A new class of nonsymmetric algebraic Riccati equations ⋆

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Abstract

Motivated by the study of linear quadratic differential games, we introduce a new class of nonsymmetric algebraic Riccati equations. It is shown that every equation in this class has a unique stabilizing solution, which is the solution required to find the open-loop Nash equilibrium for the differential game. We show that the doubling algorithm can be used to find this solution efficiently. The solution may also be found by the Schur method, and under further assumptions by Newton's method and a basic fixed-point iteration.

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1 Introduction

For a two-player linear quadratic differential game, on an infinite time horizon, the problem of finding all open-loop Nash equilibria boils down to the problem of finding all stabilizing solutions $X \in \mathbb{R}^{2n\times n}$ of a nonsymmetric algebraic Riccati equation (NARE) of the form

$$XCX - XD - AX + B = 0, \tag{1}$$

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where $D \in \mathbb{R}^{n\times n}$,

$$A = \begin{bmatrix} D^{\mathrm{T}} & 0 \\ 0 & D^{\mathrm{T}} \end{bmatrix}, \quad C = [S_1\ S_2], \quad B = -\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

and the matrices $S_1, S_2, Q_1, Q_2 \in \mathbb{R}^{n\times n}$ are all real symmetric and positive semidefinite. See [6] for more details. Here a solution X is said to be stabilizing if the matrix $D - CX$ is stable (a square matrix is stable if all its eigenvalues are in the open left half-plane).

It is important to find conditions on the coefficient matrices in (1) from which one can know *a priori* the existence and uniqueness of stabilizing solutions. One result in this direction is given in Theorem 15 of [7].

Theorem 1 [7] *If $S_1 = \alpha S_2$ or $Q_1 = \alpha Q_2\,(\alpha > 0)$, and D is stable, then the NARE (1) has a unique stabilizing solution.*

In this paper we will introduce and study a new class of NAREs. In particular, a new sufficient condition is given to guarantee the existence of a unique stabilizing solution of (1).

For any matrices $A, B \in \mathbb{R}^{m\times n}$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ for all i, j. A real square matrix A is called a Z-matrix if all its off-diagonal elements are nonpositive. Any Z-matrix A can be written as $sI - B$ with $B \geq 0$. A Z-matrix A is called a nonsingular M-matrix if $s > \rho(B)$ and a singular M-matrix if $s = \rho(B)$, where $\rho(\cdot)$ is the spectral radius. For a matrix $A = [a_{ij}] \in \mathbb{C}^{m\times n}$, its absolute value is $|A| = [|a_{ij}|]$. For a matrix $A \in \mathbb{C}^{n\times n}$, its comparison matrix is $B = [b_{ij}]$ with $b_{ii} = |a_{ii}|$ and $b_{ij} = -|a_{ij}|$ for $i \neq j$; A is called a nonsingular H-matrix if its comparison matrix B is a nonsingular M-matrix. Given a square matrix A, we will denote by $\sigma(A)$ the set of the eigenvalues of A. The open right half-plane and the open left half-plane will be denoted by $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.

We now consider the NARE

$$XCX - XD - AX + B = 0, \tag{2}$$

where A, B, C, D are complex matrices of sizes $m \times m, m \times n, n \times m, n \times n$, respectively.

Definition 2 The NARE (2) is in class H^+ if

$$T = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \tag{3}$$

is a nonsingular H-matrix with positive diagonal elements. The NARE (2) is in class H^- if T is a nonsingular H-matrix with negative diagonal elements. The NARE (2) is in class M if T is a nonsingular M-matrix.

Clearly class M is a subclass of class H^+. There is no essential difference between class H^+ and class H^- however: a NARE in class H^+ is changed to a NARE in class H^- by replacing the variable X with $-X$.

Class H^- includes many, but certainly not all, cases of the NARE (1). We can make a comparison between class H^- and the class of NARE (1) described in Theorem 1. When the NARE (1) is in class H^-, the matrix D is necessarily stable. However, we do not need the stringent assumption in Theorem 1 about the matrix $C = [S_1 \; S_2]$ or about the matrix $B = -\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$. We would say that class H^- is large enough to warrant a systematic study.

The NARE (2) in class M has applications in transport theory and Markov models [17,22], where the solution of practical interest is the minimal nonnegative solution. The equation has been studied in [9,12,14–16,18]. In the study of Markov chains, of particular interest is the NARE (2) for which the matrix T is an irreducible singular M-matrix. This NARE has been studied in [4,9–13]. We could have defined a NARE in class H^+ to be a NARE for which T has positive diagonal elements and the comparison matrix of T is a nonsingular M-matrix or an irreducible singular M-matrix. We did not choose to do so since this will expand the class H^+ only slightly and there is no special importance attached to the NAREs allowed in this expansion.

Although the NARE (1) often has additional sign patterns for the coefficient matrices, it cannot be written as a NARE in class M unless one of the matrices B and C is the zero matrix. The extension from class M to class H^+ (which is closely related to class H^-) is thus useful.

The outline of the paper is as follows. In Section 2, we collect some basic results about nonsingular M-matrices and H-matrices, and review some basic results about the NAREs in class M. In Section 3, we show the existence and uniqueness of stabilizing solutions for the NAREs in class H^-. In Section 4, we show that the doubling algorithm can be used to find the unique stabilizing solution for the NARE in class H^- (through a NARE in class H^+). The unique stabilizing solution can also be found by the Schur method in general, and by Newton's method or a basic fixed-point iteration under additional assumptions. These are discussed in Section 5. Our conclusions are given in Section 6.

2 Preliminaries

We will need some basic results about nonsingular M-matrices.

Theorem 3 [3] *For a Z-matrix A, the following are equivalent:*

(a) *A is a nonsingular M-matrix.*
(b) $A^{-1} \geq 0$.
(c) *$Av > 0$ for some vector $v > 0$.*
(d) *All eigenvalues of A have positive real parts.*

The equivalence of (a) and (c) in Theorem 3 implies the next result.

Lemma 4 *Let A be a nonsingular M-matrix. If $B \geq A$ is a Z-matrix, then B is also a nonsingular M-matrix.*

We will also need the following two results about nonsingular H-matrices. The first one can be found in [8].

Lemma 5 *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix. If $B \in \mathbb{C}^{n\times n}$ is such that*

$$|\mathrm{diag}(B)| \geq \mathrm{diag}(A), \quad |\mathrm{offdiag}(B)| \leq |\mathrm{offdiag}(A)|,$$

where $\mathrm{diag}(Z)$ *and* $\mathrm{offdiag}(Z)$ *are the diagonal part and the off-diagonal part of a square matrix Z, respectively, then B is a nonsingular H-matrix and $|B^{-1}| \leq A^{-1}$.*

The next result is also well known (see [20], for example).

Lemma 6 *Let A be a nonsingular H-matrix. Partition A as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. Then A_{11} and A_{22} are nonsingular H-matrices. The Schur complement of A_{11} (or A_{22}) in A is also a nonsingular H-matrix.

When we study the NARE (2), we also need the dual equation of (2)

$$YBY - YA - DY + C = 0. \tag{4}$$

When the NARE (2) is in class M, so is the NARE (4). Some properties of the NAREs (2) and (4) are summarized below.

Theorem 7 [9] *Let the NARE (2) be in class M. Then the NARE (2) has a minimal nonnegative solution X and $D - CX$ is a nonsingular M-matrix; the*

NARE (4) *has a minimal nonnegative solution* Y *and* $A-BY$ *is a nonsingular* M*-matrix. Moreover,* $I - XY$ *and* $I - YX$ *are nonsingular* M*-matrices.*

The last statement in the theorem was not given explicitly in [9], but it follows readily. Indeed, let $v_1 \in \mathbb{R}^n$ and $v_2 \in \mathbb{R}^m$ be positive vectors such that

$$\begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} > 0, \text{ or equivalently } \begin{bmatrix} A & -B \\ -C & D \end{bmatrix} \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} > 0.$$

Then it is shown in [9] that $Xv_1 < v_2$ and $Yv_2 < v_1$. Thus $XYv_2 < v_2$ and $YXv_1 < v_1$. So $(I - XY)v_2 > 0$ and $(I - YX)v_1 > 0$. Therefore, $I - XY$ and $I - YX$ are nonsingular M-matrices by Theorem 3.

3 Existence and uniqueness

We now consider the NARE (2) in class H^+. Let the comparison matrix of T in (3) be

$$\widehat{T} = \begin{bmatrix} \widehat{D} & -\widehat{C} \\ -\widehat{B} & \widehat{A} \end{bmatrix}, \tag{5}$$

which is a nonsingular M-matrix. By Theorem 7, the matrix equation

$$\widehat{X}\widehat{C}\widehat{X} - \widehat{X}\widehat{D} - \widehat{A}\widehat{X} + \widehat{B} = 0 \tag{6}$$

has a minimal nonnegative solution $\widehat{X}$ and $\widehat{D} - \widehat{C}\widehat{X}$ is a nonsingular M-matrix. Similarly, the matrix equation

$$\widehat{Y}\widehat{B}\widehat{Y} - \widehat{Y}\widehat{A} - \widehat{D}\widehat{Y} + \widehat{C} = 0 \tag{7}$$

has a minimal nonnegative solution $\widehat{Y}$ and $\widehat{A} - \widehat{B}\widehat{Y}$ is a nonsingular M-matrix. We also know that $I - \widehat{X}\widehat{Y}$ and $I - \widehat{Y}\widehat{X}$ are both nonsingular M-matrices.

Theorem 8 *Let the NARE* (2) (*and thus the NARE* (4)) *be in class* H^+*. Then the NARE* (2) *has a unique solution* X *such that* $|X| \le \widehat{X}$*. Moreover,* $D - CX$ *is a nonsingular* H*-matrix whose diagonal elements have positive real parts, and* X *is the unique solution of* (2) *such that* $\sigma(D - CX) \subset \mathbb{C}_+$*. Similarly, the NARE* (4) *has a unique solution* Y *such that* $|Y| \le \widehat{Y}$*. Moreover,* $A - BY$ *is a nonsingular* H*-matrix whose diagonal elements have positive real parts, and* Y *is the unique solution of* (4) *such that* $\sigma(A - BY) \subset \mathbb{C}_+$*.*

PROOF. We only need to prove the statements about the NARE (2) since the NARE (4) is also in class H^+.

Let the linear operators $\mathcal{L}, \widehat{\mathcal{L}} : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ be defined by

$$\mathcal{L}(Z) = ZD + AZ, \quad \widehat{\mathcal{L}}(Z) = Z\widehat{D} + \widehat{A}Z,$$

respectively. Both operators are invertible since A and D are nonsingular H-matrices with positive diagonal elements. We first show that the mapping $f : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ given by

$$f(Z) = \mathcal{L}^{-1}(ZCZ + B) \tag{8}$$

has a fixed point in the set $\mathcal{W} = \{Z : |Z| \le \widehat{X}\}$. It is clear that $\mathcal{W}$ is a compact convex set. We can rewrite (8) as

$$\mathrm{vec}(f(Z)) = (I \otimes A + D^{\mathrm{T}} \otimes I)^{-1}\mathrm{vec}(ZCZ + B),$$

where the vec operator stacks the columns of a matrix into a long vector and $\otimes$ is the Kronecker product (see [19], for example). Since $\widehat{A}$ and $\widehat{D}$ are nonsingular M-matrices, so is $I\otimes\widehat{A}+\widehat{D}^{\mathrm{T}}\otimes I$. By Lemma 5, we have $|(I\otimes A+D^{\mathrm{T}}\otimes I)^{-1}| \le (I \otimes \widehat{A} + \widehat{D}^{\mathrm{T}} \otimes I)^{-1}$. Thus,

$$\mathrm{vec}(|f(Z)|) \le (I \otimes \widehat{A} + \widehat{D}^{\mathrm{T}} \otimes I)^{-1}\mathrm{vec}(|Z||C||Z| + |B|).$$

Therefore, for any $Z \in \mathcal{W}$,

$$|f(Z)| \le \widehat{\mathcal{L}}^{-1}(|Z||C||Z| + |B|) \le \widehat{\mathcal{L}}^{-1}(\widehat{X}\widehat{C}\widehat{X} + \widehat{B}) = \widehat{X}.$$

In other words, $f(\mathcal{W}) \subset \mathcal{W}$. By the Brouwer fixed-point theorem, f has a fixed point X in $\mathcal{W}$. Thus the NARE (2) has a solution X such that $|X| \le \widehat{X}$. Write $\widehat{D} = sI - \widehat{N}$ with $\widehat{N} \ge 0$. Accordingly, $D = sI - N$ with $|N| = \widehat{N}$. Since $\widehat{D} - \widehat{C}\widehat{X} = sI - (\widehat{N} + \widehat{C}\widehat{X})$ is a nonsingular M-matrix, $s > \rho(\widehat{N} + \widehat{C}\widehat{X})$. Now, $D - CX = sI - (N + CX)$ and $|N + CX| \le \widehat{N} + \widehat{C}\widehat{X}$. By Lemma 5, $D - CX$ is a nonsingular H-matrix. It is also clear that the diagonal elements of $D - CX$ have positive real parts. We also have (see [23]) $\rho(N + CX) \le \rho(\widehat{N} + \widehat{C}\widehat{X}) < s$. Therefore, $\sigma(D - CX) \subset \mathbb{C}_+$. Finally, we show the uniqueness of solutions X with $\sigma(D - CX) \subset \mathbb{C}_+$. Let X be any such solution. Then

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (D - CX),$$

where

$$H = \begin{bmatrix} D & -C \\ B & -A \end{bmatrix}. \tag{9}$$

So X is determined by an invariant subspace of H corresponding to n eigenvalues in $\mathbb{C}_+$. To show the uniqueness, we just need to show H has exactly n eigenvalues in $\mathbb{C}_+$. Let v be a positive vector such that $\widehat{T}v > 0$ and let

$V = \text{diag}(v)$. It is easily seen that $W = V^{-1}HV$ is strictly diagonally dominant with n positive diagonal elements and m negative diagonal elements. It follows from Gershgorin's theorem that W (and hence H) has exactly n eigenvalues in $\mathbb{C}_+$. $\square$

Corollary 9 *The NARE* (2) *has a unique stabilizing solution if it is in class* H^-.

PROOF. Let the NARE (2) be in class H^-. Then the NARE

$$ZCZ - Z(-D) - (-A)Z + B = 0$$

is in class H^+. By Theorem 8 it has a unique solution Z such that $\sigma(-D - CZ) \subset \mathbb{C}_+$. In other words, $X = -Z$ is the unique solution of (2) such that $D - CX$ is stable. $\square$

Since the NARE (1) is a special case of the NARE (2), the above result has given a new condition for the existence of a unique stabilizing solution of (1).

Remark 10 It is noted in [6] that when a discount factor r is introduced to the quadratic performance criteria for the Nash game, the matrix D in (1) is replaced by $D - \frac{1}{2}rI$. Therefore, by using a large enough discount factor, we can assume that the NARE (1) is in class H^- and thus has a unique stabilizing solution.

4 The doubling algorithm

A structure preserving doubling algorithm has been presented and analyzed in [16] for the NARE (2) in class M. The doubling algorithm itself is not new; it was studied in [1], for example. However, the presentation in [16] provides some new information about the algorithm, which makes its analysis easier for the NARE (2).

In this section we will apply the doubling algorithm to the NARE (2) in class H^+, and show that the algorithm is well defined and quadratically convergent. As we have seen in the proof of Corollary 9, the unique stabilizing solution of a NARE in class H^- can be found readily from the solution described in Theorem 8 for a related NARE in class H^+. We start with a presentation of the doubling algorithm when the NARE (2) is in class H^+. The presentation here is a minor adaptation of the one in [16], but it is needed to fix the notation.

For the solution X of the NARE (2) described in Theorem 8, we have

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} R, \tag{10}$$

where H is defined in (9) and $R = D - CX$. Using the Cayley transform

$$\mathcal{C}_\gamma : z \to \frac{z - \gamma}{z + \gamma}, \tag{11}$$

with a scalar $\gamma > 0$, we can transform (10) into

$$(H - \gamma I) \begin{bmatrix} I \\ X \end{bmatrix} = (H + \gamma I) \begin{bmatrix} I \\ X \end{bmatrix} R_\gamma, \tag{12}$$

where

$$R_\gamma = \mathcal{C}_\gamma(R) = (R + \gamma I)^{-1}(R - \gamma I).$$

Note that $R + \gamma I$ is nonsingular since $\sigma(R) \subset \mathbb{C}_+$ by Theorem 8. For any $\gamma > 0$, the matrix $T_\gamma = T + \gamma I$ is a nonsingular H-matrix. So

$$A_\gamma = A + \gamma I, \quad D_\gamma = D + \gamma I$$

are nonsingular H-matrices. Let

$$W_\gamma = A_\gamma - BD_\gamma^{-1}C, \quad V_\gamma = D_\gamma - CA_\gamma^{-1}B \tag{13}$$

be the Schur complements of D_γ and A_γ, respectively, in T_γ. They are both nonsingular H-matrices by Lemma 6. As is shown in [16], (12) can be reduced to

$$K \begin{bmatrix} I \\ X \end{bmatrix} = L \begin{bmatrix} I \\ X \end{bmatrix} R_\gamma, \tag{14}$$

by premultiplying both sides of (12) with a proper nonsingular matrix, where

$$K = \begin{bmatrix} E_\gamma & 0 \\ -H_\gamma & I \end{bmatrix}, \quad L = \begin{bmatrix} I & -G_\gamma \\ 0 & F_\gamma \end{bmatrix},$$

with

$$\begin{aligned} E_\gamma &= I - 2\gamma V_\gamma^{-1}, \quad F_\gamma = I - 2\gamma W_\gamma^{-1}, \\ G_\gamma &= 2\gamma D_\gamma^{-1} C W_\gamma^{-1}, \quad H_\gamma = 2\gamma W_\gamma^{-1} B D_\gamma^{-1}. \end{aligned} \tag{15}$$

Similarly, for the solution Y of the NARE (4) described in Theorem 8, we have

$$(H - \gamma I) \begin{bmatrix} Y \\ I \end{bmatrix} S_\gamma = (H + \gamma I) \begin{bmatrix} Y \\ I \end{bmatrix} \tag{16}$$

and then

$$K \begin{bmatrix} Y \\ I \end{bmatrix} S_\gamma = L \begin{bmatrix} Y \\ I \end{bmatrix}, \tag{17}$$

where $S_\gamma = (S+\gamma I)^{-1}(S-\gamma I)$ with $S = A-BY$ being a nonsingular H-matrix with $\sigma(S) \subset \mathbb{C}_+$. By the property of Cayley transform, we have $\rho(R_\gamma) < 1$ and $\rho(S_\gamma) < 1$.

We then have the following doubling algorithm as presented in [16], where the sequences $\{H_k\}$ and $\{G_k\}$ are going to approximate X and Y, respectively.

Algorithm 1

$$\begin{aligned}
&E_0 = E_\gamma, \quad F_0 = F_\gamma, \quad G_0 = G_\gamma, \quad H_0 = H_\gamma,\\
&E_{k+1} = E_k(I - G_kH_k)^{-1}E_k,\\
&F_{k+1} = F_k(I - H_kG_k)^{-1}F_k,\\
&G_{k+1} = G_k + E_k(I - G_kH_k)^{-1}G_kF_k,\\
&H_{k+1} = H_k + F_k(I - H_kG_k)^{-1}H_kE_k.
\end{aligned}$$

We will use a comparison procedure to analyze Algorithm 1 for the NAREs (2) and (4). So we apply the whole process leading to Algorithm 1 with T in (3) replaced by $\widehat{T}$ in (5), but with the same value γ, and put a "hat" on all matrices obtained. Thus we have $\widehat{R}, \widehat{R}_\gamma, \widehat{A}_\gamma, \widehat{V}_\gamma, \widehat{E}_\gamma$, etc. Note that the diagonal elements of $\widehat{T}$ are the same as those of T.

The next result is a slight modification of the main results in [16], since we allow equality in (18).

Theorem 11 *Let $\widehat{T}$ be a nonsingular M-matrix and*

$$\gamma \geq \max\{\max_{1\leq i\leq m} a_{ii}, \max_{1\leq i\leq n} d_{ii}\}, \tag{18}$$

where a_{ii} and d_{ii} are the diagonal elements of A and D, respectively. Then the sequences $\{\widehat{E}_k\}, \{\widehat{F}_k\}, \{\widehat{G}_k\}, \{\widehat{H}_k\}$ are well defined, and

(a) $\widehat{E}_0, \widehat{F}_0 \leq 0$*;* $\widehat{E}_k, \widehat{F}_k \geq 0$ *for* $k \geq 1$.
(b) $I - \widehat{H}_k\widehat{G}_k$ *and* $I - \widehat{G}_k\widehat{H}_k$ *are nonsingular M-matrices.*
(c) $0 \leq \widehat{H}_k \leq \widehat{H}_{k+1} \leq \widehat{X}$ *for* $k \geq 0$ *and*

$$\widehat{X} - \widehat{H}_k = (I - \widehat{H}_k\widehat{Y})\widehat{S}_\gamma^{2^k}\widehat{X}\widehat{R}_\gamma^{2^k}.$$

(d) $0 \leq \widehat{G}_k \leq \widehat{G}_{k+1} \leq \widehat{Y}$ *for* $k \geq 0$ *and*

$$\widehat{Y} - \widehat{G}_k = (I - \widehat{G}_k\widehat{X})\widehat{R}_\gamma^{2^k}\widehat{Y}\widehat{S}_\gamma^{2^k}.$$

PROOF. The proof is largely the same as in [16]. So we only point out the differences. Since equality is allowed in (18), we can no longer prove that

$$\widehat{E}_\gamma e,\ \widehat{F}_\gamma e,\ \widehat{R}_\gamma e,\ \widehat{S}_\gamma e < 0,\ \widehat{E}_k e,\ \widehat{F}_k e > 0\ (k \geq 1),$$

where e is the vector of ones. So we need to give a new proof of the fact that $I - \widehat{H}_k\widehat{G}_k$ and $I - \widehat{G}_k\widehat{H}_k$ are nonsingular M-matrices after $0 \leq \widehat{H}_k \leq \widehat{X}$ and $0 \leq \widehat{G}_k \leq \widehat{Y}$ have been obtained. Indeed, $I - \widehat{Y}\widehat{X}$ and $I - \widehat{X}\widehat{Y}$ are nonsingular M-matrices by Theorem 7. Since $0 \leq \widehat{H}_k\widehat{G}_k \leq \widehat{X}\widehat{Y}$ and $0 \leq \widehat{G}_k\widehat{H}_k \leq \widehat{Y}\widehat{X}$, $I - \widehat{H}_k\widehat{G}_k$ and $I - \widehat{G}_k\widehat{H}_k$ are nonsingular M-matrices by Lemma 4. □

We made the effort to allow $\gamma = \max\{\max a_{ii}, \max d_{ii}\}$ in Theorem 11 since this γ is optimal in the following sense.

Lemma 12 *For $\gamma \geq \max\{\max a_{ii}, \max d_{ii}\}$, $\rho(\widehat{R}_\gamma)$ and $\rho(\widehat{S}_\gamma)$ are nondecreasing functions of γ.*

PROOF. This result is proved in [13] under the assumption that $\widehat{T}$ is irreducible. When $\widehat{T}$ is a reducible nonsingular M-matrix, we can replace each zero element in $\widehat{T}$ by $-\epsilon$ with $\epsilon > 0$ small enough to get an irreducible nonsingular M-matrix $\widehat{T}(\epsilon)$. The two NAREs corresponding to $\widehat{T}(\epsilon)$ have minimal nonnegative solutions $\widehat{X}(\epsilon)$ and $\widehat{Y}(\epsilon)$, respectively, which are continuous functions of ϵ [12]. The lemma is then proved for the reducible case by a proper limit procedure. □

When Algorithm 1 is applied to the NARE (2), the following is true.

Theorem 13 *Let the NARE* (2) *be in class H^+ and*

$$\gamma \geq \max\{\max_{1\leq i\leq m} a_{ii}, \max_{1\leq i\leq n} d_{ii}\}.$$

Then the sequences $\{E_k\}, \{F_k\}, \{G_k\}, \{H_k\}$ are well defined, and for each $k \geq 0$

(a) $|E_k| \leq |\widehat{E}_k|$, $|F_k| \leq |\widehat{F}_k|$, $|G_k| \leq \widehat{G}_k$, $|H_k| \leq \widehat{H}_k$.
(b) $I - H_kG_k$ *and* $I - G_kH_k$ *are nonsingular H-matrices.*
(c) $X - H_k = (I - H_kY)S_\gamma^{2^k}XR_\gamma^{2^k}, \quad Y - G_k = (I - G_kX)R_\gamma^{2^k}YS_\gamma^{2^k}$.

PROOF. We compare the matrices obtained by Algorithm 1 from the matrix T and the matrix $\widehat{T}$, respectively. Note that A_γ is a nonsingular H-matrix with $|A_\gamma^{-1}| \leq \widehat{A}_\gamma^{-1}$ (see Lemma 5). Now,

$$
\begin{aligned}
|\mathrm{diag}(V_\gamma)| &\geq |\mathrm{diag}(D_\gamma)| - |\mathrm{diag}(CA_\gamma^{-1}B)| \\
&\geq \mathrm{diag}(\widehat{D}_\gamma) - \mathrm{diag}(\widehat{C}\widehat{A}_\gamma^{-1}\widehat{B}) \\
&= \mathrm{diag}(\widehat{V}_\gamma),
\end{aligned}
$$

$$
\begin{aligned}
|\mathrm{offdiag}(V_\gamma)| &\leq |\mathrm{offdiag}(D_\gamma)| + |\mathrm{offdiag}(CA_\gamma^{-1}B)| \\
&\leq -\mathrm{offdiag}(\widehat{D}_\gamma) + \mathrm{offdiag}(\widehat{C}\widehat{A}_\gamma^{-1}\widehat{B}) \\
&= -\mathrm{offdiag}(\widehat{V}_\gamma) \\
&= |\mathrm{offdiag}(\widehat{V}_\gamma)|.
\end{aligned}
$$

Since $\widehat{V}_\gamma$ is a nonsingular M-matrix, V_γ is a nonsingular H-matrix with $|V_\gamma^{-1}| \leq \widehat{V}_\gamma^{-1}$ by Lemma 5. Note that $E_\gamma = I - 2\gamma V_\gamma^{-1} = V_\gamma^{-1}(-\gamma I + D - CA_\gamma^{-1}B)$. Thus,

$$
\begin{aligned}
|E_\gamma| &\leq |V_\gamma^{-1}||-\gamma I + D - CA_\gamma^{-1}B| \\
&\leq \widehat{V}_\gamma^{-1}|-\gamma I + \widehat{D} - \widehat{C}\widehat{A}_\gamma^{-1}\widehat{B}| \\
&= |\widehat{V}_\gamma^{-1}(-\gamma I + \widehat{D} - \widehat{C}\widehat{A}_\gamma^{-1}\widehat{B})| \\
&= |\widehat{E}_\gamma|,
\end{aligned}
$$

where we have used $\gamma \geq \max_{1\leq i\leq n} d_{ii}$.

Similarly, we have $|D_\gamma^{-1}| \leq \widehat{D}_\gamma^{-1}$, $|W_\gamma^{-1}| \leq \widehat{W}_\gamma^{-1}$, and $|F_\gamma| \leq |\widehat{F}_\gamma|$ (using $\gamma \geq \max_{1\leq i\leq m} a_{ii}$). It follows immediately that $|G_\gamma| \leq \widehat{G}_\gamma$ and $|H_\gamma| \leq \widehat{H}_\gamma$. Since $|H_\gamma G_\gamma| \leq \widehat{H}_\gamma \widehat{G}_\gamma$ and $I - \widehat{H}_\gamma \widehat{G}_\gamma$ is a nonsingular M-matrix, we know that $I - H_\gamma G_\gamma$ is a nonsingular H-matrix. Similarly, $I - G_\gamma H_\gamma$ is a nonsingular H-matrix. By now we have proved that E_0, F_0, G_0, H_0 are well defined and (a) and (b) are true for $k = 0$. Assume that E_i, F_i, G_i, H_i are well defined and (a) and (b) are true for $k = i$ ($i \geq 0$). Then $E_{i+1}, F_{i+1}, G_{i+1}, H_{i+1}$ are well defined. It is easy to prove that (a) and (b) are true for $k = i + 1$. For example, since $|(I - H_iG_i)^{-1}| \leq (I - \widehat{H}_i\widehat{G}_i)^{-1}$ by Lemma 5, $|H_{i+1}| \leq |H_i| + |F_i||(I - H_iG_i)^{-1}||H_i||E_i| \leq \widehat{H}_i + \widehat{F}_i(I - \widehat{H}_i\widehat{G}_i)^{-1}\widehat{H}_i\widehat{E}_i = \widehat{H}_{i+1}$. Therefore, by the induction principle, the sequences $\{E_k\}, \{F_k\}, \{G_k\}, \{H_k\}$ are well defined, and (a) and (b) are true for each $k \geq 0$. The proof of (c) is exactly the same as in [16]. $\square$

Theorem 14 *The sequence $\{H_k\}$ converges to X quadratically and the sequence $\{G_k\}$ converges to Y quadratically. Moreover, for any matrix norm $\|\cdot\|$*

$$
\limsup_{k\to\infty} \sqrt[2^k]{\|H_k - X\|} \leq \rho(R_\gamma)\rho(S_\gamma) < 1,
$$

$$
\limsup_{k\to\infty} \sqrt[2^k]{\|G_k - Y\|} \leq \rho(R_\gamma)\rho(S_\gamma) < 1.
$$

PROOF. Since $|H_k| \le \widehat{H}_k \le \widehat{X}$, we have by Theorem 13 (c)

$$\begin{aligned}\limsup_{k\to\infty} \sqrt[2^k]{\|H_k - X\|} &= \limsup_{k\to\infty} \sqrt[2^k]{\|H_k - X\|_\infty} \\ &\le \limsup_{k\to\infty} \sqrt[2^k]{(1+\|\widehat{X}\|_\infty \|Y\|_\infty)\|X\|_\infty \|R_\gamma^{2^k}\|_\infty \|S_\gamma^{2^k}\|_\infty} \\ &= \rho(R_\gamma)\rho(S_\gamma) < 1.\end{aligned}$$

Thus, $\{H_k\}$ converges to X quadratically. The proof of the statement about $\{G_k\}$ is similar. □

The next result shows that the convergence of the doubling algorithm applied to the NARE (2) in class H^+ is expected to be faster than the convergence of the doubling algorithm applied to the corresponding NARE (6) in class M.

Proposition 15 *For $\gamma \ge \max\{\max_{1\le i\le m} a_{ii}, \max_{1\le i\le n} d_{ii}\}$, $\rho(R_\gamma) \le \rho(\widehat{R}_\gamma)$ and $\rho(S_\gamma) \le \rho(\widehat{S}_\gamma)$.*

PROOF. We will prove $\rho(R_\gamma) \le \rho(\widehat{R}_\gamma)$. The other inequality can be proved similarly. Recall that $R_\gamma = (R+\gamma I)^{-1}(R-\gamma I)$ with $R = D - CX$ and $\widehat{R}_\gamma = (\widehat{R}+\gamma I)^{-1}(\widehat{R}-\gamma I)$ with $\widehat{R} = \widehat{D} - \widehat{C}\widehat{X}$. As shown in the proof of Theorem 8, we can write $R = sI - W$ and $\widehat{R} = sI - \widehat{W}$ with $s = \max_{1\le i\le n} d_{ii} \le \gamma$ and $|W| \le \widehat{W}$. When the matrix $\widehat{T}$ in (5) is irreducible, it is already shown in [13] that $\rho(\widehat{R}_\gamma) = \frac{\gamma-(s-\hat{t})}{\gamma+(s-\hat{t})}$, where $\hat{t} = \rho(\widehat{W})$. However, the irreducibility condition on $\widehat{T}$ can be removed using a continuity argument as in the proof of Lemma 12. Since $\rho(W) \le \rho(\widehat{W}) = \hat{t}$, The absolute values of the eigenvalues of R_γ have the form $p(\eta,\theta) = \left|\frac{\gamma-(s-t)}{\gamma+(s-t)}\right|$, where $t = \eta e^{i\theta}$ with $0 \le \eta \le \hat{t}$ and $0 \le \theta < 2\pi$. It is easily found that

$$p(\eta,\theta) = \sqrt{\frac{(\gamma-s)^2 + 2\eta(\gamma-s)\cos\theta + \eta^2}{(\gamma+s)^2 - 2\eta(\gamma+s)\cos\theta + \eta^2}}.$$

Since $\gamma \ge s$, we have $p(\eta,\theta) \le p(\eta,0) = \frac{\gamma-(s-\eta)}{\gamma+(s-\eta)} \le \frac{\gamma-(s-\hat{t})}{\gamma+(s-\hat{t})}$. Thus $\rho(R_\gamma) \le \rho(\widehat{R}_\gamma)$. □

5 Other algorithms

We mentioned earlier that there is no essential difference between a NARE in class H^+ and a NARE in class H^-. In this section we consider the NARE

(2) in class H^- since the introduction of the class H^- is motivated by the special NARE (1), where D is often a stable matrix with negative diagonal elements. The unique stabilizing solution of the NARE (2) in class H^- can be found efficiently by applying the doubling algorithm to a NARE in class H^+. This solution can also be found by other algorithms: the Schur method, Newton's method, and a basic fixed-point iteration. However, for Newton's method and the basic fixed-point iteration, convergence is guaranteed only under additional assumptions for the NARE (2) in class H^-.

5.1 The Schur method

When the NARE (2) is in class H^-, the stabilizing solution X can be found by the Schur method. If $H = URU^*$ is the Schur decomposition of the matrix H in (9), where

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

is unitary and $U_{11} \in \mathbb{C}^{n\times n}$, and the first n diagonal elements of the triangular matrix R are the eigenvalues of H in $\mathbb{C}_-$, then the existence of the stabilizing solution X ensures the nonsingularity of the matrix U_{11} and X is obtained from $X = U_{21}U_{11}^{-1}$. However, the matrix U_{11} may be ill-conditioned and iterative refinement of the approximation obtained from the Schur method may be necessary. When H is real, we can use the real Schur decomposition instead to avoid complex arithmetic. When H is real and $m = n$, the Schur method requires a little over $200n^3$ flops to get an approximation for X. For the doubling algorithm, we need $\frac{64}{3}n^3$ flops each iteration. So the doubling algorithm is simpler, and usually more efficient and more accurate. Proposition 15 shows that the comparison is more favorable for the doubling algorithm when the NARE is in class H^+ rather than in class M.

5.2 Fixed-point iteration

The coefficient matrices in the NARE (1) often have special sign patterns. In fact, all examples of (1) given in [6,7] have the following properties: $-D$ is a nonsingular M-matrix, $B \le 0$, and $C \ge 0$. This has led to a study in [2] of the NARE (2) for which

$$-(I \otimes A + D^{\mathrm{T}} \otimes I) \text{ is a nonsingular } M\text{-matrix, } C \ge 0,\ B \le 0. \tag{19}$$

(It is assumed in [2] that $B < 0$, but actually the weaker assumption $B \le 0$ is sufficient for their discussions.)

The fixed-point iteration

$$(-A)X_{k+1} + X_{k+1}(-D) = -B - X_k C X_k, \quad X_0 = 0 \tag{20}$$

is studied in [2] for the NARE (2) under the assumption (19), where the authors assume that $X_2 \geq 0$, and try to find a nonnegative solution of the NARE (2), hoping that this would be a stabilizing solution.

While for most NAREs in class H^- and with the assumption (19) the unique stabilizing solution happens to be nonnegative, there are examples for which this is not true.

One such example is the special NARE (1) with

$$A = \begin{bmatrix} -1.97 & 0 & 0 & 0 \\ 0 & -1.98 & 0 & 0 \\ 0 & 0 & -1.97 & 0 \\ 0 & 0 & 0 & -1.98 \end{bmatrix}, \quad B = -\begin{bmatrix} 1.00 & 0.64 \\ 0.64 & 0.50 \\ 0.66 & 0.05 \\ 0.05 & 0.97 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.83 & 0.71 & 0.12 & 0.19 \\ 0.71 & 0.69 & 0.19 & 0.38 \end{bmatrix}, \quad D = \begin{bmatrix} -1.97 & 0 \\ 0 & -1.98 \end{bmatrix}.$$

The unique stabilizing solution is found, by the doubling algorithm or by the Schur method, to be

$$X = \begin{bmatrix} 0.2261 & 0.1391 \\ 0.1429 & 0.1104 \\ 0.1552 & 0.0031 \\ -0.0038 & 0.2299 \end{bmatrix}.$$

So X fails to be nonnegative, albeit by a small margin.

The fixed-point iteration (20) for the NARE (2) requires $14n^3$ flops each iteration when $m = n$, starting with the second iteration (the first iteration is more expensive since proper decompositions of the matrices A and D are to be computed and re-used in subsequent iterations). It is difficult for the fixed-point iteration to compete with the doubling algorithm when the NARE is in class H^-. First of all, the theory for the fixed-point iteration is incomplete. It is shown in [2] that the sequence $\{X_{2i}\}$ is decreasing and the sequence $\{X_{2i+1}\}$ is increasing and that both are convergent. However, the two limits may be different. It is not known when the limits are the same. If the limits are the same, then it is shown in [2] that the common limit is the unique nonnegative solution. But then one still needs to check whether the nonnegative solution is stabilizing. Suppose everything is fine with the fixed-point iteration, it is still more expensive than the doubling algorithm in general, since its convergence is only linear.

5.3 Newton's method

It is also tempting to apply Newton's method to the NARE (2) in class H^- when the assumption (19) is satisfied. However, the convergence of Newton's method is still difficult to establish without further assumptions. The Newton iteration applied to the NARE (2) can be rewritten as

$$(A - X_kC)X_{k+1} + X_{k+1}(D - CX_k) = B - X_kCX_k, \ k = 0, 1, \ldots. \tag{21}$$

If we take $X_0 = 0$, then numerical experiments suggest that the convergence of the Newton iteration is like the one described in Theorem 4.2 of [5]. In particular, we observe that $X_k \geq X_{k+1}$ for $k \geq 1$ and $\lim X_k$ is the unique stabilizing solution. There is no guarantee of this though since some assumptions for Theorem 4.2 in [5] are not satisfied (for conceivable choices of the sets D and D_+ in that paper). We also note that the limit of $\{X_k\}$ is not necessarily nonnegative although we always have $X_1 \geq 0$. Indeed, for the example in the previous subsection, we apply the Newton iteration with $X_0 = 0$ and observe that $X_1 \geq X_2 \geq \cdots$ and $\lim X_i = K$ is the unique stabilizing solution, which is not nonnegative.

With one additional assumption, however, the nice convergence behaviour of the Newton iteration can be proved.

Theorem 16 *Consider the NARE* (2) (*not necessarily in class* H^-) *with assumption* (19)*. For the Newton iteration* (21) *with* $X_0 = 0$*, we have* $X_1 \geq 0$*. Assume that* $-(I \otimes (A - X_1C) + (D - CX_1)^{\mathrm{T}} \otimes I)$ *is a* Z*-matrix. Then* $X_k \geq X_{k+1} \geq 0$ *for* $k \geq 1$ *and* $\lim_{k\to\infty} X_k = X_* \geq 0$ *is a solution of the NARE* (2) *such that* $-(I \otimes (A - X_*C) + (D - CX_*)^{\mathrm{T}} \otimes I)$ *is a nonsingular* M*-matrix. If the NARE* (2) *is also in class* H^- *then* X_* *is the unique stabilizing solution.*

PROOF. Since $-(I \otimes A + D^{\mathrm{T}} \otimes I)$ is a nonsingular M-matrix and $B \leq 0$, we have $X_1 \geq 0$. Since $-(I \otimes (A - X_1C) + (D - CX_1)^{\mathrm{T}} \otimes I) = -(I \otimes A + D^{\mathrm{T}} \otimes I) + (I \otimes (X_1C) + (CX_1)^{\mathrm{T}} \otimes I)$ is a Z-matrix by assumption, it is a nonsingular M-matrix by Lemma 4. It follows from (21) that $X_2 \geq 0$. As in the proof of Theorem 2.1 of [14], we have

$$(A - X_1C)(X_1 - X_2) + (X_1 - X_2)(D - CX_1) = -(X_1 - X_0)C(X_1 - X_0).$$

It follows that $X_1 \geq X_2$. So $X_k \geq X_{k+1} \geq 0$ is true for $k = 1$. Assume that $X_k \geq X_{k+1} \geq 0$ is true for $1 \leq k \leq i$ ($i \geq 1$). Then $-(I \otimes (A - X_{i+1}C) + (D -$

$CX_{i+1})^{\mathrm{T}} \otimes I)$ is a Z-matrix since $-(I \otimes (A - X_1 C) + (D - CX_1)^{\mathrm{T}} \otimes I)$ is so and $X_{i+1} \leq X_1$. It follows as before that $-(I \otimes (A - X_{i+1}C) + (D - CX_{i+1})^{\mathrm{T}} \otimes I)$ is a nonsingular M-matrix. We then have $X_{i+2} \geq 0$ by (21). Since

$$\begin{aligned}&(A - X_{i+1}C)(X_{i+1} - X_{i+2}) + (X_{i+1} - X_{i+2})(D - CX_{i+1})\\ &= -(X_{i+1} - X_i)C(X_{i+1} - X_i),\end{aligned}$$

as in the proof of Theorem 2.1 in [14], we have $X_{i+1} \geq X_{i+2}$. Thus $X_k \geq X_{k+1} \geq 0$ is true for all $k \geq 1$ by the induction principle. So $\lim_{k\to\infty} X_k = X_* \geq 0$ exists and X_* is a solution of the NARE (2). As before, $-(I \otimes (A - X_*C) + (D - CX_*)^{\mathrm{T}} \otimes I)$ is a Z-matrix since $X_* \leq X_1$, and it is a nonsingular M-matrix by Lemma 4.

It is well known that for any solution S of the NARE (2)

$$\begin{bmatrix} I & 0 \\ S & I \end{bmatrix}^{-1} \begin{bmatrix} D & -C \\ B & -A \end{bmatrix} \begin{bmatrix} I & 0 \\ S & I \end{bmatrix} = \begin{bmatrix} D - CS & -C \\ 0 & -(A - SC) \end{bmatrix}$$

(see [21] for example). Thus, the eigenvalues of $D - CX_*$ are eigenvalues of H and the eigenvalues of $A - X_*C$ are the negative of the remaining eigenvalues of H. When the NARE (2) is in class H^- the eigenvalues of $D - CX_*$ must be the n eigenvalues of H in $\mathbb{C}_-$ since $\sigma(I \otimes (A - X_*C) + (D - CX_*)^{\mathrm{T}} \otimes I) \subset \mathbb{C}_-$ by Theorem 3. Thus, X_* is the unique stabilizing solution of the NARE (2) when it is in class H^-. $\square$

We remark that the additional assumption in the theorem, namely $-(I \otimes (A - X_1C) + (D - CX_1)^{\mathrm{T}} \otimes I)$ is a Z-matrix, is unlikely to be satisfied when at least one of the matrices A and D have some zero elements (unless $B = 0$ or $C = 0$). However, Theorem 16 shows that the Newton iteration is applicable to some NAREs outside class H^- (and the convergence is quadratic). If the NARE is in class H^-, the doubling algorithm is usually more efficient since Newton's method requires about $60n^3$ flops each iteration when $m = n$.

6 Conclusions

We have introduced a new class of NAREs, the class H^- or the closely related class H^+. The class H^+ is an extension of the class M studied earlier in the literature. Every equation in class H^- has a unique stabilizing solution, which is the solution required in applications. The stabilizing solution of a NARE in class H^- can be found by computing a special solution of a related NARE in class H^+. That special solution can be computed efficiently by the quadratically convergent doubling algorithm. The stabilizing solution may also be found by the Schur method, and under further assumptions by Newton's method and a basic fixed-point iteration. Generally speaking, these methods are not as efficient as the doubling algorithm for a NARE in class H^-. However, it should be noted that being in class H^- is only a sufficient condition for the existence of a unique stabilizing solution for the NARE. The applicability of the Schur method and Newton's method is not limited to the class H^-. While the doubling algorithm may still be used in practice for many NAREs not in class H^+ or class H^-, there are currently no theoretical results about its applicability for those situations.

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