On the Numerical Solution of a Nonlinear Matrix Equation in Markov Chains

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Abstract

We consider iterative methods for the minimal nonnegative solution of the matrix equation $G = \sum_{i=0}^{\infty} A_i G^i$, where the matrices A_i are nonnegative and $\sum_{i=0}^{\infty} A_i$ is stochastic. Convergence theory for an inversion free algorithm is established. The convergence rate of this algorithm is shown to be comparable with that of the fastest iteration among three fixed point iterations.

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1 Introduction

Consider the infinite block matrix

$$P = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ A_0 & A_1 & A_2 & A_3 & \cdots \\ 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

where $A_i, B_i \in \mathbf{R}^{m\times m} (i = 0, 1, \ldots)$ are (elementwise) nonnegative, $\sum_{i=0}^{\infty} A_i$ and $\sum_{i=0}^{\infty} B_i$ are stochastic matrices. A nonnegative matrix A is called

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stochastic if $Ae = e$, with e being the column vector with all components equal to one. Matrices of the structure (1) are known in the literature as stochastic matrices of $M/G/1$ type and arise in a wide variety of queueing problems modelled by an irreducible Markov chain, where P is the transition matrix associated with the Markov chain.

The matrix equation

$$G = \sum_{i=0}^{\infty} A_i G^i \tag{2}$$

plays an important role in the study of the Markov chain. It is known that (2) has at least one solution in the matrix set $\{G \geq 0 \mid Ge \leq e\}$. The desired solution G is the minimal nonnegative solution.

Assume that $A = \sum_{i=0}^{\infty} A_i$ is irreducible. Then, by the Perron-Frobenius Theorem (see [14]), there exists a unique vector $\alpha > 0$ with $\alpha^T e = 1$ and $\alpha^T A = \alpha^T$. A natural assumption for the Markov chain is that $\sum_{i=1}^{\infty} iA_i < \infty$. The Markov chain is called transient if $\sigma = \alpha^T \beta > 1$, where $\beta = \sum_{i=1}^{\infty} iA_i e$. In this case, the matrix G is substochastic (i.e., $Ge \leq e$) and its spectral radius $\rho(G)$ is strictly less than one. The Markov chain is called positive recurrent if $\sigma < 1$, and null recurrent if $\sigma = 1$. In both cases, G is stochastic and $\rho(G) = 1$. For more details, see [11].

In [2] and [8], the equation (2) was considered in a transposed form, i.e., they considered the equation $G = \sum_{i=0}^{\infty} G^i A_i$ with $A_i \geq 0$ and $e^T \sum_{i=0}^{\infty} A_i = e^T$. Their results can easily be restated for (2). The equation $G = \sum_{i=0}^{\infty} G^i A_i$ with $A_i \geq 0$ and $\sum_{i=0}^{\infty} A_i e = e$ is called the dual equation of (2) (see [12]). We will consider equation (2) only. The discussion of its dual equation is similar.

Several algorithms have been proposed in the literature for the numerical solution of (2). For example, the minimal nonnegative solution of (2) can be found by any of the following three fixed point iterations (see [9], [6] and [7]):

$$G_{n+1} = \sum_{i=0}^{\infty} A_i G_n^i, \quad G_0 = 0, \tag{3}$$

$$G_{n+1} = (I - A_1)^{-1}(A_0 + \sum_{i=2}^{\infty} A_i G_n^i), \quad G_0 = 0, \tag{4}$$

or

$$G_{n+1} = (I - \sum_{i=1}^{\infty} A_i G_n^{i-1})^{-1} A_0, \quad G_0 = 0. \tag{5}$$

The infinite series in these iterations will be truncated in numerical computations. Among the three iterations, the iteration (5) is the fastest and also the most expensive. A new linear system (with multiple right hand sides) has to be solved in each iteration. The linear system may be solved by Gaussian elimination with partial pivoting. However, when m is large, there may be difficulty in maintaining the accuracy of the solution of the linear systems during a large number of iterations.

In [1], Bai noted that inversion can be avoided by incorporating the Schulz algorithm for the inverse of a matrix into the iteration (5), as follows:

Algorithm 1 [1] *Given a proper nonnegative matrix B_0, compute for $n \geq 0$*

$$G_{n+1} = B_n A_0, \tag{6}$$

$$B_{n+1} = 2B_n - B_n Q_{n+1} B_n, \tag{7}$$

where for $n = 1, 2, \ldots,$

$$Q_n = I - \sum_{i=1}^{\infty} A_i G_n^{i-1}.$$

Some choices of the matrix B_0 will be given later. Algorithm 1 is particularly useful on a parallel computing system, since matrix-matrix multiplication can be carried out in parallel very efficiently (see [4]). The main purpose of this note is to show that the convergence rate of Algorithm 1 is competitive with the iteration (5).

In Section 2, we give some convergence results for the iterations (3)–(5). Convergence results for Algorithm 1 are then established in Section 3. Numerical results for two simple examples are reported in Section 4 to illustrate the theoretical results established.

2 Convergence results for the fixed point iterations

The convergence rate of iterations (3)–(5) has been studied in detail in [8] when the Markov chain is positive recurrent. The next result is valid no matter whether the Markov chain is transient, positive recurrent or null recurrent.

Theorem 2 *Let G be the minimal nonnegative solution of the matrix equation* (2) *and* $\bar{A}_i = \sum_{j=i}^{\infty} A_j G^{j-i} (i = 1, 2, \ldots)$. *Then we have the following convergence rate estimates: For iteration* (3),

$$\limsup_{n\to\infty} \sqrt[n]{\|G_n - G\|} \le \rho(T_1),$$

where the operator $T_1 : \mathbf{R}^{m\times m} \to \mathbf{R}^{m\times m}$ *is given by*

$$T_1(H) = \sum_{j=1}^{\infty} \bar{A}_j H G^{j-1}. \tag{8}$$

For iteration (4),

$$\limsup_{n\to\infty} \sqrt[n]{\|G_n - G\|} \le \rho(T_2),$$

where the operator $T_2 : \mathbf{R}^{m\times m} \to \mathbf{R}^{m\times m}$ *is given by*

$$T_2(H) = (I - A_1)^{-1}(\bar{A}_1 - A_1)H + \sum_{j=2}^{\infty}(I - A_1)^{-1}\bar{A}_j H G^{j-1}. \tag{9}$$

For iteration (5),

$$\limsup_{n\to\infty} \sqrt[n]{\|G_n - G\|} \le \rho(T_3),$$

where the operator $T_3 : \mathbf{R}^{m\times m} \to \mathbf{R}^{m\times m}$ *is given by*

$$T_3(H) = \sum_{j=2}^{\infty}(I - \bar{A}_1)^{-1}\bar{A}_j H G^{j-1}. \tag{10}$$

Proof. The results can be proved using the arguments in the proof of Theorem 7 of the next section. □

Using the Kronecker product (for basic properties of the Kronecker product, see [5], for example), we have

Proposition 3 *For* $i = 1, 2, 3$, $\rho(T_i) = \rho(M_i)$, *where*

$$M_1 = \sum_{j=1}^{\infty}(G^{j-1})^T \otimes \bar{A}_j,$$

$$M_2 = I \otimes (I - A_1)^{-1}(\bar{A}_1 - A_1) + \sum_{j=2}^{\infty}(G^{j-1})^T \otimes (I - A_1)^{-1}\bar{A}_j,$$

and

$$M_3 = \sum_{j=2}^{\infty} (G^{j-1})^T \otimes (I - \bar{A}_1)^{-1} \bar{A}_j.$$

To compare the spectral radii of the matrices M_1, M_2 and M_3, we need the following lemma.

Lemma 4 [9] *Let*

$$T^* = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} (I \otimes A_i)((G^{i-1-j})^T \otimes G^j).$$

If the Markov chain is transient or positive recurrent, then $\rho(T^*) < 1$. *If the Markov chain is null recurrent, then* $\rho(T^*) = 1$.

It is easy to see that the matrix M_1 in Proposition 3 is precisely the matrix T^* in the above lemma.

Theorem 5 *If the Markov chain is transient or positive recurrent, then* $\rho(M_3) \leq \rho(M_2) \leq \rho(M_1) < 1$. *If the Markov chain is null recurrent, then* $\rho(M_3) = \rho(M_2) = \rho(M_1) = 1$.

Proof. If the Markov chain is transient or positive recurrent, then $\rho(M_1) = \rho(T^*) < 1$ by Lemma 4. Thus $C = I \otimes I - M_1$ is an M-matrix (see [14]). Let $N_2 = I \otimes (I - A_1) - C$ and $N_3 = I \otimes (I - \bar{A}_1) - C$. We then have two regular splittings of C: $C = I \otimes (I - A_1) - N_2$ and $C = I \otimes (I - \bar{A}_1) - N_3$. Note that $(I \otimes (I - A_1))^{-1} N_2 = M_2$ and $(I \otimes (I - \bar{A}_1))^{-1} N_3 = M_3$. Since $M_1 \geq N_2 \geq N_3 \geq 0$, we get $\rho(M_3) \leq \rho(M_2) \leq \rho(M_1) < 1$ (see [14]). If the Markov chain is null recurrent, then $\rho(M_1) = 1$ by Lemma 4. Thus $M_1 v = v$ for some $v \neq 0$. Therefore, $Cv = 0$, which implies $M_2 v = v$ and $M_3 v = v$. Hence $\rho(M_3) = \rho(M_2) = \rho(M_1) = 1$. □

From the above results we know that the convergence of the iterations (3)–(5) is linear when the Markov chain is transient or positive recurrent. If the Markov chain is null recurrent, the convergence of these iterations is typically sublinear.

3 Convergence results for Algorithm 1

The convergence of Algorithm 1 has been discussed in [1] and includes a monotone convergence result. There are, however, some inaccuracies in the statement and proof of that result. We will first give a restatement of the result and a brief revised proof.

Theorem 6 (cf. [1]) *Let G be the minimal nonnegative solution of the matrix equation* (2)*, and B_0 be a given nonnegative matrix such that B_0 has no zero columns and $(I - A_1)B_0 \le I$. Then for Algorithm* 1*, $G_1 \le G_2 \le \cdots$, $B_0 \le B_1 \le \cdots$, and $\lim_{n\to\infty} G_n = G$, $\lim_{n\to\infty} B_n = Q^{-1}$ with*

$$Q = I - \sum_{i=1}^{\infty} A_i G^{i-1}. \tag{11}$$

Proof. Since $(I - A_1)B_0 \le I$, we have $B_0 \le (I - A_1)^{-1} \le Q^{-1}$. Moreover, $G_1 = B_0 A_0 \le Q^{-1} A_0 = G$ and $Q_1 B_0 \le (I - A_1)B_0 \le I$. It can be proved by induction that for all $n \ge 1$,

$$0 \le B_0 \le B_1 \le \cdots \le B_{n-1} \le Q^{-1},$$

$$G_1 \le G_2 \le \cdots \le G_n \le G, \quad Q_n B_{n-1} \le I.$$

Let $\lim_{n\to\infty} G_n = \tilde{G}$, and $\lim_{n\to\infty} B_n = B$. We have $\lim_{n\to\infty} Q_n = \tilde{Q} = I - \sum_{i=1}^{\infty} A_i \tilde{G}^{i-1}$ and

$$\tilde{G} = B A_0, \quad B = 2B - B\tilde{Q}B. \tag{12}$$

From the second equality in (12), we get $B(I - \tilde{Q}B) = 0$. Since $I - \tilde{Q}B \ge 0$ and $B \ge B_0$ is nonnegative with no zero columns, we conclude that $I - \tilde{Q}B = 0$. Therefore, $\tilde{G} = BA_0 = \tilde{Q}^{-1}A_0$, which means that $\tilde{G}$ is a nonnegative solution of (2). Since $\tilde{G} \le G$ and G is the minimal nonnegative solution, we have $\tilde{G} = G$. □

Note that any matrix B_0 satisfying the conditions in the theorem must be nonsingular. If B_0 were singular, all matrices B_n would be singular by (7). This contradicts the fact that $\{B_n\}$ converges to a nonsingular matrix. Some common choices of B_0 have already been given in [1], e.g., $B_0 = \omega I (0 < \omega \le 1)$ or $B_0 = I + A_1 + \cdots + A_1^p$ $(p \ge 1)$. Of course, we can also take $B_0 = (I - A_1)^{-1}$. This latter choice gives the best G_1, but requires an inversion in this initial step.

A convergence rate result is given in [1] for Algorithm 1. However, the conditions needed there for the linear convergence of the algorithm are never satisfied when the solution G is stochastic. When G is stochastic, we have $(A_0 + \sum_{i=1}^{\infty} A_i G^{i-1})e = (\sum_{i=0}^{\infty} A_i)e = e$. Thus $Q^{-1}A_0 e = e$. In [1, Theorem 2], it is required that, among several other conditions, $\|Q^{-1}\|_\infty \|A_0\|_\infty < 1$. We have, however, $\|Q^{-1}\|_\infty \|A_0\|_\infty \geq \|Q^{-1}A_0\|_\infty = \|Q^{-1}A_0 e\|_\infty = \|e\|_\infty = 1$. Since the solution G is involved in the conditions of [1, Theorem 2], there is no easy way to see whether those conditions can be consistent when the solution G is substochastic. Consequently, the convergence rate of Algorithm 1 has not been determined.

We will now give a fairly complete analysis for the convergence rate of Algorithm 1.

Theorem 7 *Let $\mathcal{M} = \{X \in \mathbf{R}^{m\times m} \mid X = PA_0$ for some $P \in \mathbf{R}^{m\times m}\}$. If the conditions in Theorem 6 are satisfied, then for Algorithm 1,*

$$\limsup_{n\to\infty} \sqrt[n]{\|G_n - G\|} \leq \rho(T),$$

where the operator $T : \mathcal{M} \to \mathcal{M}$ is given by

$$T(H) = H - GG^+H + \sum_{j=2}^{\infty} GA_0^+ \bar{A}_j H G^{j-1},$$

where X^+ denotes the Moore-Penrose generalized inverse of a matrix X.

Proof. For Algorithm 1, we have for $n = 0, 1, \ldots,$

$$\begin{aligned} G_{n+2} &= B_{n+1}A_0 \\ &= 2B_nA_0 - B_nQ_{n+1}B_nA_0 \\ &= 2G_{n+1} - G_{n+1}A_0^+(I - \sum_{i=1}^{\infty} A_i G_{n+1}^{i-1})G_{n+1}, \end{aligned}$$

where $B_n = G_{n+1}A_0^+$ is obtained from (6) (for basic properties of the Moore-Penrose inverse, see [5], for example). Therefore, for $n = 1, 2, \ldots$, $G_{n+1} = F(G_n)$ with the map $F : \mathcal{D} \subset \mathcal{M} \to \mathcal{M}$ defined by

$$F(X) = 2X - XA_0^+(I - \sum_{i=1}^{\infty} A_i X^{i-1})X,$$

where $\mathcal{D}$ is the set of all substochastic matrices. Note that all stochastic matrices are boundary points of $\mathcal{D}$.

Let $H = G_n - G$. Using $(G+H)^i - G^i = \sum_{j=0}^{i-1} G^{i-1-j} H G^j$ (see, e.g., [5, p. 22]), we have

$$
\begin{aligned}
G_{n+1} - G &= F(G+H) - F(G) \\
&= 2H - HA_0^+(G+H) - GA_0^+H \\
&\quad + \sum_{i=1}^{\infty} [HA_0^+ A_i (G+H)^i + GA_0^+ A_i (\sum_{j=0}^{i-1} G^{i-1-j} H G^j)] \\
&= H - HA_0^+ G + \sum_{i=1}^{\infty} HA_0^+ A_i G^i \\
&\quad + H - GA_0^+ H + GA_0^+ \sum_{i=1}^{\infty} A_i G^{i-1} H \\
&\quad + \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} GA_0^+ A_i G^{i-1-j} H G^j + o(\|H\|),
\end{aligned}
$$

where $o(\|H\|)$ denotes a term $W(H)$ such that $\lim_{H \to 0} \|W(H)\|/\|H\| = 0$ for any norm $\|\cdot\|$. Since

$$
\begin{aligned}
H - HA_0^+ G + \sum_{i=1}^{\infty} HA_0^+ A_i G^i &= H - HA_0^+ (G - \sum_{i=1}^{\infty} A_i G^i) \\
&= H - HA_0^+ A_0 \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
H - GA_0^+ H + GA_0^+ \sum_{i=1}^{\infty} A_i G^{i-1} H &= H - GA_0^+ (I - \sum_{i=1}^{\infty} A_i G^{i-1}) H \\
&= H - GA_0^+ A_0 G^+ H \\
&= H - GG^+ H,
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=2}^{\infty} \sum_{j=1}^{i-1} GA_0^+ A_i G^{i-1-j} H G^j &= \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} GA_0^+ A_i G^{i-1-j} H G^j \\
&= \sum_{j=1}^{\infty} GA_0^+ (\sum_{i=j+1}^{\infty} A_i G^{i-1-j}) H G^j
\end{aligned}
$$

$$= \sum_{j=1}^{\infty} GA_0^+ \bar{A}_{j+1} HG^j,$$

we have

$$G_{n+1} - G = T(G_n - G) + o(\|G_n - G\|). \tag{13}$$

Now for any $\epsilon > 0$, we choose a norm on $\mathcal{M}$ such that the induced operator norm $\|\cdot\|$ of T satisfies $\|T\| \le \rho(T) + \epsilon$. Note that this can be done even for an operator from a general Banach space into itself (see [3, p. 77]). By (13), we have for $n \ge n_0$

$$\|G_{n+1} - G - T(G_n - G)\| \le \epsilon \|G_n - G\|.$$

Thus

$$\begin{aligned} \|G_{n+1} - G\| &\le \|G_{n+1} - G - T(G_n - G)\| + \|T(G_n - G)\| \\ &\le (\rho(T) + 2\epsilon)\|G_n - G\| \\ &\le (\rho(T) + 2\epsilon)^{n-n_0+1}\|G_{n_0} - G\|. \end{aligned}$$

This implies $\limsup_{n\to\infty} \sqrt[n]{\|G_n - G\|} \le \rho(T) + 2\epsilon$. The conclusion in the theorem follows since $\epsilon > 0$ is arbitrary. □

The matrix A_0 is often nonsingular in applications. For example, for a class of Markov chains considered in [10], the matrix A_0 is an upper triangular Toeplitz matrix with a positive main diagonal.

Corollary 8 *If A_0 is nonsingular in Theorem 6, then for Algorithm 1,*

$$\limsup_{n\to\infty} \sqrt[n]{\|G_n - G\|} \le \rho(T_3),$$

where the operator T_3 is defined by (10).

Proof. When A_0 is nonsingular, G is also nonsingular. Note also that $GA_0^{-1} = (I - \bar{A}_1)^{-1}$. □

From the previous discussions, we conclude that the convergence of Algorithm 1 is linear whenever A_0 is nonsingular and the Markov chain is not null recurrent. The convergence rate is comparable with that of iteration (5), which is faster than iterations (3) and (4).

The next two results further confirm that the convergence of Algorithm 1 is often not slower than that of iteration (5). The first result can be seen readily from the proof of Theorem 5 in [8].

Theorem 9 *If the matrix G is stochastic, then for iteration* (5),

$$\limsup_{n\to\infty} \sqrt[n]{\|G_n - G\|} \le \rho((I-\bar{A}_1)^{-1}\sum_{j=2}^{\infty}\bar{A}_j). \tag{14}$$

Equality holds in (14) *if the matrix $(I-A_1)^{-1}\sum_{i=2}^{\infty}A_i$ has no zero rows.*

Proposition 10 *If A_0 is nonsingular and G is stochastic, then for Algorithm 1,*

$$\limsup_{n\to\infty} \sqrt[n]{\|G_n - G\|} \le \rho((I-\bar{A}_1)^{-1}\sum_{j=2}^{\infty}\bar{A}_j).$$

Proof. In view of Corollary 8, we only need to show $\rho(T_3) \le \rho((I-\bar{A}_1)^{-1}\sum_{j=2}^{\infty}\bar{A}_j)$. By a generalized Perron-Frobenius Theorem (see [13]), we have

$$\rho(T_3)H = \sum_{j=2}^{\infty}(I-\bar{A}_1)^{-1}\bar{A}_j H G^{j-1}$$

for some nonzero $H \ge 0$. Since G is stochastic, we have

$$\rho(T_3)He = \sum_{j=2}^{\infty}(I-\bar{A}_1)^{-1}\bar{A}_j He$$

Since $He \ne 0$, $\rho(T_3)$ is an eigenvalue of $(I-\bar{A}_1)^{-1}\sum_{j=2}^{\infty}\bar{A}_j$. □

4 Numerical results

We will perform some numerical experiments on two simple examples. The first one is given in [1]. The second one is a particular case of the Markov chains considered in [10].

Example 1. Consider the Markov chain of the $M/G/1$ type for which

$$A_0 = \frac{4}{3}(1-p)\begin{pmatrix} 0.05 & 0.1 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.05 & 0.1 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.05 & 0.1 \\ 0.1 & 0.05 & 0.2 & 0.1 & 0.3 \\ 0.3 & 0.1 & 0.1 & 0.2 & 0.05 \end{pmatrix}$$

and

$$A_n = p^n A_0, \quad n = 1, 2, \dots,$$

where $p \in (0,1)$. The Markov chain is positive recurrent, null recurrent or transient according as $p < 0.5$, $p = 0.5$ or $p > 0.5$.

We will use Algorithm 1 and iterations (3), (4) and (5) to find the minimal nonnegative solution G of the matrix equation (2) for some values of p. For Algorithm 1, we take $B_0 = I$. All iterations will be stopped as soon as $\|G_n - \sum_{i=0}^{\infty} A_i G_n^i\|_\infty \le 10^{-8}$. In our numerical computations, the matrices A_i are treated as zero matrix for $i > 50$ when $p \le 0.6$ and they are treated as zero matrix for $i > 100$ when $p > 0.6$. The numerical results are reported in Table 1.

Example 2. Consider the Markov chain of the $M/G/1$ type for which

$$A_0 = \begin{pmatrix} a_0 & \cdots & a_4 \\ & \ddots & \vdots \\ & & a_0 \end{pmatrix}, \quad A_j = \begin{pmatrix} a_{5j} & \cdots & a_{5j+4} \\ \vdots & & \vdots \\ a_{5j-4} & \cdots & a_{5j} \end{pmatrix}, \quad j \ge 1$$

are all 5×5 Toeplitz matrices, where $a_i = (1-p)p^i$ with $p \in (0,1)$. The Markov chain is positive recurrent, null recurrent or transient according as $p < 5/6$, $p = 5/6$ or $p > 5/6$.

For Algorithm 1, we take $B_0 = I + A_1$. For all iterations, we use the same stopping criterion as in Example 1. In our numerical computations, the matrices A_i are treated as zero matrix for $i > 20$ when $p \le 0.6$ and they are treated as zero matrix for $i > 50$ when $p > 0.6$. The numerical results are reported in Table 2.

From Tables 1 and 2 we can see that the convergence rate of Algorithm 1 is comparable with that of iteration (5), which is in general considerably faster than iterations (3) and (4). These observations are consistent with the theory in Sections 2 and 3. We note that all these iterations are very slow when the Markov chain is null recurrent or "nearly null recurrent". The cyclic reduction method proposed in [2] may offer significant improvement in these situations.

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Table 1: Number of Iterations for Example 1

p	Algorithm1	Iteration(5)	Iteration(4)	Iteration(3)
0.10	6	5	5	9
0.20	8	7	8	13
0.30	12	11	14	21
0.40	22	21	28	41
0.45	39	38	54	75
0.48	85	84	122	167
0.49	151	150	222	299
0.50	5001	5000	7497	9999
0.51	151	149	221	298
0.52	85	83	122	166
0.55	39	37	53	74
0.60	21	20	27	40
0.70	12	10	13	20
0.80	8	6	8	12
0.90	5	4	5	8

Table 2: Number of Iterations for Example 2

p	Algorithm1	Iteration(5)	Iteration(4)	Iteration(3)
0.10	2	2	2	3
0.20	2	2	2	4
0.30	3	3	3	5
0.40	4	3	3	7
0.50	4	4	4	9
0.60	6	5	6	12
0.70	9	8	11	20
0.80	27	26	39	63
0.83	179	177	290	441
$0.8\dot{3}$	5470	5468	9046	13607
0.84	98	97	157	241
0.90	12	11	15	27

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