A note on the minimal nonnegative solution of a nonsymmetric algebraic Riccati equation

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Abstract

For the nonsymmetric algebraic Riccati equation for which the four coefficient matrices form an M-matrix, the solution of practical interest is often the minimal nonnegative solution. In this note we prove that the minimal nonnegative solution is positive when the M-matrix is irreducible.

Key words: Nonsymmetric algebraic Riccati equations; M-matrices; Minimal nonnegative solution

1 Introduction

In this note we consider the nonsymmetric algebraic Riccati equation (ARE)

$$XCX - XD - AX + B = 0, \tag{1}$$

where A, B, C, D are real matrices of sizes $m \times m, m \times n, n \times m, n \times n$, respectively, and

$$K = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \tag{2}$$

is a nonsingular M-matrix or an irreducible singular M-matrix. The relevant definitions are as follows.

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Definition 1 [1] *A square matrix* A *is called an* M*-matrix if* $A = sI - B$ *with* $B \geq 0$ *(elementwise order) and* $s \geq \rho(B)$*, where* $\rho(\cdot)$ *is the spectral radius. It is called a singular* M*-matrix if* $s = \rho(B)$*; it is called a nonsingular* M*-matrix if* $s > \rho(B)$.

Definition 2 [8] *For* $n \geq 2$*, an* $n \times n$ *matrix* A *is reducible if there exists an* $n \times n$ *permutation matrix* P *such that*

$$PAP^T = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B *and* D *are square matrices. Otherwise,* A *is irreducible.*

Nonsymmetric AREs of this type appear in transport theory (see [3–5]) and Wiener–Hopf factorization of Markov chains (see [6,7]). The solution of practical interest in these applications is the minimal nonnegative solution.

The following general result about the minimal nonnegative solution of (1) has been established in [2].

Theorem 3 *If* K *is a nonsingular* M*-matrix, then* (1) *has a minimal nonnegative solution* S *and* $D - CS$ *is a nonsingular* M*-matrix. If* K *is an irreducible singular* M*-matrix, then* (1) *has a minimal nonnegative solution* S *and* $D - CS$ *is an* M*-matrix.*

For the nonsymmetric ARE studied in [4,5], the matrix K has no zero elements and the minimal nonnegative solution S is actually positive. For the nonsymmetric ARE arising in the Wiener–Hopf factorization of Markov chains, however, a reasonable assumption would be the irreducibility of the matrix K. In this note we prove that $S > 0$ whenever K is an irreducible M-matrix.

When K is an irreducible M-matrix, so is the matrix

$$\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

Thus, we also have $\hat{S} > 0$, where $\hat{S}$ is the minimal nonnegative solution of

$$XBX - XA - DX + C = 0,$$

the dual equation of (1).

The assumption that $S, \hat{S} > 0$ is needed in several main results in [2]. With the result in this note, we see that the assumption is always satisfied when K is an irreducible M-matrix.

2 The result

Since the matrix K in (2) is a nonsingular M-matrix or an irreducible singular M-matrix, the matrices A and D are both nonsingular M-matrices (see [2]). In particular, the diagonal elements of A and D are positive. Let $A = A_1 - A_2, D = D_1 - D_2$, where $A_1 = \mathrm{diag}(A)$ and $D_1 = \mathrm{diag}(D)$. We then have the fixed-point iteration for (1)

$$X_{k+1} = \mathcal{L}^{-1}(X_k C X_k + X_k D_2 + A_2 X_k + B), \tag{3}$$

where the linear operator $\mathcal{L}$ is given by $\mathcal{L}(X) = A_1 X + X D_1$.

Lemma 4 [2] *For* (3) *with* $X_0 = 0$*, we have* $X_0 \le X_1 \le \cdots$, $\lim_{k\to\infty} X_k = S$*, the minimal nonnegative solution of* (1).

Theorem 5 *If* K *is an irreducible* M*-matrix, then* $S > 0$.

PROOF. For the iteration (3) with $X_0 = 0$, we claim that for each $k \ge 0$, X_{k+1} has at least one more positive element than X_k does, unless X_k is already a positive matrix. Once this claim is proved, we have $S \ge X_{m\cdot n} > 0$ by Lemma 4.

Since $B \ne 0$ by the irreducibility of K, the claim is true if $X_k = 0$. So, we let G be a nontrivial subset of $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ and assume that $(X_k)_{ij} > 0$ for all $(i, j) \in G$ and $(X_k)_{ij} = 0$ for all $(i, j) \notin G$ (In the proof we denote by Y_{ij} the (i, j) element of a matrix Y.) We will show by contradiction that X_{k+1} has at least one more positive element than X_k does.

Suppose that $(X_{k+1})_{ij} = 0$ for all $(i, j) \notin G$. Then, by iteration (3),

$$B_{ij} = 0, \quad (A_2 X_k)_{ij} = 0, \quad (X_k D_2)_{ij} = 0, \quad (X_k C X_k)_{ij} = 0 \tag{4}$$

for all $(i, j) \notin G$. Note that

$$(A_2 X_k)_{ij} = \sum_{q=1}^{m} (A_2)_{iq} (X_k)_{qj}, \quad (X_k D_2)_{ij} = \sum_{p=1}^{n} (X_k)_{ip} (D_2)_{pj},$$

$$(X_k C X_k)_{ij} = \sum_{q=1}^{m} \sum_{p=1}^{n} (X_k)_{ip} C_{pq} (X_k)_{qj}.$$

It follows from (4) that the following four assertions hold:

$$\text{If } (i,j) \notin G, \text{ then } B_{ij} = 0. \tag{5}$$

$$\text{If } (i,j) \notin G \text{ and } (q,j) \in G, \text{ then } (A_2)_{iq} = 0. \tag{6}$$

$$\text{If } (i,j) \notin G \text{ and } (i,p) \in G, \text{ then } (D_2)_{pj} = 0. \tag{7}$$

$$\text{If } (i,j) \notin G, (i,p) \in G \text{ and } (q,j) \in G, \text{ then } C_{pq} = 0. \tag{8}$$

Now we define the sets

$$G_l = \{r \mid 1 \le r \le m, (r,l) \in G\}, \quad l = 1, 2, \dots, n.$$

If G_l is empty for some l, we suppose that G_l is empty for $l = l_1, l_2, \dots, l_s$ only. Then, for each $p \notin \{l_1, l_2, \dots, l_s\}$ we can find i such that $(i,p) \in G$. Since $(i,j) \notin G$ for each $j \in \{l_1, l_2, \dots, l_s\}$, it follows form (7) that $(D_2)_{pj} = 0$. Thus, all elements in the columns $l_1, l_2, \dots, l_s$ of the matrix

$$\begin{pmatrix} D_2 & C \\ B & A_2 \end{pmatrix} \tag{9}$$

are zero except those in the rows $l_1, l_2, \dots, l_s$. It follows from Definition 2 that the matrix (9) is reducible. Thus, the matrix K is also reducible.

We can then assume that none of the sets G_l is empty. Let $1 \le l_1 < l_2 < \dots < l_s \le n$ be such that

$$G_{l_1} = G_{l_2} = \dots = G_{l_s} = \{r_1, r_2, \dots, r_t\}$$

(where $1 \le r_1 < r_2 < \dots < r_t \le m$) and for $l \in \{1, 2, \dots, n\} \setminus \{l_1, l_2, \dots, l_s\}$,

$$|G_l| \ge |G_{l_1}| \text{ and } G_l \ne G_{l_1}.$$

Since G is a proper subset of $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$, we necessarily have $t < m$. Now, by (5) we have that $B_{ij} = 0$ if $i \notin \{r_1, r_2, \dots, r_t\}$ and $j \in \{l_1, l_2, \dots, l_s\}$. For the matrix A_2, it follows from (6) with $j = l_1$ that $(A_2)_{iq} = 0$ if $i \notin \{r_1, r_2, \dots, r_t\}$ and $q \in \{r_1, r_2, \dots, r_t\}$. For the matrix D_2, we claim that $(D_2)_{pj} = 0$ if $p \notin \{l_1, l_2, \dots, l_s\}$ and $j \in \{l_1, l_2, \dots, l_s\}$. In fact, for each

$p \notin \{l_1, l_2, \ldots, l_s\}$, we can find $i \notin \{r_1, r_2, \ldots, r_t\}$ such that $(i,p) \in G$ since otherwise we would have $|G_p| < |G_{l_1}|$ or $G_p = G_{l_1}$. Since $(i,j) \notin G$ for this i, $(D_2)_{pj} = 0$ by (7). Finally, we claim that $C_{pq} = 0$ if $p \notin \{l_1, l_2, \ldots, l_s\}$ and $q \in \{r_1, r_2, \ldots, r_t\}$. In fact, for each $p \notin \{l_1, l_2, \ldots, l_s\}$ we can find, as before, $i \notin \{r_1, r_2, \ldots, r_t\}$ such that $(i,p) \in G$. For this i and each $q \in \{r_1, r_2, \ldots, r_t\}$, $(i,j) \notin G$ and $(q,j) \in G$ for $j = l_1$. Thus, $C_{pq} = 0$ by (8).

Therefore, for the matrix (9) all elements in the columns $l_1, l_2, \ldots, l_s, n{+}r_1, n{+}r_2, \ldots, n{+}r_t$ are zero except those in the rows $l_1, l_2, \ldots, l_s, n{+}r_1, n{+}r_2, \ldots, n{+}r_t$. It follows as before that the matrix K is reducible. The contradiction shows that X_{k+1} has at least one more positive element than X_k does. $\square$

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