CONVERGENCE RATE OF AN ITERATIVE METHOD FOR A NONLINEAR MATRIX EQUATION*

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Abstract. We prove a convergence result for an iterative method, proposed recently by B. Meini, for finding the maximal Hermitian positive definite solution of the matrix equation $X+A^*X^{-1}A=Q$, where Q is Hermitian positive definite.

Key words. matrix equation, maximal Hermitian solution, cyclic reduction, iterative methods, convergence rate

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1. Introduction.

Nonlinear matrix equations occur in many applications. Examples of these equations are algebraic Riccati equations of continuous or discrete type, which have been studied extensively and have been the subject of the monographs [12] and [13]. Another example is the quadratic matrix equation

$$AX^2 + BX + C = 0, \tag{1.1}$$

where A, B, C are given coefficient matrices. This equation has also been the topic of many papers, including two recent papers by Higham and Kim (see [9] and [10]). In this paper, our interest is in the matrix equation

$$X + A^*X^{-1}A = Q, \tag{1.2}$$

where $A, Q \in \mathbb{C}^{m\times m}$ with Q Hermitian positive definite and a Hermitian positive definite solution is required. This equation has been studied recently by several authors (see [1], [4], [5], [8], [14], [16], [17]). For the application areas in which the equation arises, see the references given in [1]. Note also that a solution X of (1.2) is such that the Schur complement of X in the matrix

$$\begin{pmatrix} X & A \\ A^* & Q \end{pmatrix}$$

is X itself (see [1]).

There is some connection between (1.2) and (1.1). For example, if X is a solution of (1.2), then $X^{-1}A$ is a solution of $A^*Y^2 - QY + A = 0$. Equation (1.2) is also a special case of the discrete algebraic Riccati equation

$$-X + C^*XC + Q - (A + B^*XC)^*(R + B^*XB)^{-1}(A + B^*XC) = 0, \tag{1.3}$$

with $C = R = 0$ and $B = I$.

For Hermitian matrices X and Y, we write $X \geq Y (X > Y)$ if $X - Y$ is positive semidefinite (definite). A Hermitian solution X_+ of a matrix equation is called maximal if $X_+ \geq X$ for any Hermitian solution X of the matrix equation.

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For algebraic Riccati equations, it is well known that the desirable solution is the maximal solution. For (1.2), the maximal solution is also the right choice in view of the presence of X^{-1} in the equation.

The main purpose of the paper is to prove a convergence result for an iterative method proposed by Meini [14] for finding the maximal solution of (1.2). Roughly speaking, our new result together with the results obtained in [14] shows that the convergence of Meini's method is no slower than Newton's method. Meini's method is thus preferable when we try to find the maximal solution of (1.2), since the computational work per iteration for Newton's method is $5 \sim 10$ times that for Meini's method. To put our result in a proper setting, we review in Section 2 some theoretical results for the solution of (1.2) and present in Section 3 three iterative methods, with emphasis on Meini's method. Our convergence result for Meini's method is then presented in Section 4. The paper ends with some discussions in Section 5.

2. Theoretical background. Necessary and sufficient conditions for the existence of a positive definite solution of (1.2) have been given in [5].

THEOREM 2.1. *Equation* (1.2) *has a positive definite solution if and only if the rational matrix function* $\psi(\lambda) = \lambda A + Q + \lambda^{-1} A^*$ *is regular (i.e., the determinant of* $\psi(\lambda)$ *is not identically zero) and* $\psi(\lambda) \geq 0$ *for all* λ *on the unit circle.*

The existence of the maximal solution of (1.2) has also been established in [5], along with a characterization of the maximal solution.

THEOREM 2.2. *If equation* (1.2) *has a positive definite solution, then it has a maximal solution* X_+. *Moreover,* X_+ *is the unique positive definite solution such that* $X + \lambda A$ *is nonsingular for all* λ *with* $|\lambda| < 1$.

This result has the following immediate corollary, where $\rho(\cdot)$ is the spectral radius.

COROLLARY 2.3. *For the maximal solution* X_+ *of* (1.2), $\rho(X_+^{-1}A) \leq 1$; *for any other positive definite solution* X, $\rho(X^{-1}A) > 1$.

We also have the following characterization for the eigenvalues of the matrix $X_+^{-1}A$ (see [8]).

THEOREM 2.4. *For equation* (1.2), *the eigenvalues of* $X_+^{-1}A$ *are precisely the eigenvalues of the matrix pencil*

$$\lambda \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -I & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -I \\ Q & -I & A^* \\ -A & 0 & 0 \end{pmatrix}$$

inside or on the unit circle, with half of the partial multiplicities for each eigenvalue on the unit circle.

3. Iterative methods. The maximal solution X_+ of (1.2) can be found by the following basic fixed point iteration:

ALGORITHM 3.1.

$$X_0 = Q,$$
$$X_{n+1} = Q - A^* X_n^{-1} A, \quad n = 0, 1, \ldots.$$

For Algorithm 3.1, we have $X_0 \geq X_1 \geq \cdots$, and $\lim_{n\to\infty} X_n = X_+$ (see, e.g., [5]). Moreover, the following result is proved in [8].

THEOREM 3.2. *Let* $\{X_n\}$ *be given by Algorithm* 3.1. *Then*

$$\limsup_{n\to\infty} \sqrt[n]{\|X_n - X_+\|} \leq (\rho(X_+^{-1}A))^2,$$

where $\|\cdot\|$ is any matrix norm.

Note that $\rho(X_+^{-1}A) \leq 1$ is always true by Corollary 2.3. From the above result, we know that the convergence of the fixed point iteration is R-linear whenever $\rho(X_+^{-1}A) < 1$. For detailed definitions of the rates of convergence, see [15]. If $\rho(X_+^{-1}A) = 1$, the convergence of the fixed point iteration is typically sublinear. Therefore, the convergence of Algorithm 3.1 would be excruciatingly slow when $X_+^{-1}A$ has eigenvalues on, or near, the unit circle. Naturally, one would turn to Newton's method for help with this situation.

Newton's method is studied in [7] for the discrete algebraic Riccati equation of the form (1.3). For equation (1.2), a special case of (1.3), Newton's method is as follows (see [8]).

ALGORITHM 3.3 (Newton's method for (1.2)). *Take $X_0 = Q$. For $n = 1, 2, \dots,$ compute $L_n = X_{n-1}^{-1}A$, and solve*

$$X_n - L_n^* X_n L_n = Q - 2L_n^* A. \tag{3.1}$$

Note that the Stein equation (3.1) is uniquely solvable when $\rho(L_n) < 1$. The convergence behavior of Algorithm 3.3 is described in [8].

THEOREM 3.4. *If (1.2) has a positive definite solution, then Algorithm 3.3 determines a sequence of Hermitian matrices $\{X_n\}_{n=0}^{\infty}$ for which $\rho(L_n) < 1$ for $n = 0, 1, \dots, X_0 \geq X_1 \geq \cdots$, and $\lim_{n\to\infty} X_n = X_+$. The convergence is quadratic if $\rho(X_+^{-1}A) < 1$. If $\rho(X_+^{-1}A) = 1$ and all eigenvalues of $X_+^{-1}A$ on the unit circle are semisimple (i.e., all elementary divisors associated with these eigenvalues are linear), then the convergence is either quadratic or linear with rate $1/2$.*

Recently, Meini proposed a new algorithm by following the strategy successfully devised in [2, 3] for solving nonlinear matrix equations arising in Markov chains. Her algorithm is described below.

For the maximal solution X_+ of (1.2),

$$-I + QX_+^{-1} - A^* X_+^{-1} A X_+^{-1} = 0, \tag{3.2}$$

and the matrix $G_+ = X_+^{-1}A$ satisfies

$$-A + QG_+ - A^* G_+^2 = 0. \tag{3.3}$$

The equations (3.2) and (3.3) can be rewritten as

$$\begin{pmatrix} Q & -A^* & & 0 \\ -A & Q & -A^* & \\ & -A & Q & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I \\ G_+ \\ G_+^2 \\ \vdots \end{pmatrix} X_+^{-1} = \begin{pmatrix} I \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{3.4}$$

The cyclic reduction algorithm is then applied to (3.4). This consists in performing an even-odd permutation of the block rows and columns, followed by one step of block Gaussian elimination on the resulting 2×2 block system. This results in a reduced system with a structure similar to (3.4). Repeated application of the cyclic reduction algorithm generates the sequence of systems:

$$
(3.5)\quad \begin{pmatrix} X_n & -A_n^* & & 0 \\ -A_n & Q_n & -A_n^* & \\ & -A_n & Q_n & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I \\ G_+^{2^n} \\ G_+^{2\cdot 2^n} \\ \vdots \end{pmatrix} X_+^{-1} = \begin{pmatrix} I \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad n = 0, 1, \ldots,
$$

where the matrices A_n, Q_n, and X_n are recursively defined as follows:

ALGORITHM 3.5.

$$
\begin{aligned}
&A_0 = A, \quad Q_0 = X_0 = Q, \\
&A_{n+1} = A_n Q_n^{-1} A_n, \\
&Q_{n+1} = Q_n - A_n Q_n^{-1} A_n^* - A_n^* Q_n^{-1} A_n, \\
&X_{n+1} = X_n - A_n^* Q_n^{-1} A_n, \quad n = 0, 1, \ldots.
\end{aligned}
$$

Meini proposed using the above algorithm to find the maximal solution X_+ and she proved the following result (see [14]):

THEOREM 3.6. *For the matrices Q_n and X_n in Algorithm 3.5, it holds that $Q_n \geq Q_{n+1} > 0$, $X_n \geq X_{n+1} > 0$ $(n = 0, 1, \ldots)$. Moreover, if $\rho(X_+^{-1}A) < 1$ then the sequence $\{X_n\}$ converges to X_+ quadratically.*

Meini's method and Newton's method are most useful when $\rho(X_+^{-1}A)$ is close to 1, since otherwise the basic fixed point iteration is adequate. It is therefore important to investigate the convergence behavior of Meini's method when $\rho(X_+^{-1}A) = 1$.

4. Convergence rate. In this section, we prove a convergence result for Algorithm 3.5 when $\rho(X_+^{-1}A) = 1$. In our proof, we will need the following two equations from (3.5):

$$
X_n - X_+ = A_n^* G_+^{2^n}, \tag{4.1}
$$

$$
-A_n + Q_n G_+^{2^n} - A_n^* G_+^{2\cdot 2^n} = 0. \tag{4.2}
$$

These two equations were also used in Meini's proof of Theorem 3.6.

THEOREM 4.1. *If $\rho(X_+^{-1}A) = 1$ and all eigenvalues of $X_+^{-1}A$ on the unit circle are semisimple, then the sequence $\{X_n\}$ produced by Algorithm 3.5 converges to X_+ and the convergence is at least linear with rate 1/2.*

Proof. By Theorem 3.6, the sequence $\{X_n\}$ is monotonically decreasing and bounded below and hence has a limit. Therefore, $\lim_{n\to\infty} A_n^* Q_n^{-1} A_n = 0$ by the last equation in Algorithm 3.5. Since $\|Q\|_2 I \geq Q \geq Q_n > 0$, $Q_n^{-1} \geq I/\|Q\|_2$. Thus,

$$
0 \leq A_n^* A_n / \|Q\|_2 \leq A_n^* Q_n^{-1} A_n.
$$

Therefore, $\lim_{n\to\infty} A_n = 0$. Since all eigenvalues of $G_+ = X_+^{-1}A$ on the unit circle are semisimple by assumption, the sequence $\{G_+^{2^n}\}$ is bounded. It follows from (4.1) that $\lim_{n\to\infty} X_n = X_+$. As a result, $X_n \geq X_+$ for each $n \geq 0$.

To prove the assertion about the convergence rate, we need to make some simplifications. Let $P^{-1}G_+P = J$ be the Jordan Canonical form of G_+. Accordingly, let

$$
B_n = P^* A_n P, \quad R_n = P^* Q_n P, \quad Y_n = P^* X_n P, \quad Y_+ = P^* X_+ P.
$$

It is easily verified that for $n = 0, 1, \dots,$

$$B_{n+1} = B_n R_n^{-1} B_n, \tag{4.3}$$

$$R_{n+1} = R_n - B_n R_n^{-1} B_n^* - B_n^* R_n^{-1} B_n, \tag{4.4}$$

$$Y_{n+1} = Y_n - B_n^* R_n^{-1} B_n, \tag{4.5}$$

and

$$Y_n - Y_+ = B_n^* J^{2^n}, \tag{4.6}$$

$$-B_n + R_n J^{2^n} - B_n^* J^{2\cdot 2^n} = 0. \tag{4.7}$$

We may assume that

$$J = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_k}, J_<),$$

where $e^{i\theta_1}, \dots, e^{i\theta_k}$ are the eigenvalues of G_+ on the unit circle (not necessarily distinct) and $J_<$ consists of Jordan blocks associated with the eigenvalues of G_+ inside the unit disk. Let the corresponding block diagonals of $Y_n - Y_+$ and B_n be

$$\operatorname{diag}(\alpha_n^{(1)}, \dots, \alpha_n^{(k)}, Z_n) \tag{4.8}$$

and

$$\operatorname{diag}(\beta_n^{(1)}, \dots, \beta_n^{(k)}, C_n),$$

respectively. Since $Y_n - Y_+ = P^*(X_n - X_+)P$ is positive semidefinite, $\alpha_n^{(i)} \geq 0$ for $i = 1, \dots, k$.

We first examine how fast $\alpha_n^{(1)}$ converges to zero. By (4.6) we have

$$\alpha_n^{(1)} = \overline{\beta_n^{(1)}} e^{i2^n\theta_1}. \tag{4.9}$$

To find the relation between $\beta_{n+1}^{(1)}$ and $\beta_n^{(1)}$ from (4.3), we let

$$B_n = \begin{pmatrix} \beta_n^{(1)} & t_n^* \\ s_n & U_n \end{pmatrix}, \quad R_n = \begin{pmatrix} \gamma_n & v_n^* \\ v_n & W_n \end{pmatrix}.$$

where $s_n, t_n, v_n \in \mathbb{C}^{m-1}$ and $U_n, W_n \in \mathbb{C}^{(m-1)\times(m-1)}$. Since R_n is positive definite, it is well known (see [11], for example) that $H_n = W_n - \frac{1}{\gamma_n} v_n v_n^*$, the Schur complement of γ_n in R_n, is also positive definite, and

$$R_n^{-1} = \begin{pmatrix} \frac{1}{\gamma_n} + \frac{1}{\gamma_n^2} v_n^* H_n^{-1} v_n & -\frac{1}{\gamma_n} v_n^* H_n^{-1} \\ -\frac{1}{\gamma_n} H_n^{-1} v_n & H_n^{-1} \end{pmatrix}.$$

Now, a straightforward computation shows

$$\beta_{n+1}^{(1)} = \frac{(\beta_n^{(1)})^2}{\gamma_n} + \frac{(\beta_n^{(1)})^2}{\gamma_n^2} v_n^* H_n^{-1} v_n - \frac{\beta_n^{(1)}}{\gamma_n} v_n^* H_n^{-1} s_n - \frac{\beta_n^{(1)}}{\gamma_n} t_n^* H_n^{-1} v_n + t_n^* H_n^{-1} s_n.$$

By (4.7), we have

$$-\beta_n^{(1)} + \gamma_n e^{i2^n\theta_1} - \overline{\beta_n^{(1)}} e^{i2\cdot 2^n\theta_1} = 0, \tag{4.10}$$

and

$$-s_n + v_n e^{i2^n\theta_1} - t_n e^{i2\cdot 2^n\theta_1} = 0. \tag{4.11}$$

Since $\alpha_n^{(1)}$ is real, we have by (4.9) $\overline{\beta_n^{(1)}} e^{i2^n\theta_1} = \beta_n^{(1)} e^{-i2^n\theta_1}$. Thus, $\overline{\beta_n^{(1)}} e^{i2\cdot 2^n\theta_1} = \beta_n^{(1)}$. It follows from (4.10) that

$$\frac{\beta_n^{(1)}}{\gamma_n} = \frac{1}{2} e^{i2^n\theta_1}.$$

The relation between $\beta_{n+1}^{(1)}$ and $\beta_n^{(1)}$ is thus simplified to

$$\begin{aligned}\beta_{n+1}^{(1)} = \frac{1}{2} e^{i2^n\theta_1}\beta_n^{(1)} + \frac{1}{4} e^{i2\cdot 2^n\theta_1} v_n^* H_n^{-1} v_n - \frac{1}{2} e^{i2^n\theta_1} v_n^* H_n^{-1} s_n \\ -\frac{1}{2} e^{i2^n\theta_1} t_n^* H_n^{-1} v_n + t_n^* H_n^{-1} s_n.\end{aligned}$$

Multiplying both sides by $e^{-i2^{n+1}\theta_1}$, we get

$$\begin{aligned}\alpha_{n+1}^{(1)} = \frac{1}{2}\alpha_n^{(1)} + \frac{1}{4} v_n^* H_n^{-1} v_n - \frac{1}{2} e^{-i2^n\theta_1} v_n^* H_n^{-1} s_n \\ -\frac{1}{2} e^{-i2^n\theta_1} t_n^* H_n^{-1} v_n + e^{-i2\cdot 2^n\theta_1} t_n^* H_n^{-1} s_n.\end{aligned}$$

Substituting $s_n = v_n e^{i2^n\theta_1} - t_n e^{i2\cdot 2^n\theta_1}$ (from (4.11)) into the above identity, we get after some manipulations that

$$\alpha_{n+1}^{(1)} = \frac{1}{2}\alpha_n^{(1)} - (t_n - \frac{1}{2} e^{-i2^n\theta_1} v_n)^* H_n^{-1} (t_n - \frac{1}{2} e^{-i2^n\theta_1} v_n).$$

Therefore, $\alpha_{n+1}^{(1)} \le \frac{1}{2}\alpha_n^{(1)}$ for each $n \ge 0$. Thus,

$$\alpha_n^{(1)} \le \frac{1}{2^n}\alpha_0^{(1)}, \quad n = 0, 1, \ldots.$$

Using appropriate permutations, we can show that, for $i = 2, \ldots, k$,

$$\alpha_n^{(i)} \le \frac{1}{2^n}\alpha_0^{(i)}, \quad n = 0, 1, \ldots.$$

The matrices Z_n in (4.8) are positive semidefinite and it is shown below that $\{Z_n\}$ converges to the zero matrix quadratically. Note that $Z_n = C_n^* J_<^{2^n}$ by (4.6). Fix an $\epsilon > 0$ such that $\rho(J_<) + \epsilon < 1$ and choose a norm $\|\cdot\|_\epsilon$ such that $\|J_<\|_\epsilon \le \rho(J_<) + \epsilon$. Since $\lim_{n\to\infty} C_n = 0$ and all matrix norms are equivalent, $\|Z_n\|_2 \le c_1(\rho(J_<) + \epsilon)^{2^n}$ for some constant c_1.

Now, noting that $\mathrm{trace}(Z_n) \le (m-k)\|Z_n\|_2$,

$$\begin{aligned}\|X_n - X_+\|_2 &= \|P^{-*}(Y_n - Y_+)P^{-1}\|_2 \\ &\le \|P^{-*}\|_2 \|Y_n - Y_+\|_2 \|P^{-1}\|_2 \\ &\le \|P^{-*}\|_2 \|P^{-1}\|_2 \,\mathrm{trace}(Y_n - Y_+) \\ &= \|P^{-*}\|_2 \|P^{-1}\|_2 \big(\alpha_n^{(1)} + \cdots + \alpha_n^{(k)} + \mathrm{trace}(Z_n)\big) \\ &\le c_2\big(\frac{1}{2^n} + (\rho(J_<) + \epsilon)^{2^n}\big)\end{aligned}$$

for some constant c_2. Thus,

$$\limsup_{n\to\infty} \sqrt[n]{\|X_n - X_+\|} \le \frac{1}{2}$$

for any matrix norm $\|\cdot\|$. □

5. Discussions. In Section 3, we reviewed three iterative methods for finding the maximal solution of (1.2). For Newton's method, the equation (3.1) can be solved by a complex version of the algorithm described in [6]. Meini's method and the basic fixed point iteration can be implemented easily. The computational work per iteration for Meini's method is roughly twice that for the fixed point iteration, while the computational work per iteration for Newton's method is about 15 times that for the fixed point iteration. With the establishment of Theorem 4.1, we have had a much better picture for the convergence behavior of all three methods. When $\rho(X_+^{-1}A) < 1$, the convergence of the fixed point iteration is linear and the convergence of Newton's method and Meini's method is quadratic. When $\rho(X_+^{-1}A) = 1$, the convergence of the fixed point iteration is typically sublinear, while the convergence of Newton's method and Meini's method is at least linear with rate 1/2 provided that all unimodular eigenvalues of $X_+^{-1}A$ are semisimple (we conjecture that the convergence is *exactly* linear with rate 1/2 for both methods). When $X_+^{-1}A$ has non-semisimple unimodular eigenvalues, Newton's method is still convergent but the rate of convergence is only conjectured to be $1/\sqrt[p]{2}$, where p is the size of the largest Jordan blocks associated with unimodular eigenvalues of $X_+^{-1}A$. The conjecture was made in [7] for equation (1.3), which includes equation (1.2) as a special case. Also, when $X_+^{-1}A$ has non-semisimple unimodular eigenvalues, it is still not known whether the sequence $\{X_n\}$ produced by Meini's method will converge to X_+.

When $\rho(X_+^{-1}A) = 1$, we cannot expect Newton's method to approximate X_+ with full accuracy since the linear system (3.1) is eventually nearly singular. Meini's method has the same problem in this case: $\tilde{Q} = \lim_{n\to\infty} Q_n$ is necessarily singular in this case. In fact, it follows easily from (4.7) that $\tilde{R} = \lim_{n\to\infty} R_n$ is singular. So, $\tilde{Q} = P^{-*}\tilde{R}P^{-1}$ is singular as well.

Our final comments are about test examples for the iterative methods we discussed. In [14], Meini gives an example of equation (1.2) with $Q = I$ and A Hermitian, in which case an analytical expression is available for the maximal solution. More informative examples can be generated as follows. First note that we may assume that $X_+ = I$ (otherwise we can pre-multiply and post-multiply the equation by $X_+^{-1/2}$). The test examples will thus be of the form $X + A^*X^{-1}A = I + A^*A$. By Corollary 2.3, I is the maximal solution of this equation if and only if $\rho(A) \leq 1$. We can then produce a lot of test examples by taking $A = S/r$ with $r \geq \rho(S)$ and S a random matrix. For test examples generated in this way, the convergence behavior of the three methods is very much similar to that reported by Meini [14] for her example. In fact, it is all but certain that all those examples are covered by the theory we have available by now. Note, however, that all three methods will run into difficulties when A is a large Jordan block with eigenvalue 1, for example.

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