Complex symmetric stabilizing solution of the matrix equation $X + A^\top X^{-1} A = Q$

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Abstract

We study the matrix equation $X + A^\top X^{-1} A = Q$, where A is a complex square matrix and Q is complex symmetric. Special cases of this equation appear in Green's function calculation in nano research and also in the vibration analysis of fast trains. In those applications, the existence of a unique complex symmetric stabilizing solution has been proved using advanced results on linear operators. The stabilizing solution is the solution of practical interest. In this paper we provide an elementary proof of the existence for the general matrix equation, under an assumption that is satisfied for the two special applications. Moreover, our new approach here reveals that the unique complex symmetric stabilizing solution has a positive definite imaginary part. The unique stabilizing solution can be computed efficiently by the doubling algorithm.

AMS classification: 15A24, 65F30

Key words: nonlinear matrix equation, complex symmetric solution, stabilizing solution, doubling algorithm

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[1] The work of this author was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

[2] The work of this author was partially supported by the National Science Council in Taiwan.

[3] The work of this author was partially supported by the National Science Council and the National Center for Theoretical Sciences in Taiwan.

1 Introduction

The matrix equation $X + A^*X^{-1}A = Q$, where Q is Hermitian positive definite, arises in several applications. The corresponding real case is the matrix equation $X + A^\top X^{-1}A = Q$, where A is real and Q is real symmetric positive definite. In both cases, we may assume without loss of generality that $Q = I$, the identity matrix. These equations have been studied in [1,3,5,10,13,15], for example.

Recently, there arises the need to consider the matrix equation

$$X + A^\top X^{-1}A = Q, \tag{1}$$

where A is *complex* and Q is *complex symmetric*. First, it is explained in [6] that the computation of the surface Green's function in nano research [2,8,9] can be reduced to the problem of solving the matrix equation (1), where $Q = Q_1 + iQ_2$ with Q_1 real symmetric and $Q_2 = \eta I$ for a positive scalar η, but the matrix A is still a real matrix. And then it is shown in [7] that a quadratic eigenvalue problem arising from the vibration analysis of fast trains [11] can be solved efficiently and accurately by solving a matrix equation of the form (1), where A is complex and Q is complex symmetric.

In those two applications, the existence of a unique complex symmetric stabilizing solution has been proved using advanced results on linear operators (see [4, Chapter XXIV, Theorem 4.1] and [12]). The stabilizing solution is the solution of practical interest. In Section 2 we provide an elementary proof of the existence for the general matrix equation (1), under an assumption that is satisfied for the two special applications. Moreover, our new approach reveals that the unique complex symmetric stabilizing solution has a positive definite imaginary part. In Section 3 we make some concluding remarks. In particular, we mention that the unique stabilizing solution can be computed efficiently by the doubling algorithm, as for the special case studied in [7].

2 Existence of complex symmetric stabilizing solution

For equation (1) we write

$$A = A_1 + iA_2, \quad Q = Q_1 + iQ_2 \tag{2}$$

with A_1, A_2, $Q_1 = Q_1^\top$, $Q_2 = Q_2^\top \in \mathbb{R}^{n\times n}$. A solution X of (1) is said to be stabilizing if $\rho(X^{-1}A) < 1$, where $\rho(\cdot)$ denotes the spectral radius. The assumption we need to guarantee the existence of a stabilizing solution is

$$Q_2 + e^{i\theta} A_2^\top + e^{-i\theta} A_2 > 0, \text{ for } \theta \in [0, 2\pi]. \tag{3}$$

Here $W > 0$ denotes the positive definiteness of a Hermitian matrix W. This assumption is satisfied for the two applications we mentioned earlier. In particular, the assumption is trivially satisfied for the nano application since $A_2 = 0$ and $Q_2 = \eta I$ with $\eta > 0$ there. Note that we do not need any further assumptions on the matrices A_1 and Q_1. Also, if (3) has been verified for the matrices A_2 and Q_2, then it also holds when any positive semi-definite matrix is added to Q_2. From [3] we also know that (3) holds if and only if the matrix equation $Y + A_2^\top Y^{-1} A_2 = Q_2$ has a real symmetric positive definite stabilizing solution Y. So one way to verify the assumption (3) is to use the doubling algorithm in [10] or the equivalent cyclic reduction algorithm in [13] to find the stabilizing solution Y.

We now assume (3) and let

$$M = \begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix}, \quad L = \begin{bmatrix} 0 & I \\ A^\top & 0 \end{bmatrix}. \tag{4}$$

It is easily seen that the matrix pair (M, L) satisfies the relation $MJM^\top = LJL^\top$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. The matrix pair (M, L) or the matrix pencil $M - \lambda L$ is called $\top-$symplectic. It holds that λ is an eigenvalue of (M, L) if and only if $1/\lambda$ is an eigenvalue of (M, L), with the same multiplicity. Here λ can be 0 or ∞.

Lemma 1 *The $\top-$symplectic pencil $M - \lambda L$ has no eigenvalues on the unit circle.*

PROOF. We show that $M - e^{i\theta} L$ is nonsingular for all $\theta \in [0, 2\pi]$. Suppose there are a $\theta_0 \in [0, 2\pi]$ and a nonzero vector $x = (x_1^\top, x_2^\top)^\top$ with $x_1, x_2 \in \mathbb{C}^n$ such that $(M - e^{i\theta_0} L)x = 0$. This implies that

$$Ax_1 = e^{i\theta_0} x_2, \quad Qx_1 - x_2 = e^{i\theta_0} A^\top x_1. \tag{5}$$

By eliminating x_2 in (5) we have

$$Hx_1 \equiv \left(e^{i\theta_0} A^\top - Q + e^{-i\theta_0} A \right) x_1 = 0. \tag{6}$$

Write $H = H_1 + iH_2$, where $H_1 = e^{i\theta_0} A_1^\top - Q_1 + e^{-i\theta_0} A_1$ and $H_2 = e^{i\theta_0} A_2^\top - Q_2 + e^{-i\theta_0} A_2$. It is easily seen that H_1 and H_2 are Hermitian. From assumption (3) it holds that H_2 is negative definite. By the classical Bendixson theorem (see [14] for example) $H_1 + iH_2$ is invertible. From (6) and (5) it follows that $x_1 = 0$ and $x_2 = 0$. Thus, $M - e^{i\theta}L$ is nonsingular for all $\theta \in [0, 2\pi]$. $\square$

From Lemma 1 we see that there is a matrix $\begin{bmatrix} U \\ V \end{bmatrix} \in \mathbb{C}^{2n\times n}$ of full rank spanning the stable invariant subspace of $M - \lambda L$ corresponding to the stable eigenvalue matrix $S \in \mathbb{C}^{n\times n}$, i.e.,

$$\begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 & I \\ A^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} S, \tag{7}$$

where $\rho(S) < 1$. From (7) we get

$$AU = VS, \tag{8}$$

$$QU - V = A^\top US. \tag{9}$$

Multiplying (9) by U^* from the left we get

$$U^*QU - U^*V = U^*A^\top US = U^*(A^* + 2iA_2^\top)US. \tag{10}$$

Substituting (8) into (10), we have

$$U^*QU - U^*V = S^*V^*US + 2iU^*A_2^\top US. \tag{11}$$

Taking conjugate transposes in (11) and subtracting the result from (11) we obtain

$$2iU^*Q_2U + (V^*U - U^*V) = S^*(V^*U - U^*V)S + 2i(U^*A_2^\top US + S^*U^*A_2U). \tag{12}$$

Let

$$K = i(V^*U - U^*V). \tag{13}$$

Then K is Hermitian. From (12) it follows that K satisfies the equation

$$K - S^*KS = 2(U^*Q_2U - U^*A_2^\top US - S^*U^*A_2U). \tag{14}$$

Lemma 2 *The matrix K in* (13) *is positive definite.*

PROOF. From (14), for any positive integer ℓ we have

$$K-(S^*)^{\ell}KS^{\ell}=(K-S^*KS)+S^*(K-S^*KS)S+\cdots+(S^*)^{\ell-1}(K-S^*KS)S^{\ell-1} \tag{15}$$

$$=2\Big[U^*Q_2U+S^*U^*Q_2US+\cdots+(S^*)^{\ell-1}U^*Q_2US^{\ell-1} - U^*A_2^{\top}US-S^*U^*A_2^{\top}US^2-\cdots-(S^*)^{\ell-1}U^*A_2^{\top}US^{\ell} -S^*U^*A_2U-(S^*)^2U^*A_2US-\cdots-(S^*)^{\ell}U^*A_2US^{\ell-1}\Big]. \tag{16}$$

Since $\rho(S)<1$, $S^{\ell}\to 0$ as $\ell\to\infty$. Hence from (16) we have

$$K=2(\widetilde{Q}_2-\widetilde{A}_2^*S-S^*\widetilde{A}_2), \tag{17}$$

where

$$\widetilde{Q}_2=\sum_{\ell=0}^{\infty}(S^*)^{\ell}U^*Q_2US^{\ell},\quad \widetilde{A}_2=\sum_{\ell=0}^{\infty}(S^*)^{\ell}U^*A_2US^{\ell}. \tag{18}$$

Note that $Q_2+e^{i\theta}A_2^{\top}+e^{-i\theta}A_2>0$ for all $\theta\in[0,2\pi]$ is equivalent to that

$$\mathcal{A}_2=\begin{bmatrix} Q_2 & -A_2^{\top} & & \\ -A_2 & Q_2 & -A_2^{\top} & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix} \tag{19}$$

is positive definite. From (17) and (18) it is easy to check that

$$K=2\,[U^*,S^*U^*,\cdots]\,\mathcal{A}_2\begin{bmatrix} U \\ US \\ \vdots \end{bmatrix}. \tag{20}$$

We need to show that $z^*Kz>0$ for all $z\neq 0$. Since $\mathcal{A}_2$ is positive definite, it is enough to show $Wz\neq 0$ for all $z\neq 0$, where W is the rightmost block matrix in (20). Suppose $Wz=0$. Then $Uz=0$ and $USz=0$. It follows from (9) that $Vz=QUz-A^{\top}USz=0$. Thus $\begin{bmatrix} U \\ V \end{bmatrix}z=0$ and then $z=0$ since $\begin{bmatrix} U \\ V \end{bmatrix}$ is of full rank. □

The next result follows readily.

Theorem 3 *The matrix U in (7) is invertible.*

PROOF. Suppose $Ux = 0$ with $x \in \mathbb{C}^n$. From (13) we have

$$x^* K x = x^* \left[i \left(V^* U - U^* V \right) \right] x = 0.$$

So $x = 0$ since K is positive definite by Lemma 2. Thus U is invertible. $\square$

Since U is invertible, we can define $X = VU^{-1}$.

Theorem 4 *Let $X = VU^{-1}$. Then*

(a) *X is complex symmetric;*
(b) *X is invertible;*
(c) *X is a stabilizing solution of (1);*
(d) *$X_2 \equiv \mathrm{Im}(X)$ is positive definite.*

PROOF. (a) Multiplying (9) by $U^\top$ from the left we get

$$U^\top Q U - U^\top V = U^\top A^\top U S. \tag{21}$$

Subtracting the transpose of (21) from (21) and using (8) we have

$$\begin{aligned} U^\top V - V^\top U &= S^\top U^\top A U - U^\top A^\top U S \\ &= S^\top U^\top V S - S^\top V^\top U S = S^\top \left(U^\top V - V^\top U \right) S. \end{aligned} \tag{22}$$

Since $\rho(S) < 1$, $U^\top V = V^\top U$. Then $X = VU^{-1} = U^{-\top}(U^\top V)U^{-1}$ is a complex symmetric matrix.

(b) From (8) and (9), and noting that $U^\top V = V^\top U$, we have

$$\begin{aligned} &\lambda^2 A^\top - \lambda Q + A \\ =&\lambda^2 \left(U^{-\top} S^\top U^\top V U^{-1} \right) - \lambda \left(V U^{-1} + U^{-\top} S^\top U^\top V S U^{-1} \right) + V S U^{-1} \\ =&\left(I - \lambda U S U^{-1} \right)^\top V U^{-1} \left(-\lambda I + U S U^{-1} \right). \end{aligned} \tag{23}$$

Since $\det(\lambda^2 A^\top - \lambda Q + A) = \det(M - \lambda L) \neq 0$ for every unimodular λ (by Lemma 1), we know that $X = VU^{-1}$ is nonsingular.

(c) From (8) and (9) we have

$$A = X(USU^{-1}), \quad Q - X = A^\top (USU^{-1}). \tag{24}$$

Eliminating USU^{-1} in (24) gives $X + A^\top X^{-1} A = Q$ and we also have $\rho(X^{-1}A) = \rho(USU^{-1}) = \rho(S) < 1$.

(d) From (13) it follows that

$$U^{-*}KU^{-1} = i\,(X^* - X) = 2\mathrm{Im}(X). \tag{25}$$

So $X_2 \equiv \mathrm{Im}(X)$ is positive definite by Lemma 2. $\square$

We have shown that the (unique) stabilizing solution of (1) must be complex symmetric, and that it has a positive definite imaginary part. When A is not a real matrix, it is quite possible that some other complex symmetric solutions of the equation (1) also have a positive definite imaginary part. In fact, for a real matrix A_2 and a real symmetric positive definite matrix Q_2 satisfying the assumption (3), the equation $Y + A_2^T Y^{-1} A_2 = Q_2$ may have many positive definite solutions Y (see [3]). So for each such Y, $X = iY$ is a solution of $X + (iA_2)X^{-1}(iA_2) = iQ_2$ with a positive definite imaginary part.

We can also provide an elementary proof for the following statement proved in [3] using advanced results in operator theory: for a real matrix A_2 and a real symmetric positive definite matrix Q_2 satisfying the assumption (3), the equation $Y + A_2^T Y^{-1} A_2 = Q_2$ has a positive definite stabilizing solution Y. In fact, we have already proved that the equation $X + (iA_2)^T X^{-1}(iA_2) = iQ_2$ has a complex symmetric stabilizing solution X with a positive definite imaginary part. We only need to show that the real part of X must be zero. Since $A = iA_2$ and $Q = iQ_2$ now, we have from (10) and (8) that

$$U^*QU - U^*V = -U^*A^*US = -S^*V^*US. \tag{26}$$

Taking conjugate transpose on (26) gives

$$-U^*QU - V^*U = -S^*U^*VS. \tag{27}$$

It follows from (26) and (27) that

$$(U^*V + V^*U) - S^*(U^*V + V^*U)S = 0. \tag{28}$$

So $U^*V + V^*U = 0$ since $\rho(S) < 1$. Now $2\mathrm{Re}(X) = X + X^* = U^{-*}(U^*V + V^*U)U^{-1} = 0$.

3 Conclusions

We have provided an elementary proof of the existence of a (unique) complex symmetric stabilizing solution X for the nonlinear matrix equation (1) with assumption (3). Our new approach here has revealed that the imaginary part of X is positive definite. We also mention that the solution X can be found efficiently by a doubling algorithm, as presented in [7, Algorithm 4.1]. A convergence result for the algorithm is given in [7, Theorem 4.1] for the equation (1) with the matrices A and Q having special block structures. However, those special structures were not used in the proof of convergence in [7]. So the statements in that theorem are also valid for our general equation (1) with assumption (3).

Acknowledgement

The authors thank Beatrice Meini and two referees for their helpful comments. In particular, comments by one referee significantly simplified the arguments leading to the conclusion in Theorem 3.

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