $t > 0$; and (ii) $\phi(x \cdot y) = \phi(x)\phi(y)$, $x \in E(s)$, $y \in E(t)$, $s,\, t > 0$, extends to a contractive representation of the Banach algebra $L^1(E)$ on H via the formula

$$\phi(f) := \int_{\mathbb{R}_+} \phi(f(t))dt,\ \ f \in L^1(E). \tag{1.1}$$

The spectral C^*-algebra $C^*(E) \subseteq \mathscr{B}(L^2(E))$ is then defined as

$$C^*(E) = \overline{\text{span}}\{\ell_f \ell_g^* \,|\, f,\, g \in L^1(E)\},$$

where $\ell : E \to \mathscr{B}(L^2(E))$ is the regular representation of E,

$$(\ell_x f)(s) \;=\; \begin{cases} x \cdot f(s-t), & s > t, \\ 0, & 0 < s \le t, \end{cases}$$

for $x \in E(t)$, $t > 0$, and $f \in L^2(E)$, and ℓ_f, ℓ_g are as in (1.1) for $f,\, g \in L^1(E)$. The C^*-algebra $C^*(E)$ admits a natural one-parameter automorphism group, the gauge group $\{\gamma_\lambda\}_{\lambda\in\mathbb{R}}$, acting as

$$\gamma_\lambda(\ell_f \ell_g^*) = \ell_{e^{i\lambda} f} \ell_{e^{i\lambda} g}^*,\ \ \lambda \in \mathbb{R}$$

where $(e^{i\lambda} f)(t) = e^{i\lambda t} f(t)$, for $f \in L^1(E)$, $t > 0$, and $\lambda \in \mathbb{R}$.

2. Proof of the theorem

Let ω be a gauge-invariant positive linear functional on $C^*(E)$, and $\pi : C^*(E) \to \mathscr{B}(H)$ be the corresponding GNS representation with cyclic vector $\xi \in H$. Using [3, Theorem 4.1.7], one can find a representation $\phi : E \to \mathscr{B}(H)$ satisfying

$$\pi(\ell_f \ell_g^*) = \phi(f)\phi(g)^*, \quad f,\, g \in L^1(E).$$

We then have

$$\begin{aligned} \omega(\ell_f \ell_g^*) &= \left\langle \pi(\ell_f \ell_g^*)\xi, \xi \right\rangle = \langle \phi(g)^*\xi, \phi(f)^*\xi \rangle \\ &= \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \langle \phi(g(t))^*\xi, \phi(f(s))^*\xi \rangle \; dsdt. \end{aligned} \tag{2.1}$$

For every $f,\, g \in L^1(E)$, we consider the complex-valued measure $\mu_{f,g}$ on $\mathbb{R}_+ \times \mathbb{R}_+$, defined by

$$d\mu_{f,g} = \langle \phi(g(t))^*\xi, \phi(f(s))^*\xi \rangle \; dsdt.$$

Is is readily seen that $\|\mu_{f,g}\| \leq \|\omega\| \|f\|_1 \|g\|_1$, and that $\mu_{f,g}$ is absolutely continuous with respect to the Lebesgue measure.

We claim that $\mu_{f,g} = 0$, for every $f,\, g \in L^1(E)$. For this purpose, fix $f,\, g \in L^1(E)$, and consider the sections $uf,\, vg \in L^1(E)$,

$$(uf)(t) = u(t)f(t), \; (vg)(t) = v(t)g(t), \; t > 0,$$

associated with continuous functions with compact support $u,\, v \in C_{00}([0,\infty))$. Since ω is gauge invariant, we have

$$\iint_{\mathbb{R}_+ \times \mathbb{R}_+} e^{i\lambda(s-t)} u(s)\overline{v}(t)\, d\mu_{f,g}(s,t) = \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(s)\overline{v}(t)\, d\mu_{f,g}(s,t),$$

for all $u,\, v \in C_{00}([0,\infty))$ and $\lambda \in \mathbb{R}$. Indeed

$$\begin{aligned} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(s)\overline{v}(t)\, d\mu_{f,g}(s,t) &= \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \langle \phi((vg)(t))^*\xi, \phi((uf)(s))^*\xi \rangle \; dsdt \\ &= \omega(\ell_{uf}\ell_{vg}^*) = \omega(\ell_{e^{i\lambda}uf}\ell_{e^{i\lambda}vg}^*) \\ &= \iint_{\mathbb{R}_+ \times \mathbb{R}_+} e^{i\lambda(s-t)} u(s)\overline{v}(t)\, d\mu_{f,g}(s,t). \end{aligned}$$

Let now $\eta \in L^1(\mathbb{R})$ satisfying $\hat{\eta}(0) = 0$, where $\hat{\eta}$ is the Fourier transform of η. Multiplying the previous equation by $\eta(\lambda)$ and integrating with respect to λ, we obtain

$$\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \hat{\eta}(s-t) u(s)\overline{v}(t)\, d\mu_{f,g}(s,t) = \hat{\eta}(0) \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(s)\overline{v}(t)\, d\mu_{f,g}(s,t) = 0.$$

Using the Stone-Weierstrass theorem, one can easily see that the set of all linear combinations of functions of the form $\hat{\eta}(s-t)u(s)\overline{v}(t)$, where

u, v, η are as before, is uniformly dense in the space of all continuous functions $D(s,t)$ on $[0,\infty)\times[0,\infty)$ that vanish both at ∞ and along the diagonal

$$\mathscr{D} = \{(s,s)\,|\,s\in[0,\infty)\}.$$

It then follows that for any such function D, we have

$$\iint_{\mathbb{R}_+\times\mathbb{R}_+} D(s,t)\,d\mu_{f,g}(s,t) = 0.$$

Consequently, the measure $\mu_{f,g}$ vanishes on the complement of the set $\mathscr{D}$. Since $\mathscr{D}$ is a set of Lebesgue measure zero and $\mu_{f,g}$ is absolutely continuous with respect to Lebesgue measure, we obtain that $\mu_{f,g}=0$ as claimed. Finally, we infer from (2.1) that $\omega(\ell_f\ell_g^*)=0$ for all f, $g\in L^1(E)$, i.e., $\omega=0$. The theorem is proved.

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