CONVERGENCE ANALYSIS OF THE LATOUCHE-RAMASWAMI ALGORITHM FOR NULL RECURRENT QUASI-BIRTH-DEATH PROCESSES*

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Abstract. The minimal nonnegative solution G of the matrix equation $G = A_0 + A_1 G + A_2 G^2$, where the matrices A_i $(i = 0, 1, 2)$ are nonnegative and $A_0 + A_1 + A_2$ is stochastic, plays an important role in the study of quasi-birth-death processes (QBDs). The Latouche-Ramaswami algorithm is a highly efficient algorithm for finding the matrix G. The convergence of the algorithm has been shown to be quadratic for positive recurrent QBDs and for transient QBDs. In this paper, we show that the convergence of the algorithm is linear with rate 1/2 for null recurrent QBDs under mild assumptions. This new result explains the experimental observation that the convergence of the algorithm is still quite fast for nearly null recurrent QBDs.

Key words. matrix equations, minimal nonnegative solution, Markov chains, cyclic reduction, iterative methods, convergence rate

AMS subject classifications. 15A24, 15A51, 60J10, 60K25, 65U05

1. Introduction.

A discrete-time quasi-birth-death process (QBD) is a Markov chain with state space $\{(i,j) \,|\, i \geq 0, 1 \leq j \leq m\}$, which has a transition probability matrix of the form

$$P = \begin{pmatrix} B_0 & B_1 & 0 & 0 & \cdots \\ A_0 & A_1 & A_2 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where B_0, B_1, A_0, A_1, and A_2 are $m \times m$ nonnegative matrices such that P is stochastic. In particular, $(A_0 + A_1 + A_2)e = e$, where e is the column vector with all components equal to one. The matrix P is also assumed to be irreducible. Thus, $A_0 \neq 0$ and $A_2 \neq 0$.

The matrix equation

$$G = A_0 + A_1 G + A_2 G^2 \tag{1.1}$$

plays an important role in the study of the QBD (see [12] and [16]). It is known that (1.1) has at least one solution in the set $\{G \geq 0 \mid Ge \leq e\}$ (i.e., the set of substochastic matrices). The desired solution G is the minimal nonnegative solution.

We assume that $A = A_0 + A_1 + A_2$ is irreducible. Then, by the Perron-Frobenius Theorem (see [17]), there exists a unique vector $\alpha > 0$ with $\alpha^T e = 1$ and $\alpha^T A = \alpha^T$. The vector α is called the stationary probability vector of A. By Theorem 7.2.3 in [12], the QBD is null recurrent if $\alpha^T A_0 e = \alpha^T A_2 e$; positive recurrent if $\alpha^T A_0 e > \alpha^T A_2 e$; and transient if $\alpha^T A_0 e < \alpha^T A_2 e$. For our purpose, we may use this criterion as an alternative definition for the three classes of QBDs.

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The minimal nonnegative solution of (1.1) can be found by any of the following three fixed-point iterations (see [3], [5], [9], [10], [14], [15], [18]):

$$G_{n+1} = A_0 + A_1 G_n + A_2 G_n^2, \quad G_0 = 0, \tag{1.2}$$

$$G_{n+1} = (I - A_1)^{-1}(A_0 + A_2 G_n^2), \quad G_0 = 0, \tag{1.3}$$

$$G_{n+1} = (I - A_1 - A_2 G_n)^{-1} A_0, \quad G_0 = 0. \tag{1.4}$$

Among the three iterations, iteration (1.4) has the fastest rate of convergence. An inversion free version of (1.4) has also been proposed in [1] and analysed in [1] and [5]. These four iterations are adequate for most situations. However, the convergence of all four iterations is sublinear when the QBD is null recurrent (see [5]). The convergence of these methods is also extremely slow if the QBD is nearly null recurrent.

The algorithm proposed by Latouche and Ramaswami [11] is a little more complicated. However, it works very well even for nearly null recurrent QBDs.

The algorithm is as follows:

ALGORITHM 1.1. *Set*

$$\begin{aligned} H_0 &= (I - A_1)^{-1} A_2; \\ L_0 &= (I - A_1)^{-1} A_0; \\ G_0 &= L_0; \\ T_0 &= H_0. \end{aligned}$$

For $k = 0, 1, \ldots,$ *compute*

$$\begin{aligned} U_k &= H_k L_k + L_k H_k; \\ H_{k+1} &= (I - U_k)^{-1} H_k^2; \\ L_{k+1} &= (I - U_k)^{-1} L_k^2; \\ G_{k+1} &= G_k + T_k L_{k+1}; \\ T_{k+1} &= T_k H_{k+1}. \end{aligned}$$

It is shown in [11] that the matrices H_k and L_k are well defined and nonnegative and that the sequence $\{G_k\}$ converges quadratically to the matrix G for positive recurrent QBDs and for transient QBDs. The algorithm is called a logarithmic reduction algorithm in [11]. We will call it the LR algorithm (for Logarithmic Reduction or for Latouche-Ramaswami). A similar method is proposed in [2] for positive recurrent QBDs.

Since the LR algorithm has the greatest advantage over the fixed-point iterations when the QBD is nearly null recurrent, it is important to know the convergence rate of the LR algorithm when the QBD is null recurrent.

Before we can determine the convergence rate, we will take a closer look into the LR algorithm and present some preliminary results.

2. Preliminaries. It was mentioned in [11] that G. W. Stewart pointed out that the LR algorithm is related to the cyclic reduction technique. We will make this point more transparent and derive two equations relating H_k and L_k.

Let G and F be the minimal nonnegative solution of (1.1) and

$$F = A_2 + A_1 F + A_0 F^2, \tag{2.1}$$

respectively. We have the following fundamental result (see [12], for example).

THEOREM 2.1. *If the QBD is positive recurrent, then G is stochastic and F is substochastic with spectral radius $\rho(F) < 1$. If the QBD is transient, then F is stochastic and G is substochastic with $\rho(G) < 1$. If the QBD is null recurrent, then G and F are both stochastic.*

It is clear that the matrix G is also the minimal nonnegative solution of $G = L_0 + H_0 G^2$. Thus, we have the infinite system

$$\begin{pmatrix} W_0 & -H_0 & & 0 \\ -L_0 & I & -H_0 & \\ & -L_0 & I & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I \\ G \\ G^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} K_0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{2.2}$$

for appropriate K_0 and W_0.

As in [2], we apply the cyclic reduction algorithm to (2.2) and get a reduced system. Multiplying both sides of the reduced system by a proper block diagonal matrix, we get an infinite system with the same structure as (2.2), but with G replaced by G^2. After repeated application of the cyclic reduction algorithm and the block diagonal scaling, we obtain for each $n \geq 0$,

$$\begin{pmatrix} W_n & -H_n & & 0 \\ -L_n & I & -H_n & \\ & -L_n & I & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I \\ G^{2^n} \\ G^{2\cdot 2^n} \\ \vdots \end{pmatrix} = \begin{pmatrix} K_n \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{2.3}$$

where H_n and L_n are as in the LR algorithm.

From equation (2.3), we have

$$-L_n + G^{2^n} - H_n G^{2\cdot 2^n} = 0 \tag{2.4}$$

for each $n \geq 0$. Therefore,

$$G = L_0 + H_0 G^2 = L_0 + H_0(L_1 + H_1 G^4) = L_0 + H_0 L_1 + H_0 H_1 (L_2 + H_2 G^8) = \cdots .$$

In general,

$$G = G_k + \Big(\prod_{0 \leq i \leq k} H_i \Big) G^{2\cdot 2^k}, \tag{2.5}$$

where G_k is as in the LR algorithm.

It is clear that the matrix F is also the minimal nonnegative solution of $F = H_0 + L_0 F^2$. By repeating the whole process leading to the equation (2.4), we get for each $n \geq 0$,

$$-H_n + F^{2^n} - L_n F^{2\cdot 2^n} = 0. \tag{2.6}$$

From (2.6), we can see that $H_n \leq F^{2^n}$ for each $n \geq 0$. Thus, we have by (2.5)

$$0 \leq G - G_k \leq F^{2\cdot 2^k - 1} G^{2\cdot 2^k}. \tag{2.7}$$

Therefore, if the QBD is positive recurrent or transient, the quadratic convergence of $\{G_k\}$ is an immediate consequence of Theorem 2.1. In this situation, it is also very easy to determine the limits of the sequences $\{H_k\}$ and $\{L_k\}$. The following result is necessary.

THEOREM 2.2. *Let Q be a stochastic matrix. If Q^r has a positive column for some integer $r \geq 1$, then there is a unique vector $q \geq 0$ such that $q^T Q = q^T$ and $q^T e = 1$ (the vector q is called the stationary probability vector of Q). Moreover, there are constants $K > 0$ and $\beta \in (0, 1)$ such that*

$$\|Q^n - eq^T\| \leq K\beta^n$$

for all $n \geq 0$. In particular, $\lim_{n\to\infty} Q^n = eq^T$.

For a proof of this result, see [6]. See also [7] for the special case when Q^r is positive for some integer $r \geq 1$. Obviously, the condition that Q^r has a positive column for some $r \geq 1$ is necessary for $\lim_{n\to\infty} Q^n = eq^T$.

If the QBD is positive recurrent, then G is stochastic and $\rho(F) < 1$. Assuming that G^p has a positive column for some integer $p \geq 1$, we get from (2.4) and (2.6) that $\lim_{n\to\infty} H_n = 0$ and $\lim_{n\to\infty} L_n = eg^T$, where g is the stationary probability vector of G. If the QBD is transient, then $\rho(G) < 1$ and F is stochastic. Assuming that F^p has a positive column for some integer $p \geq 1$, we have $\lim_{n\to\infty} L_n = 0$ and $\lim_{n\to\infty} H_n = ef^T$, where f is the stationary probability vector of F. The limits of $\{H_n\}$ and $\{L_n\}$ were determined in [11] in a different way.

If the QBD is null recurrent, then $\rho(G) = 1$ and $\rho(F) = 1$. In this case, (2.7) tells us nothing about the convergence rate of the LR algorithm. It is also much more difficult to determine the limits of the sequences $\{H_n\}$ and $\{L_n\}$. These issues will be resolved in the next section.

3. Convergence rate of the LR algorithm for the null recurrent case.

We start with an algebraic proof of a basic result about the LR algorithm. An probabilistic proof was given in [11].

LEMMA 3.1. *For each $k \geq 0$, $(H_k + L_k)e = e$.*

Proof. First, $(H_0+L_0)e = (I-A_1)^{-1}(A_0+A_2)e = e$. Assuming that $(H_k+L_k)e = e\,(k \geq 0)$, we have $(H_k+L_k)^2 e = e$. So, $(I-H_kL_k-L_kH_k)e = (H_k^2+L_k^2)e$. Therefore, $(H_{k+1} + L_{k+1})e = (I - H_kL_k - L_kH_k)^{-1}(H_k^2 + L_k^2)e = e$. We have thus proved the result by induction. ∎

In the above proof, we have used the fact that the sequences $\{H_k\}$ and $\{L_k\}$ are well defined (i.e., the matrices $I - H_kL_k - L_kH_k$ are nonsingular).

It is noted in [11] that, when the QBD is null recurrent, it is not true in general that one of the two sequences $\{H_k\}$ and $\{L_k\}$ converges to 0. Our next result shows that neither of the two sequences can converge to 0 for null recurrent QBDs.

LEMMA 3.2. *For the null recurrent QBD, there is a sequence $\{\alpha_k\}$ such that for all $k \geq 0$, $\alpha_k \geq 0$, $\alpha_k^T e = 1$, $\alpha_k^T(H_k + L_k) = \alpha_k^T$, and $\alpha_k^T H_k e = \alpha_k^T L_k e = \frac{1}{2}$.*

Proof. Recall that α is the stationary probability vector of $A_0 + A_1 + A_2$. So,

$$\alpha^T(I - A_1) = \alpha^T(A_0 + A_2) = \alpha^T(I - A_1)(H_0 + L_0).$$

Let $\hat{\alpha}^T = \alpha^T(I - A_1)$. Since $\alpha > 0$ and $A_0 \neq 0$, $\hat{\alpha}^T = \alpha^T(A_0 + A_2) \geq 0$ and $c_0 = \hat{\alpha}^T e > 0$. Since the QBD is null recurrent, we have $\alpha^T A_2 e = \alpha^T A_0 e$ and thus $\hat{\alpha}^T H_0 e = \hat{\alpha}^T L_0 e$. Let $\alpha_0 = \hat{\alpha}/c_0$. It is clear that α_0 has all the properties in the lemma, noting that $\alpha_0^T H_0 e + \alpha_0^T L_0 e = \alpha_0^T e = 1$.

Assuming that an α_i $(i \geq 0)$, with all the properties in the lemma, has been found, we are going to find an α_{i+1} satisfying these properties.

Since

$$\alpha_i^T = \alpha_i^T(H_i + L_i) = \alpha_i^T(H_i + L_i)^2 = \alpha_i^T(H_i^2 + L_i^2 + H_iL_i + L_iH_i),$$

we have

$$\alpha_i^T(I - H_iL_i - L_iH_i) = \alpha_i^T(H_i^2 + L_i^2) = \alpha_i^T(I - H_iL_i - L_iH_i)(H_{i+1} + L_{i+1}).$$

Since $\alpha_i \neq 0$ and $I - H_iL_i - L_iH_i$ is nonsingular, $\alpha_i^T(I - H_iL_i - L_iH_i) = \alpha_i^T(H_i^2 + L_i^2) \neq 0$. Thus, $\alpha_i^T(H_i^2 + L_i^2)e > 0$ and we can define

$$\alpha_{i+1}^T = \frac{\alpha_i^T(H_i^2 + L_i^2)}{\alpha_i^T(H_i^2 + L_i^2)e} = \frac{\alpha_i^T(I - H_iL_i - L_iH_i)}{\alpha_i^T(H_i^2 + L_i^2)e}.$$

It remains to prove $\alpha_{i+1}^T H_{i+1}e = \alpha_{i+1}^T L_{i+1}e$, which is equivalent to $\alpha_i^T H_i^2 e = \alpha_i^T L_i^2 e$. Note that

$$\begin{aligned}
\alpha_i^T H_i^2 e - \alpha_i^T L_i^2 e &= \alpha_i^T H_i(e - L_ie) - \alpha_i^T L_i(e - H_ie) \\
&= -\alpha_i^T H_iL_ie + \alpha_i^T L_iH_ie \\
&= -\alpha_i^T(I - L_i)L_ie + \alpha_i^T(I - H_i)H_ie \\
&= \alpha_i^T L_i^2 e - \alpha_i^T H_i^2 e.
\end{aligned}$$

Thus, $\alpha_i^T H_i^2 e = \alpha_i^T L_i^2 e$. □

Remark 3.1. The result in the above lemma has also been obtained independently by Ye [19]. In [19] it is assumed that $\alpha_i^T(H_i^2 + L_i^2)e \neq 0$ for each i. In the first version of this paper, the author used the assumption that $(H_i^2 + L_i^2)e > 0$ for each i. Without this assumption, the short argument (in the proof of the lemma) showing $\alpha_i^T(H_i^2 + L_i^2)e > 0$ for each i was pointed out by two referees.

Our further analysis will rely on Theorem 2.2. In order to apply Theorem 2.2, we make the following assumption:

(3.1) $\det(A_0 + zA_1 + z^2A_2 - zI)$ has no zeros on the unit circle other than $z = 1$.

This assumption may be verified easily when the matrices A_0, A_1, A_2 have special structures (see [13], for example). From [4] we know that, in the null recurrent case, assumption (3.1) is equivalent to the assumption that $\lambda = 1$ is a simple eigenvalue of G and F and there are no other eigenvalues of G or F on the unit circle. It is easy to show that the latter assumption is in turn equivalent to the next assumption.

(3.2) G^p and F^q have each a positive column for some $p \geq 1$ and some $q \geq 1$.

Note that assumption (3.2) for G is satisfied if G_k in the LR algorithm has a positive column for some $k \geq 0$, since $G \geq G_k$. In particular, assumption (3.2) for G is satisfied if L_0 has a positive column. Similar comments can be made on assumption (3.2) for F. Since assumptions (3.1) and (3.2) are equivalent, Theorem 2.2 can be applied to G and F under assumption (3.1). We let f and g be the unique stationary probability vector of F and G, respectively.

Since $(H_k + L_k)e = e$ for all $k \geq 0$, the sequences $\{H_k\}$ and $\{L_k\}$ are bounded and hence have convergent subsequences. Let $\{H_{n_k}\}$ and $\{L_{n_k}\}$ be convergent with

$$\lim_{k\to\infty} H_{n_k} = H, \quad \lim_{k\to\infty} L_{n_k} = L.$$

Then, by equations (2.4) and (2.6) and Theorem 2.2,

$$-L + eg^T - Heg^T = 0, \quad -H + ef^T - Lef^T = 0.$$

Therefore, $H = af^T$ with $a = e - Le$, and $L = bg^T$ with $b = e - He$. Note that $a + b = 2e - (H + L)e = e$.

We have thus proved the following result.

LEMMA 3.3. *For the null recurrent QBD with assumption* (3.1)*, if* (H, L) *is a limit point of* $\{(H_k, L_k)\}$*, then* $H = af^T$ *and* $L = bg^T$ *with* $a \geq 0$*,* $b \geq 0$*, and* $a + b = e$*.*

To prove that the convergence of the LR algorithm is linear with rate 1/2, we will need to show that

$$\lim_{k\to\infty} H_k = \frac{1}{2}ef^T, \quad \lim_{k\to\infty} L_k = \frac{1}{2}eg^T.$$

Lemma 3.3 is only one small step towards this goal. Many other auxiliary results will be needed.

Although we are unable to show the convergence of the sequences $\{H_k\}$ and $\{L_k\}$ at the moment, it is fairly easy to show that the sequence $\{\alpha_k\}$ in Lemma 3.2 converges.

LEMMA 3.4. *For the null recurrent QBD with assumption* (3.1)*,*

$$\lim_{k\to\infty} \alpha_k = \frac{1}{2}(f + g).$$

Proof. Let α^* be any limit point of $\{\alpha_k\}$ and $\lim_{k\to\infty} \alpha_{n_k} = \alpha^*$. We will prove that $\alpha^* = \frac{1}{2}(f + g)$. We may assume without loss of generality that

$$\lim_{k\to\infty} H_{n_k} = af^T, \quad \lim_{k\to\infty} L_{n_k} = bg^T$$

for some $a, b \geq 0$ with $a + b = e$. By taking limits in

$$\alpha_{n_k}^T = \alpha_{n_k}^T (H_{n_k} + L_{n_k}), \quad \alpha_{n_k}^T H_{n_k} e = \alpha_{n_k}^T L_{n_k} e,$$

we get

$$(\alpha^*)^T = (\alpha^*)^T (af^T + bg^T), \quad (\alpha^*)^T a = (\alpha^*)^T b.$$

Thus, $(\alpha^*)^T a = (\alpha^*)^T (e - a) = 1 - (\alpha^*)^T a$. So, $(\alpha^*)^T a = (\alpha^*)^T b = 1/2$ and $(\alpha^*)^T = \frac{1}{2}(f^T + g^T)$, or $\alpha^* = \frac{1}{2}(f + g)$. □

As we have already seen, in the null recurrent case, the two equations (2.4) and (2.6) are not sufficient to determine the convergence of the sequences $\{H_n\}$ and $\{L_n\}$. We have to seek additional information from the recursions for the sequences $\{H_n\}$ and $\{L_n\}$. The next result is one such finding.

LEMMA 3.5. *For the null recurrent QBD with assumption* (3.1)*, if*

$$\lim_{k\to\infty} H_{n_k} = af^T, \quad \lim_{k\to\infty} L_{n_k} = bg^T,$$

and $g^T a \neq 1$*, then*

$$\lim_{k\to\infty} H_{n_k+1} = \hat{a}f^T, \quad \lim_{k\to\infty} L_{n_k+1} = \hat{b}g^T,$$

with

$$\hat{a} = \frac{1+g^Ta}{1+2g^Ta}a + \frac{g^Ta}{1+2g^Ta}b, \quad \hat{b} = \frac{g^Ta}{1+2g^Ta}a + \frac{1+g^Ta}{1+2g^Ta}b.$$

Proof. Let $(\tilde{a}f^T, \tilde{b}g^T)$ be any limit point of $\{(H_{n_k+1}, L_{n_k+1})\}$ and let

$$\lim_{k\to\infty} H_{n_{s_k}+1} = \tilde{a}f^T, \quad \lim_{k\to\infty} L_{n_{s_k}+1} = \tilde{b}g^T.$$

Since

$$(I - H_{n_{s_k}}L_{n_{s_k}} - L_{n_{s_k}}H_{n_{s_k}})H_{n_{s_k}+1} = (H_{n_{s_k}})^2,$$

we get by letting $k \to \infty$,

$$(I - af^Tbg^T - bg^Taf^T)\tilde{a}f^T = af^Taf^T.$$

Post-multiplying the above equality by e gives

$$\tilde{a} = (f^Ta + f^Tbg^T\tilde{a})a + (g^Taf^T\tilde{a})b \equiv \lambda a + \mu b.$$

By Lemma 3.2,

$$\alpha_{n_k}^T H_{n_k} e = \alpha_{n_k}^T L_{n_k} e = \frac{1}{2}, \quad (\alpha_{n_{s_k}+1})^T H_{n_{s_k}+1} e = (\alpha_{n_{s_k}+1})^T L_{n_{s_k}+1} e = \frac{1}{2}.$$

By taking limits in the above identities and using Lemma 3.4, we have

$$(f^T+g^T)a = (f^T+g^T)b = (f^T+g^T)\tilde{a} = (f^T+g^T)\tilde{b} = 1.$$

So, $f^Ta = 1 - g^Ta = g^Te - g^Ta = g^Tb$. Similarly, $f^Tb = g^Ta$, $f^T\tilde{a} = g^T\tilde{b}$, $f^T\tilde{b} = g^T\tilde{a}$. Thus,

$$\lambda + \mu = f^Ta + f^Tbg^T\tilde{a} + g^Taf^T\tilde{a} = f^Ta + f^Tb(g^T\tilde{a} + f^T\tilde{a}) = f^Ta + f^Tb = f^Te = 1.$$

Now,

$$\begin{aligned}\mu &= g^Taf^T\tilde{a} = g^Taf^T(\lambda a + \mu b)\\ &= (1-\mu)g^Taf^Ta + \mu g^Taf^Tb = (1-\mu)g^Ta(1-g^Ta) + \mu(g^Ta)^2.\end{aligned}$$

Thus,

$$(1+2g^Ta)(1-g^Ta)\mu = g^Ta(1-g^Ta).$$

Since $g^Ta \neq 1$, we have $\mu = g^Ta/(1+2g^Ta)$ and $\lambda = 1-\mu = (1+g^Ta)/(1+2g^Ta)$. So,

$$\tilde{a} = \frac{1+g^Ta}{1+2g^Ta}a + \frac{g^Ta}{1+2g^Ta}b,$$

and

$$\tilde{b} = e - \tilde{a} = a + b - \tilde{a} = \frac{g^Ta}{1+2g^Ta}a + \frac{1+g^Ta}{1+2g^Ta}b.$$

The proof is completed since the limit point is uniquely determined by a and b. ☐

We can now move a little closer to our goal.

LEMMA 3.6. *For the null recurrent QBD with assumptions* (3.1) *and*

(3.3) *Each limit point* af^T *of the sequence* $\{H_n\}$ *is such that* $0 < g^T a < 1$,

the sequence $\{(H_n, L_n)\}$ *has a limit point* $(\frac{1}{2}ef^T, \frac{1}{2}eg^T)$.

Proof. Take any subsequence $\{(H_{n_k}, L_{n_k})\}$ such that

$$\lim_{k\to\infty}(H_{n_k}, L_{n_k}) = (a_0 f^T, b_0 g^T).$$

By the previous lemma, for each integer $r \geq 1$,

$$\lim_{k\to\infty}(H_{n_k+r}, L_{n_k+r}) = (a_r f^T, b_r g^T),$$

where

$$a_{k+1} = \frac{1 + g^T a_k}{1 + 2g^T a_k} a_k + \frac{g^T a_k}{1 + 2g^T a_k} b_k, \tag{3.4}$$

and $b_{k+1} = e - a_{k+1}$ for each integer $k \geq 0$. Let $p_k = g^T a_k$. We have by (3.4)

$$p_{k+1} = \frac{(1 + p_k)p_k}{1 + 2p_k} + \frac{p_k(1 - p_k)}{1 + 2p_k} = \frac{2p_k}{1 + 2p_k}.$$

Since $p_0 = g^T a_0 > 0$ by assumption, it is easy to show that $\lim_{k\to\infty} p_k = \frac{1}{2}$. By (3.4) we have

$$a_{k+1} = \frac{g^T a_k}{1 + 2g^T a_k} e + \frac{1}{1 + 2g^T a_k} a_k,$$

which can be rewritten as

$$a_{k+1} - \frac{1}{2}e = \frac{1}{1 + 2g^T a_k}\big(a_k - \frac{1}{2}e\big).$$

Since $\lim_{k\to\infty} g^T a_k = \frac{1}{2}$, we have

$$\Big|a_{k+1} - \frac{1}{2}e\Big| \leq \frac{2}{3}\,\Big|a_k - \frac{1}{2}e\Big|$$

for k large enough. Thus

$$\lim_{r\to\infty} a_r = \lim_{r\to\infty} b_r = \frac{1}{2}e.$$

Therefore, we can find a subsequence $\{(H_{m_k}, L_{m_k})\}$ such that

$$\lim_{k\to\infty}(H_{m_k}, L_{m_k}) = \big(\frac{1}{2}ef^T, \frac{1}{2}eg^T\big).$$

This completes the proof. ☐

The next result is quite straightforward.

LEMMA 3.7. *Let the relation between* (H_{k+1}, L_{k+1}) *and* (H_k, L_k) *in the LR algorithm be denoted by*

$$(H_{k+1}, L_{k+1}) = \mathcal{T}(H_k, L_k).$$

Then $(\frac{1}{2}ef^T, \frac{1}{2}eg^T)$ *is a fixed point of* $\mathcal{T}$.

Proof. It is easy to verify that

$$\big(I - \frac{1}{2}ef^T\frac{1}{2}eg^T - \frac{1}{2}eg^T\frac{1}{2}ef^T\big)\frac{1}{2}ef^T = \big(\frac{1}{2}ef^T\big)^2,$$
$$\big(I - \frac{1}{2}ef^T\frac{1}{2}eg^T - \frac{1}{2}eg^T\frac{1}{2}ef^T\big)\frac{1}{2}eg^T = \big(\frac{1}{2}eg^T\big)^2.$$

The result follows since

$$M \equiv I - \frac{1}{2}ef^T\frac{1}{2}eg^T - \frac{1}{2}eg^T\frac{1}{2}ef^T = I - \frac{1}{4}e(f^T + g^T)$$

is a nonsingular M-matrix (note that $Me = e/2$). □

Thus, we have shown that the sequence $\{(H_n, L_n)\}$ defined by $(H_{k+1}, L_{k+1}) = \mathcal{T}(H_k, L_k)\,(k \geq 0)$ has a limit point $(\frac{1}{2}ef^T, \frac{1}{2}eg^T)$ that is a fixed point of $\mathcal{T}$. By a theorem on general fixed-point iterations (see [8, p. 21], for example), we can conclude that the whole sequence $\{(H_n, L_n)\}$ converges to this fixed point *if* the spectral radius of the Fréchet derivative of the operator $\mathcal{T}$ at the fixed point is less than 1. But, unfortunately, the spectral radius is *not* less than 1 in our case (the spectral radius is equal to 4 when the matrices A_0, A_1, A_2 are 1×1). However, the sequence $\{(H_n, L_n)\}$ can still converge since (H_n, L_n) may approach $(\frac{1}{2}ef^T, \frac{1}{2}eg^T)$ in a special way. A delicate analysis for the error $(H_n - \frac{1}{2}ef^T, L_n - \frac{1}{2}eg^T)$ is necessary.

For notational convenience, let $H = \frac{1}{2}ef^T$ and $L = \frac{1}{2}eg^T$. It is easy to see that

$$H^2 = LH = \frac{1}{2}H, \quad L^2 = HL = \frac{1}{2}L.$$

We start with expressing $H_{k+1} - H$ in terms of $H_k - H$ and $L_k - L$:

$$\begin{aligned}
H_{k+1} - H &= (I - H_kL_k - L_kH_k)^{-1}H_k^2 - (I - HL - LH)^{-1}H^2 \\
&= (I - H_kL_k - L_kH_k)^{-1}(H_k^2 - H^2) \\
&\quad + \big((I - H_kL_k - L_kH_k)^{-1} - (I - HL - LH)^{-1}\big)H^2 \\
&= (I - H_kL_k - L_kH_k)^{-1}(H_k^2 - H^2) + (I - H_kL_k - L_kH_k)^{-1} \\
&\quad \big((I - HL - LH) - (I - H_kL_k - L_kH_k)\big)(I - HL - LH)^{-1}H^2 \\
&= (I - H_kL_k - L_kH_k)^{-1}\big(H_k^2 - H^2 + (H_kL_k - HL + L_kH_k - LH)H\big) \\
&= (I - H_kL_k - L_kH_k)^{-1}\big(H_k(H_k - H) + (H_k - H)H \\
&\quad + (H_k(L_k - L) + (H_k - H)L + L_k(H_k - H) + (L_k - L)H)H\big).
\end{aligned}$$

To simplify the expression, observe that

$$(H_k - H + L_k - L)H = \frac{1}{2}\big((H_k + L_k)e - (H + L)e\big)f^T = 0.$$

Thus,

$$(L_k - L)H = -(H_k - H)H$$

and

$$\big((H_k - H)L + (L_k - L)H\big)H = \frac{1}{2}(H_k - H + L_k - L)H = 0.$$

Therefore,

$$H_{k+1}-H=(I-H_kL_k-L_kH_k)^{-1}\big(H_k(H_k-H)+(H_k-H)H \\ -H_k(H_k-H)H+L_k(H_k-H)H\big).$$

Similarly, we can get

$$L_{k+1}-L=(I-H_kL_k-L_kH_k)^{-1}\big(L_k(L_k-L)+(L_k-L)L \\ -L_k(L_k-L)L+H_k(L_k-L)L\big).$$

Now, for any $\epsilon\in(0,\frac{1}{4})$, we can find $\delta>0$ such that whenever $\|H_k-H\|_\infty\le\delta$ and $\|L_k-L\|_\infty\le\delta$,

$$H_{k+1}-H=(I-HL-LH)^{-1}\big(H(H_k-H)+(H_k-H)H \\ -H(H_k-H)H+L(H_k-H)H\big)+W_k \tag{3.5}$$

with $\|W_k\|_\infty\le\epsilon\|H_k-H\|_\infty$, and

$$L_{k+1}-L=(I-HL-LH)^{-1}\big(L(L_k-L)+(L_k-L)L \\ -L(L_k-L)L+H(L_k-L)L\big)+Z_k$$

with $\|Z_k\|_\infty\le\epsilon\|L_k-L\|_\infty$.

To get rid of the inverse in (3.5), we use

$$(I-HL-LH)^{-1}=\big(I-\frac{1}{2}(H+L)\big)^{-1}=I+\frac{1}{2}(H+L)+\frac{1}{2^2}(H+L)^2+\cdots. \tag{3.6}$$

Since

$$\begin{aligned}&(H+L)\big(H(H_k-H)+(H_k-H)H-H(H_k-H)H+L(H_k-H)H\big)\\ &=H(H_k-H)+(H+L)(H_k-H)H-H(H_k-H)H+L(H_k-H)H\\ &=H(H_k-H)+2L(H_k-H)H,\end{aligned}$$

and

$$(H+L)^i\big(H(H_k-H)+2L(H_k-H)H\big)=H(H_k-H)+2L(H_k-H)H$$

for all $i\ge 1$, we get by (3.5) and (3.6) that

$$\begin{aligned}H_{k+1}-H&=H(H_k-H)+(H_k-H)H-H(H_k-H)H+L(H_k-H)H\\ &\quad+H(H_k-H)+2L(H_k-H)H+W_k\\ &=(H_k-H)H+2H(H_k-H)-H(H_k-H)H+3L(H_k-H)H+W_k.\end{aligned}$$

Similarly,

$$L_{k+1}-L=(L_k-L)L+2L(L_k-L)-L(L_k-L)L+3H(L_k-L)L+Z_k.$$

For the scalar case, $H=L=\frac{1}{2}$. So, the estimate for $H_{k+1}-H$ becomes $H_{k+1}-H\approx 2(H_k-H)$. If we could replace $3L(H_k-H)H$ by $-3H(H_k-H)H$ in the estimate, we would have $H_{k+1}-H\approx\frac{1}{2}(H_k-H)$ instead. Thus, we should try to show that $3(H+L)(H_k-H)H$ is "small" in the general case. Let $\alpha^*=(f+g)/2$. We have

$$3(H+L)(H_k-H)H=\frac{3}{2}e(\alpha^*)^T(H_k-H)ef^T=\frac{3}{2}e(\alpha^*-\alpha_k)^T(H_k-H)ef^T,$$

since $\alpha_k^T(H_k - H)e = \frac{1}{2} - \alpha_k^T(\frac{1}{2}e) = 0$ by Lemma 3.2. Similarly,

$$3(H+L)(L_k - L)L = \frac{3}{2}e(\alpha^* - \alpha_k)^T(L_k - L)eg^T.$$

Since $\lim \alpha_k = \alpha^*$ by Lemma 3.4, we can find integer k_1 such that for all $k \geq k_1$,

$$3(H+L)(H_k - H)H = P_k, \quad 3(H+L)(L_k - L)L = Q_k$$

with $\|P_k\|_\infty \leq \epsilon \|H_k - H\|_\infty$ and $\|Q_k\|_\infty \leq \epsilon \|L_k - L\|_\infty$. Now we have

$$\begin{aligned} H_{k+1} - H &= (H_k - H)H + 2H(H_k - H) - 4H(H_k - H)H + P_k + W_k \\ &= \frac{1}{2}(H_k - H) - \frac{1}{2}(I - 4H)(H_k - H)(I - 2H) + P_k + W_k, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} L_{k+1} - L &= (L_k - L)L + 2L(L_k - L) - 4L(L_k - L)L + Q_k + Z_k \\ &= \frac{1}{2}(L_k - L) - \frac{1}{2}(I - 4L)(L_k - L)(I - 2L) + Q_k + Z_k. \end{aligned} \tag{3.8}$$

Next we will estimate the term $(H_k - H)(I - 2H)$ in (3.7) and the term $(L_k - L)(I - 2L)$ in (3.8). By the equations (2.4) and (2.6), we have

$$H_k(I - G^{2\cdot 2^k}F^{2\cdot 2^k}) = F^{2^k} - G^{2^k}F^{2\cdot 2^k}, \quad L_k(I - F^{2\cdot 2^k}G^{2\cdot 2^k}) = G^{2^k} - F^{2^k}G^{2\cdot 2^k}.$$

Now,

$$\begin{aligned} (H_k - H)(I - 2H) &= H_k(I - ef^T) \\ &= F^{2^k} - G^{2^k}F^{2\cdot 2^k} + H_k(G^{2\cdot 2^k}F^{2\cdot 2^k} - ef^T) \\ &= F^{2^k} - ef^T - (G^{2^k} - eg^T)F^{2\cdot 2^k} - eg^T(F^{2\cdot 2^k} - ef^T) \\ &\quad + H_k\big((G^{2\cdot 2^k} - eg^T)F^{2\cdot 2^k} + eg^T(F^{2\cdot 2^k} - ef^T)\big). \end{aligned}$$

Similarly,

$$\begin{aligned} (L_k - L)(I - 2L) &= G^{2^k} - eg^T - (F^{2^k} - ef^T)G^{2\cdot 2^k} - ef^T(G^{2\cdot 2^k} - eg^T) \\ &\quad + L_k\big((F^{2\cdot 2^k} - ef^T)G^{2\cdot 2^k} + ef^T(G^{2\cdot 2^k} - eg^T)\big). \end{aligned}$$

By Theorem 2.2, there are constants $C_1 > 0$ and $\beta \in (0,1)$ such that

$$\|(H_k - H)(I - 2H)\|_\infty \leq C_1\beta^{2^k}, \quad \|(L_k - L)(I - 2L)\|_\infty \leq C_1\beta^{2^k}$$

for all $k \geq 0$. Now, by (3.7) and (3.8), we have

$$\|H_{k+1} - H\|_\infty \leq (\frac{1}{2} + 2\epsilon)\|H_k - H\|_\infty + C_2\beta^{2^k}, \tag{3.9}$$

$$\|L_{k+1} - L\|_\infty \leq (\frac{1}{2} + 2\epsilon)\|L_k - L\|_\infty + C_2\beta^{2^k} \tag{3.10}$$

for any $k \geq k_1$ with $\|H_k - H\|_\infty < \delta$ and $\|L_k - L\|_\infty < \delta$. Let $r = \frac{1}{2} + 2\epsilon < 1$. Since (H, L) is a limit point of $\{(H_k, L_k)\}$ by Lemma 3.6, we can find $l \geq k_1$ such that $\|H_l - H\|_\infty < \delta$, $\|L_l - L\|_\infty < \delta$, $r\delta + C_2\beta^{2^l} < \delta$, and $\beta^{2^l} \leq r$. Now, it is clear that

(3.9) and (3.10) are valid for all $k \geq l$ and that $\beta^{2^{l+j-1}} \leq r^j$ for all $j \geq 0$. Thus, we can obtain for any $m \geq 1$ that

$$\begin{aligned}\|H_{l+m} - H\|_\infty &\leq r^m \|H_l - H\|_\infty + C_2(r^{m-1}\beta^{2^l} + r^{m-2}\beta^{2^{l+1}} + \cdots + \beta^{2^{l+m-1}}) \\ &\leq r^m \|H_l - H\|_\infty + C_2 m r^m,\end{aligned}$$

and that

$$\|L_{l+m} - L\|_\infty \leq r^m \|L_l - L\|_\infty + C_2 m r^m.$$

Therefore, $\lim_{k\to\infty} H_k = H$ and $\lim_{k\to\infty} L_k = L$. Moreover, since $\epsilon > 0$ can be arbitrarily small, we also have

$$\limsup_{k\to\infty} \sqrt[k]{\|H_k - H\|_\infty} \leq \frac{1}{2}, \quad \limsup_{k\to\infty} \sqrt[k]{\|L_k - L\|} \leq \frac{1}{2}.$$

In summary, we have proved the following result.

THEOREM 3.8. *For the null recurrent QBD with assumptions* (3.1) *and* (3.3), *we have*

$$\lim_{k\to\infty} H_k = \frac{1}{2} e f^T, \quad \lim_{k\to\infty} L_k = \frac{1}{2} e g^T.$$

It is clear that assumption (3.3) is necessary for the conclusion of the above theorem. Since the assumption cannot be verified directly, we will give a sufficient condition that is easier to verify.

PROPOSITION 3.9. *Let the components of f and g be f_i and g_i $(i = 1, 2, \ldots, m)$, respectively, and let*

$$S_f = \{i \,|\, 1 \leq i \leq m, f_i = 0\}, \quad S_g = \{i \,|\, 1 \leq i \leq m, g_i = 0\}.$$

If assumption (3.1) *and the assumption that*

$$S_f \subset S_g \text{ or } S_g \subset S_f \tag{3.11}$$

are satisfied, then assumption (3.3) *is also satisfied.*

Proof. Let

$$\lim_{k\to\infty} H_{n_k} = a f^T, \quad \lim_{k\to\infty} L_{n_k} = b g^T.$$

It is shown in the proof of Lemma 3.5 that

$$(f^T + g^T)a = (f^T + g^T)b = 1, \quad f^T a = g^T b, \quad f^T b = g^T a.$$

If $g^T a = 1$, then $g^T b = f^T a = 0$. By assumption (3.11), we would have $(f^T + g^T)a = 0$ or $(f^T + g^T)b = 0$, which is a contradiction. Similarly, we get a contradiction if $g^T a = 0$. ∎

Remark 3.2. Assumption (3.11) is certainly satisfied if one of F and G is irreducible (in particular, if one of H_0 and L_0 is irreducible) since one of S_f and S_g is an empty set in this case.

We are now ready to determine the convergence rate of the LR algorithm for the null recurrent case. Recall that, for the sequence $\{G_k\}$ generated by the LR algorithm,

$$G - G_k = H_0 H_1 \cdots H_k G^{2^{k+1}}. \tag{3.12}$$

PROPOSITION 3.10. *For each* $k \geq 0$, $H_0 H_1 \cdots H_k \neq 0$.

Proof. Equation (8.47) in [12] states that $(H_0 H_1 \cdots H_k)_{ij}$ is the probability of first passage to the state $(2^{k+1}, j)$ before any of the states $(0, \cdot)$ starting from $(1, i)$. If all of these entries were equal to 0, it would be impossible to reach the states $(2^{k+1}, \cdot)$ and the transition probability matrix P would not be irreducible. □

Remark 3.3. The above proof is provided by a referee. In the first version of the paper, the author gave the statement in Proposition 3.10 as an assumption.

The next theorem is our main result. It shows that the sequence $\{G_k\}$ converges to the minimal nonnegative solution of (1.1) at precisely the rate of 1/2.

THEOREM 3.11. *For the null recurrent QBD with assumptions* (3.1) *and* (3.3),

$$\lim_{k\to\infty} \sqrt[k]{\|G_k - G\|_\infty} = \frac{1}{2}.$$

Proof. Since $\lim_{k\to\infty} H_k = \frac{1}{2} e f^T$, we have $\lim_{k\to\infty} H_k e = \frac{1}{2} e$. Therefore, for any $\epsilon \in (0, \frac{1}{2})$, we can find an integer k_0 such that $(\frac{1}{2} - \epsilon) e \leq H_k e \leq (\frac{1}{2} + \epsilon) e$ for all $k > k_0$.

Note that, by (3.12),

$$\|G - G_k\|_\infty = \|(G - G_k) e\|_\infty = \|H_0 \cdots H_{k_0} H_{k_0+1} \cdots H_k e\|_\infty$$

for $k > k_0$. Thus,

$$(\frac{1}{2} - \epsilon)^{k-k_0} \|H_0 \cdots H_{k_0} e\|_\infty \leq \|G - G_k\|_\infty \leq (\frac{1}{2} + \epsilon)^{k-k_0} \|H_0 \cdots H_{k_0} e\|_\infty.$$

In view of Proposition 3.10, it follows readily that

$$\limsup_{k\to\infty} \sqrt[k]{\|G_k - G\|_\infty} \leq \frac{1}{2} + \epsilon, \quad \liminf_{k\to\infty} \sqrt[k]{\|G_k - G\|_\infty} \geq \frac{1}{2} - \epsilon.$$

Since ϵ can be arbitrarily small, we have $\lim_{k\to\infty} \sqrt[k]{\|G_k - G\|_\infty} = \frac{1}{2}$. □

4. Improvement of the approximate solution in the null recurrent case. By (3.12) and Theorem 3.8, it is easy to get a much better approximation to the matrix G from the sequence $\{G_k\}$ generated by the LR algorithm. In fact, we have by (3.12)

$$G - G_k - 2(G - G_{k+1}) = H_0 \cdots H_k (G^{2^{k+1}} - G^{2^{k+2}}) + H_0 \cdots H_{k-1} (H_k - 2 H_k H_{k+1}) G^{2^{k+2}}.$$

The first term converges to zero quadratically by Theorem 2.2 since $G^{2^{k+1}} - G^{2^{k+2}} = (G^{2^{k+1}} - e g^T) - (G^{2^{k+2}} - e g^T)$. The second term is also much smaller than $G - G_{k+1}$ since $\lim_{k\to\infty} (H_k - 2 H_k H_{k+1}) = 0$ and $\lim_{k\to\infty} (H_k H_{k+1}) = \frac{1}{4} e f^T$ by Theorem 3.8. Therefore, $\tilde{G}_{k+1} = 2 G_{k+1} - G_k = G_{k+1} + (G_{k+1} - G_k)$ can be a much better approximation to G in the null recurrent case. Of course, improvements may also be achieved for nearly null recurrent QBDs by using the above strategy.

5. Examples. In this section, we will present a few examples to illustrate the theoretical results in Section 3 and the simple strategy described in Section 4 for the improvement of the approximate solution. For all examples, assumption (3.1) is checked through the equivalent assumption (3.2).

Example 5.1. Consider the equation (1.1) with

$$A_0 = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0 & 0.1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0.26 & 0 & 0.24 \\ 0.2 & 0 & 0.8 \end{pmatrix}.$$

It is easy to verify that the corresponding QBD is null recurrent. We also find that $G_1 = L_0 + H_0 L_1$ is irreducible and has a positive column and that $F_1 = H_0 + L_0 H_1$ has a positive column. Since $G \geq G_1$ and $F \geq F_1$, assumptions (3.1) and (3.11) are satisfied. By Proposition 3.9, assumption (3.3) is also satisfied.

For this example, the exact minimal nonnegative solutions of (1.1) and (2.1) can be found to be

$$G = \frac{1}{2300} \begin{pmatrix} 1380 & 920 & 0 \\ 1320 & 680 & 300 \\ 1357 & 828 & 115 \end{pmatrix}, \quad F = \frac{1}{25} \begin{pmatrix} 5 & 0 & 20 \\ 9 & 0 & 16 \\ 5 & 0 & 20 \end{pmatrix}.$$

Accordingly, we have

$$g^T = (1431, 874, 120)/2425, \quad f^T = (1, 0, 4)/5.$$

For the matrices H_{18} and L_{18}, found by the LR algorithm using double precision, we have

$$H_{18} - \frac{1}{2} e f^T = 10^{-5} \begin{pmatrix} -0.0916 & 0 & -0.3665 \\ 0.0037 & 0 & 0.0149 \\ 0.0991 & 0 & 0.3964 \end{pmatrix},$$

$$L_{18} - \frac{1}{2} e g^T = 10^{-5} \begin{pmatrix} 0.3091 & 0.1888 & 0.0259 \\ 0.0277 & 0.0169 & 0.0023 \\ -0.2537 & -0.1549 & -0.0213 \end{pmatrix}.$$

Note that H_{18} and L_{18} are already very close to $\frac{1}{2} e f^T$ and $\frac{1}{2} e g^T$, respectively. We also find that, for the matrices G_k computed by the LR algorithm,

$$G - G_{17} = 10^{-5} \begin{pmatrix} 0 & 0 & 0 \\ 0.474676 & 0.289914 & 0.039805 \\ 1.091754 & 0.666802 & 0.091552 \end{pmatrix}$$

and

$$G - G_{18} = 10^{-5} \begin{pmatrix} 0 & 0 & 0 \\ 0.237339 & 0.144957 & 0.019903 \\ 0.545879 & 0.333402 & 0.045776 \end{pmatrix}.$$

Note that $G - G_{18} \approx \frac{1}{2}(G - G_{17})$. For $\tilde{G}_{18} = 2G_{18} - G_{17}$, we have

$$G - \tilde{G}_{18} = 10^{-10} \begin{pmatrix} 0 & 0 & 0 \\ 0.1959 & 0.1196 & 0.0164 \\ 0.4505 & 0.2752 & 0.0378 \end{pmatrix}.$$

So, $\tilde{G}_{18}$ is a much better approximation for G.

For the next example, assumption (3.11) is satisfied although neither of S_f and S_g is empty.

Example 5.2. Consider the equation (1.1) with

$$A_0 = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}.$$

The corresponding QBD is clearly null recurrent. Since

$$H_0 = L_0 = \begin{pmatrix} 0 & 0.5 \\ 0 & 0.5 \end{pmatrix}$$

for this example, assumptions (3.1) and (3.11) are satisfied. It is easy to find that

$$G = F = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

So, we actually have $S_f = S_g = \{1\}$. For this example, we have $H_k = \frac{1}{2}ef^T$ and $L_k = \frac{1}{2}eg^T$ for all $k \geq 0$. We also have for each $k \geq 0$

$$G_k = \begin{pmatrix} 0 & 1 - 1/2^{k+1} \\ 0 & 1 - 1/2^{k+1} \end{pmatrix}.$$

So, $\{G_k\}$ converges to G linearly with rate $1/2$ and $\tilde{G}_k = 2G_k - G_{k-1} = G$ for all $k \geq 1$.

We can also find examples for which (3.11) is not satisfied.

Example 5.3. Consider the equation (1.1) with

$$A_0 = \begin{pmatrix} 0.25 & 0 \\ 0.25 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0.25 \\ 0 & 0.25 \end{pmatrix}.$$

The corresponding QBD is clearly null recurrent. Since

$$H_0 = \begin{pmatrix} 0 & 0.5 \\ 0 & 0.5 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0.5 & 0 \\ 0.5 & 0 \end{pmatrix}$$

for this example, assumption (3.1) is satisfied. It is easy to find that

$$G = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

So, we have $S_f = \{1\}$ and $S_g = \{2\}$. Therefore, assumption (3.11) is not satisfied. However, the conclusions in our main results in Section 3 still hold. In fact, we have $H_k = \frac{1}{2}ef^T$ and $L_k = \frac{1}{2}eg^T$ for all $k \geq 0$. We also have for each $k \geq 0$

$$G_k = \begin{pmatrix} 1 - 1/2^{k+1} & 0 \\ 1 - 1/2^{k+1} & 0 \end{pmatrix}.$$

So, $\{G_k\}$ converges to G linearly with rate $1/2$ and $\tilde{G}_k = 2G_k - G_{k-1} = G$ for all $k \geq 1$.

There are also examples for which assumption (3.1) is not satisfied. The next example is provided by a referee.

Example 5.4. Consider the equation (1.1) with

$$A_0 = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \quad A_1 = 0, \quad A_2 = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}.$$

The corresponding QBD is clearly null recurrent. In this case,

$$G = F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So, (3.11) is true, but (3.1) is not satisfied. It is easy to find that $H_k = L_k = \frac{1}{2}I$ for each $k \geq 1$ and that

$$G_k = \begin{pmatrix} 0 & 1 - 1/2^{k+1} \\ 1 - 1/2^{k+1} & 0 \end{pmatrix}$$

for each $k \geq 0$. Thus, $\{G_k\}$ converges to G linearly with rate $1/2$ and $\tilde{G}_k = 2G_k - G_{k-1} = G$ for all $k \geq 1$.

We do not have any examples of null recurrent QBDs for which the convergence of the LR algorithm is not linear with rate 1/2.

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