

On algebraic Riccati equations associated with M -matrices[☆]

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Abstract

We consider the algebraic Riccati equation for which the four coefficient matrices form an M -matrix K . When K is a nonsingular M -matrix or an irreducible singular M -matrix, the Riccati equation is known to have a minimal nonnegative solution and several efficient methods are available to find this solution. In this paper we are mainly interested in the case where K is a reducible singular M -matrix. Under a regularity assumption on the M -matrix K , we show that the Riccati equation still has a minimal nonnegative solution. We also study the properties of this particular solution and explain how the solution can be found by existing methods.

Keywords: Algebraic Riccati equation; Reducible singular M -matrix; Minimal nonnegative solution.

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1. Introduction

We consider the algebraic Riccati equation

$$XCX - XD - AX + B = 0, \quad (1)$$

where A, B, C, D are real matrices of sizes $m \times m, m \times n, n \times m, n \times n$, respectively, and

$$K = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \quad (2)$$

is an M -matrix.

For any matrices $A, B \in \mathbb{R}^{p \times q}$, we write $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) for all i, j . A real square matrix A is called a Z -matrix if all its off-diagonal elements are nonpositive. Note that any Z -matrix A can be written as $sI - B$ with $B \geq 0$. A Z -matrix A is called an M -matrix if $s \geq \rho(B)$, where $\rho(\cdot)$ is the spectral radius; it is a singular M -matrix if $s = \rho(B)$ and a nonsingular M -matrix if $s > \rho(B)$.

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If K is an arbitrary M -matrix, then the equation (1) may not have a solution. A simple example is the reducible singular M -matrix

$$K = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad (3)$$

for which the corresponding scalar equation (1) does not have any solutions.

It is proved in [10] that the equation (1) always has a minimal nonnegative solution if K in (2) is a nonsingular M -matrix or an irreducible singular M -matrix. This assumption on the matrix K has now become a standard assumption. In [23], the equation (1) with K being a nonsingular M -matrix or an irreducible singular M -matrix is formally called an M -matrix algebraic Riccati equation. The case where K is a reducible singular M -matrix has never been studied, due to some technical difficulties associated with it. However, this case is of both theoretical and practical interest. The most important application of the Riccati equation (1) is in the study of Markov chains [20], where $K = -Q$ and Q is the generator of a Markov chain. So all diagonal entries of Q are nonpositive, all off-diagonal entries of Q are nonnegative, and every row sum of Q is nonpositive. In this case, K is an M -matrix with $Ke \geq 0$, where e is the vector of ones. Of particular interest is the case where K is a singular M -matrix with $Ke = 0$. In the study of Markov chains, the irreducibility of Q is often assumed since Q is irreducible for most interesting Markov chains. But it is still desirable to have results available when Q happens to be reducible. In general, even if K is an irreducible singular M -matrix, it may still be close to some reducible singular M -matrix. If we run into difficulties for a reducible singular M -matrix, we would also run into difficulties for nearby irreducible singular M -matrices. It is therefore of importance to study the case where K is a reducible singular M -matrix.

2. Existence of a minimal nonnegative solution

As we have seen earlier, the equation (1) may not have a solution for an arbitrary M -matrix K . So to guarantee the existence of a solution we still need to add some assumption on K . The assumption we need is that $Kv \geq 0$ for some vector $v > 0$. Note that this assumption is always satisfied if K is from the study of Markov chains, as we mentioned above, whether it is reducible or not. We also note that the so-called M -splittings of a reducible singular M -matrix have been studied in [21], where this same assumption on the singular M -matrix also plays an important role.

For convenience, we introduce the following definition.

Definition 1. An M -matrix A is said to be regular if $Av \geq 0$ for some $v > 0$.

It is well known [2] that for any nonsingular M -matrix A there is a vector $v > 0$ such that $Av > 0$ and that for any irreducible singular M -matrix A there is a vector $v > 0$ such that $Av = 0$. Therefore, nonsingular M -matrices and irreducible singular M -matrices are always regular M -matrices. It is easily seen that the reducible singular M -matrix in (3) is not a regular M -matrix. Note that any Z -matrix A such that $Av \geq 0$ for some $v > 0$ is an M -matrix [2] and hence a regular M -matrix by definition. Note also that any principal submatrix of a regular M -matrix is still a regular M -matrix.

We are going to prove that the equation (1) has a minimal nonnegative solution whenever K is a regular M -matrix. We start with the following result.

Lemma 1. *Suppose that the matrix K in (2) is a regular M -matrix and all diagonal entries of A are positive. Then (1) has a nonnegative solution Φ such that $D - C\Phi$ is a regular M -matrix. Moreover, $\Phi \leq X$ for any nonnegative solution of the inequality $\mathcal{R}(X) \leq 0$, where*

$$\mathcal{R}(X) = XCX - XD - AX + B.$$

In particular, Φ is the minimal nonnegative solution of the equation (1).

PROOF. The proof is an adaptation of the proof of [10, Theorem 3.1]. We write $A = A_1 - A_2$ and $D = D_1 - D_2$, where $A_1 = \text{diag}(A)$ and $D_1 = \text{diag}(D)$ are the diagonal parts of A and D , respectively. We take $X_0 = 0$ and use the fixed-point iteration:

$$A_1 X_{i+1} + X_{i+1} D_1 = X_i C X_i + X_i D_2 + A_2 X_i + B, \quad i = 0, 1, \dots \quad (4)$$

The iteration is well defined since the diagonal entries of D are nonnegative and the diagonal entries of A are assumed to be positive. It is easily shown by induction that $X_i \leq X_{i+1}$ for any $i \geq 0$.

Since K is a regular M -matrix, we have $Kv \geq 0$ for some $v > 0$. Write $v = [v_1^T \ v_2^T]^T$ with $v_1 \in \mathbb{R}^n$ and $v_2 \in \mathbb{R}^m$. Then we have

$$D_1 v_1 - D_2 v_1 - C v_2 \geq 0, \quad A_1 v_2 - A_2 v_2 - B v_1 \geq 0. \quad (5)$$

We now show that $X_k v_1 \leq v_2$ for all $k \geq 0$. The inequality is trivial for $k = 0$. Assume that $X_i v_1 \leq v_2$ ($i \geq 0$). Then, by (4) and (5),

$$\begin{aligned} A_1 X_{i+1} v_1 + X_{i+1} D_1 v_1 &= X_i C X_i v_1 + X_i D_2 v_1 + A_2 X_i v_1 + B v_1 \\ &\leq X_i C v_2 + X_i D_2 v_1 + A_2 v_2 + B v_1 \\ &\leq X_i D_1 v_1 + A_1 v_2. \end{aligned}$$

It follows that $A_1 X_{i+1} v_1 \leq A_1 v_2 + (X_i - X_{i+1}) D_1 v_1 \leq A_1 v_2$. So $X_{i+1} v_1 \leq v_2$. Thus, we have proved by induction that $X_k v_1 \leq v_2$ for all $k \geq 0$. Now, the sequence $\{X_i\}$ is monotonically increasing and bounded above, and hence has a limit. Let $\Phi = \lim_{i \rightarrow \infty} X_i$. Then Φ is a nonnegative solution of (1) by (4). We also have $\Phi v_1 \leq v_2$ and then $(D - C\Phi)v_1 \geq Dv_1 - Cv_2 \geq 0$. Thus, the Z -matrix $D - C\Phi$ is a regular M -matrix. We now let X be any nonnegative solution of $\mathcal{R}(X) \leq 0$ and re-examine the iteration (4). Since $XCX + XD_2 + A_2X + B \leq A_1X + XD_1$ by $\mathcal{R}(X) \leq 0$, it is easily shown by induction that $X_i \leq X$ for any $i \geq 0$. It follows that $\Phi \leq X$. In particular, Φ is the minimal nonnegative solution of (1). \square

Theorem 2. *If the matrix K in (2) is a regular M -matrix then (1) has a minimal nonnegative solution Φ and $D - C\Phi$ is a regular M -matrix.*

PROOF. If all diagonal entries of A are positive, the result is already proved. Suppose that r diagonal entries of A are 0. Then the r rows of K containing these diagonal entries must be zero rows since K is a regular M -matrix. If $r = m$, then $A = 0$ and $B = 0$, and the equation (1) has a minimal nonnegative solution $\Phi = 0$ and $D - C\Phi$ is a regular M -matrix. So we assume $1 \leq r < m$. Then, there is an $m \times m$ permutation matrix P such that

$$PAP^T = \begin{bmatrix} \tilde{A} & \hat{A} \\ 0 & 0 \end{bmatrix}, \quad PB = \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix}, \quad CP^T = \begin{bmatrix} \tilde{C} & \hat{C} \end{bmatrix},$$

where \tilde{A} is $(m-r) \times (m-r)$, \tilde{B} is $(m-r) \times n$, and \tilde{C} is $n \times (m-r)$. The equation (1) is equivalent to

$$(PX)(CP^T)(PX) - (PX)D - (PAP^T)(PX) + PB = 0.$$

Let

$$PX = \begin{bmatrix} \tilde{X} \\ \hat{X} \end{bmatrix},$$

where \tilde{X} is $(m-r) \times n$. Then

$$\begin{bmatrix} \tilde{X} \\ \hat{X} \end{bmatrix} \begin{bmatrix} \tilde{C} & \hat{C} \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \hat{X} \end{bmatrix} - \begin{bmatrix} \tilde{X} \\ \hat{X} \end{bmatrix} D - \begin{bmatrix} \tilde{A} & \hat{A} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \hat{X} \end{bmatrix} + \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix} = 0. \quad (6)$$

We now take $\hat{X} = 0$. Then the above equation reduces to

$$\tilde{X}\tilde{C}\tilde{X} - \tilde{X}D - \tilde{A}\tilde{X} + \tilde{B} = 0. \quad (7)$$

Since the matrix

$$\tilde{K} = \begin{bmatrix} D & -\tilde{C} \\ -\tilde{B} & \tilde{A} \end{bmatrix} \quad (8)$$

is a principal submatrix of the regular M -matrix

$$\begin{bmatrix} I_n & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & P \end{bmatrix}^T,$$

it is also a regular M -matrix. Since all diagonal entries of \tilde{A} are positive, we know from Lemma 1 that the equation (7) has a minimal nonnegative solution $\tilde{\Phi}$ and $D - \tilde{C}\tilde{\Phi}$ is a regular M -matrix. The solution

$$\Phi = P^T \begin{bmatrix} \tilde{\Phi} \\ 0 \end{bmatrix} \quad (9)$$

of (1) is then the only candidate for the minimal nonnegative solution. To confirm that a minimal nonnegative solution exists, we let

$$P^T \begin{bmatrix} \tilde{X} \\ \hat{X} \end{bmatrix}$$

be any nonnegative solution of (1). Then we have by (6) that

$$\tilde{X}\tilde{C}\tilde{X} + \tilde{X}\hat{C}\hat{X} - \tilde{X}D - \tilde{A}\tilde{X} - \hat{A}\hat{X} + \tilde{B} = 0.$$

Thus

$$\tilde{X}\tilde{C}\tilde{X} - \tilde{X}D - \tilde{A}\tilde{X} + \tilde{B} = -\tilde{X}\hat{C}\hat{X} + \hat{A}\hat{X} \leq 0$$

since $\hat{A} \leq 0$. It follows from Lemma 1 that $\tilde{\Phi} \leq \tilde{X}$. Thus the matrix Φ in (9) is indeed the minimal nonnegative solution of (1). Note that $D - C\Phi = D - (CP^T)(P\Phi) = D - \tilde{C}\tilde{\Phi}$ is a regular M -matrix. \square

Associated with the matrix K in (2) is the matrix

$$H = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} K = \begin{bmatrix} D & -C \\ B & -A \end{bmatrix}. \quad (10)$$

The next factorization result follows from Theorem 2 easily.

Corollary 3. *If the matrix K in (2) is a regular M -matrix, then there exist nonnegative matrices Φ and Ψ such that*

$$\begin{bmatrix} D & -C \\ B & -A \end{bmatrix} \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} = \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -S \end{bmatrix}, \quad (11)$$

where $R = D - C\Phi$ and $S = A - B\Psi$ are regular M -matrices.

PROOF. By Theorem 2, (1) has a minimal nonnegative solution Φ and $D - C\Phi$ is a regular M -matrix. Thus

$$\begin{bmatrix} D & -C \\ B & -A \end{bmatrix} \begin{bmatrix} I_n \\ \Phi \end{bmatrix} = \begin{bmatrix} I_n \\ \Phi \end{bmatrix} R. \quad (12)$$

for $R = D - C\Phi$. Since K is a regular M -matrix, we have $Kv \geq 0$ for some $v = [v_1^T \ v_2^T]^T > 0$, where $v_1 \in \mathbb{R}^n$ and $v_2 \in \mathbb{R}^m$. Then

$$\widehat{K} = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix}$$

is a Z -matrix such that

$$\widehat{K} \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} \geq 0.$$

So \widehat{K} is also a regular M -matrix. Theorem 2 implies that the dual equation of (1)

$$YBY - YA - DY + C = 0 \quad (13)$$

has a minimal nonnegative solution Ψ such that $A - B\Psi$ is a regular M -matrix. Thus

$$\begin{bmatrix} D & -C \\ B & -A \end{bmatrix} \begin{bmatrix} \Psi \\ I_m \end{bmatrix} = \begin{bmatrix} \Psi \\ I_m \end{bmatrix} (-S) \quad (14)$$

for $S = A - B\Psi$. The factorization (11) is obtained by combining (12) and (14). \square

3. Properties of the minimal nonnegative solution

We assume that the matrix K in (2) is a regular M -matrix. So the equation (1) has a minimal nonnegative solution Φ and $D - C\Phi$ is a regular M -matrix, and the dual equation (13) has a minimal nonnegative solution Ψ and $A - B\Psi$ is a regular M -matrix. In this section we prove some additional properties of Φ and Ψ .

Proposition 4. *Suppose that K is a regular M -matrix and $Kw \geq 0$ for $w = [w_1^T \ w_2^T]^T > 0$, where $w_1 \in \mathbb{R}^n$ and $w_2 \in \mathbb{R}^m$. Then $\Phi w_1 \leq w_2$ and $\Psi w_2 \leq w_1$. Moreover, $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are both regular M -matrices.*

PROOF. If A has no zero diagonal entries, we already know from the proof of Lemma 1 that $\Phi w_1 \leq w_2$. Suppose that r diagonal entries of A are 0. If $r = m$, then $A = 0$ and $B = 0$. In this case, $\Phi = 0$ and $\Phi w_1 \leq w_2$ is trivial. So we assume $1 \leq r < m$ and use the notations in the proof of Theorem 2. Let $Pw_2 = \begin{bmatrix} \tilde{w}_2 \\ \hat{w}_2 \end{bmatrix}$, where $\tilde{w}_2 \in \mathbb{R}^{m-r}$. Then the matrix \tilde{K} in (8) satisfies $\tilde{K} \begin{bmatrix} w_1 \\ \tilde{w}_2 \end{bmatrix} \geq 0$. It follows that $\tilde{\Phi} w_1 \leq \tilde{w}_2$. Then

$$\Phi w_1 = P^T \begin{bmatrix} \tilde{\Phi} \\ 0 \end{bmatrix} w_1 \leq P^T \begin{bmatrix} \tilde{w}_2 \\ 0 \end{bmatrix} \leq P^T \begin{bmatrix} \tilde{w}_2 \\ \hat{w}_2 \end{bmatrix} = w_2.$$

By applying the result $\Phi w_1 \leq w_2$ to the dual equation (13), we get $\Psi w_2 \leq w_1$. We then have $\Phi \Psi w_2 \leq \Phi w_1 \leq w_2$ and thus $(I_m - \Phi \Psi)w_2 \geq 0$. So $I_m - \Phi \Psi$ is a regular M -matrix. Similarly, $I_n - \Psi \Phi$ is a regular M -matrix. \square

Note that the relation in (11) does not always give a similarity transformation since the matrix

$$\begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} \quad (15)$$

may be singular. The matrix (15) is nonsingular if and only if the M -matrix $I_m - \Phi \Psi$ is nonsingular, i.e., if and only if $\rho(\Phi \Psi) < 1$.

We denote by $\mathbb{C}_<$, \mathbb{C}_\leq , $\mathbb{C}_>$, and \mathbb{C}_\geq the open left half plane, the closed left half plane, the open right half plane, and the closed right half plane, respectively. We have the following information about the eigenvalues of H even if the matrix (15) is singular.

Lemma 5. *For any M -matrix K , the matrix H in (10) has $m_1(\leq m)$ eigenvalues in $\mathbb{C}_<$, $n_1(\leq n)$ eigenvalues in $\mathbb{C}_>$, and $r = (m - m_1) + (n - n_1)$ zero eigenvalues.*

PROOF. For any $\epsilon > 0$, $K_\epsilon = K + \epsilon I_{m+n}$ is a nonsingular M -matrix. Thus (see [10])

$$H_\epsilon = \begin{bmatrix} D + \epsilon I_n & -C \\ B & -(A + \epsilon I_m) \end{bmatrix}$$

has m eigenvalues in $\mathbb{C}_<$ and n eigenvalues in $\mathbb{C}_>$. The result follows by letting $\epsilon \rightarrow 0^+$ since eigenvalues are continuous functions of ϵ . \square

Since

$$H \begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix} (D - CX) \quad (16)$$

for any solution X of (1), the column space of $\begin{bmatrix} I_n \\ X \end{bmatrix}$ is an invariant subspace of H corresponding to n of its eigenvalues. For the minimal nonnegative solution Φ of (1) (when K in (2) is a regular M -matrix), $D - C\Phi$ is an M -matrix and thus all its eigenvalues are in \mathbb{C}_\geq . When H has exactly n eigenvalues in \mathbb{C}_\geq , the column space of $\begin{bmatrix} I_n \\ \Phi \end{bmatrix}$ must be the invariant subspace of H corresponding to these eigenvalues. When H has more than n eigenvalues in \mathbb{C}_\geq , the next result shows that the column space of $\begin{bmatrix} I_n \\ \Phi \end{bmatrix}$ is an invariant subspace of H corresponding to its n eigenvalues with the largest real parts.

Theorem 6. *Assume that the matrix K in (2) is a regular M -matrix. Let all eigenvalues of H in (10) be arranged in an descending order by their real parts, and be denoted by $\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_{n+m}$. Then the eigenvalues of $D - C\Phi$ are $\lambda_1, \dots, \lambda_n$ and the eigenvalues of $A - B\Psi$ are $-\lambda_{n+1}, \dots, -\lambda_{n+m}$. In particular, λ_n and λ_{n+1} are real numbers.*

PROOF. For any $\epsilon > 0$, let K_ϵ and H_ϵ be as in the proof of Lemma 5. Then the equation (1) corresponding to K_ϵ has a minimal nonnegative solution Φ_ϵ and the eigenvalues of $D + \epsilon I_n - C\Phi_\epsilon$ are the n eigenvalues of H_ϵ in $\mathbb{C}_>$. Since K is a regular M -matrix, $Kv \geq 0$ for some $v > 0$ partitioned as in Proposition 4. Then we have $K_\epsilon v > 0$ and $\Phi_\epsilon v_1 \leq v_2$. So the set $\{\Phi_\epsilon\}$ is bounded and thus there is a sequence $\{\epsilon_k\}$ with $\lim_{k \rightarrow \infty} \epsilon_k = 0$ such that $\lim_{k \rightarrow \infty} \Phi_{\epsilon_k} = \widehat{\Phi}$ exists. It is clear that $\widehat{\Phi}$ is a nonnegative solution of the equation (1) (corresponding to the matrix K) and that the eigenvalues of $D - C\widehat{\Phi}$ are $\lambda_1, \dots, \lambda_n$. Since $\Phi \leq \widehat{\Phi}$, we have $\text{trace}(D - C\Phi) \geq \text{trace}(D - C\widehat{\Phi})$. However, the trace of a matrix is equal to the sum of its eigenvalues. It follows that the eigenvalues of $D - C\Phi$ are $\lambda_1, \dots, \lambda_n$. By applying this result to the dual equation (13), we know that the eigenvalues of $A - B\Psi$ are $-\lambda_{n+1}, \dots, -\lambda_{n+m}$. The eigenvalues λ_n and λ_{n+1} are real numbers because for any M -matrix the eigenvalue with smallest real part must be a real number. \square

Corollary 7. *Let K and H be as in Theorem 6. Then the eigenvalues of $A - \Phi C$ are $-\lambda_{n+1}, \dots, -\lambda_{n+m}$ and the eigenvalues of $D - \Psi B$ are $\lambda_1, \dots, \lambda_n$. In particular, $D - \Psi B$ is nonsingular if and only if $D - C\Phi$ is nonsingular, and $A - \Phi C$ is nonsingular if and only if $A - B\Psi$ is nonsingular.*

PROOF. Since Φ is a solution of (1), it is easily verified (and is well known) that

$$\begin{bmatrix} I_n & 0 \\ \Phi & I_m \end{bmatrix}^{-1} H \begin{bmatrix} I_n & 0 \\ \Phi & I_m \end{bmatrix} = \begin{bmatrix} D - C\Phi & -C \\ 0 & -(A - \Phi C) \end{bmatrix}. \quad (17)$$

By this similarity transformation, the eigenvalues of $A - \Phi C$ are $-\lambda_{n+1}, \dots, -\lambda_{n+m}$ in view of Theorem 6. Applying this result to the dual equation (13), we know that the eigenvalues of $D - \Psi B$ are $\lambda_1, \dots, \lambda_n$. \square

When K in (2) is an irreducible singular M -matrix, the minimal nonnegative solutions Φ and Ψ are positive [11] and the matrix H has a simple zero eigenvalue except for a critical case where H has a double zero eigenvalue with only one linearly independent eigenvector [10]. Our emphasis in this paper is on regular M -matrices that are singular and reducible. We now make the following assumption.

Assumption 1. The matrix H in (10) has only one linearly independent eigenvector corresponding to the zero eigenvalue of multiplicity $r \geq 1$.

Without this assumption, the minimal nonnegative solution of (1) may not be a continuous function of K on the set of regular M -matrices. A trivial example for $m = n = 1$ is given by

$$K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_\epsilon = \begin{bmatrix} \epsilon & -\epsilon \\ -\epsilon & \epsilon \end{bmatrix}.$$

The minimal nonnegative solution corresponding to K is 0, but the minimal nonnegative solution corresponding to K_ϵ is 1 for any $\epsilon > 0$. A less trivial example for $m = n = 2$ is

$$K = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_\epsilon = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & \epsilon & 0 & -\epsilon \\ -1 & 0 & 1 & 0 \\ 0 & -\epsilon & 0 & \epsilon \end{bmatrix}.$$

The minimal nonnegative solution corresponding to K is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, but the minimal nonnegative solution corresponding to K_ϵ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for any $\epsilon > 0$.

Under Assumption 1, the null space of H is one-dimensional. In view of the relation (10), the null spaces of K and K^T are then both one-dimensional. Note that K can have at most one zero diagonal entry under this assumption. By the Perron–Frobenius theorem [2], there are nonnegative nonzero vectors $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, where $u_1, v_1 \in \mathbb{R}^n$ and $u_2, v_2 \in \mathbb{R}^m$, such that

$$K \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0, \quad [u_1^T \ u_2^T]K = 0. \quad (18)$$

They are each unique up to a scalar multiple, in view of our assumption on the null spaces.

Lemma 8. *Let K be a singular M -matrix with Assumption 1 and let u_1, u_2, v_1, v_2 be as in (18). Then $u_1^T v_1 \neq u_2^T v_2$ if and only if zero is a simple eigenvalue of H .*

PROOF. Let P be a nonsingular matrix such that

$$H = P \begin{bmatrix} J & 0 \\ 0 & W \end{bmatrix} P^{-1}, \quad (19)$$

where J is the $r \times r$ Jordan block associated with the zero eigenvalue and W is nonsingular. It follows from (10) and (18) that

$$[u_1^T \ -u_2^T]H = 0, \quad H \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0. \quad (20)$$

Let e_i be the i th column of the identity matrix I_{m+n} . Then by (19) and (20)

$$[u_1^T \ -u_2^T]P = k_1 e_r^T, \quad P^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k_2 e_1, \quad (21)$$

where k_1 and k_2 are nonzero constants. Multiplying the two equations in (21) gives $u_1^T v_1 - u_2^T v_2 = k_1 k_2 e_r^T e_1$. Thus $u_1^T v_1 \neq u_2^T v_2$ if and only if $r = 1$. \square

We now assume that the singular M -matrix K is regular. Then we have $w > 0$ such that $Kw \geq 0$, but we do not necessarily have $Kv = 0$ for some $v > 0$. A simple example is

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

So the vectors u_1, u_2, v_1, v_2 in (18) are nonnegative, but not necessarily positive. However, we still have the following result.

Lemma 9. *Let K be a regular singular M -matrix with Assumption 1 and let u_1, u_2, v_1, v_2 be as in (18). Then $\Phi v_1 \leq v_2$, $\Psi v_2 \leq v_1$, $u_2^T \Phi \leq u_1^T$, $u_1^T \Psi \leq u_2^T$.*

PROOF. If A has no zero diagonal entries, we re-examine the proof of Lemma 1 and find that we still have $X_i v_1 \leq v_2$ for all $i \geq 0$, for the new vectors v_1 and v_2 here. Since $\lim_{i \rightarrow \infty} X_i = \Phi$ was proved there, we have $\Phi v_1 \leq v_2$. When A has zero diagonal entries, we can still show that $\Phi v_1 \leq v_2$ using the procedure in the proof of Proposition 4.

By taking transpose on the equation (1), we know that Φ^T is the minimal nonnegative solution of the equation

$$ZC^T Z - ZA^T - D^T Z + B^T = 0, \quad (22)$$

and for the corresponding singular M -matrix

$$\check{K} = \begin{bmatrix} A^T & -C^T \\ -B^T & D^T \end{bmatrix},$$

we have $\check{K} \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = 0$ by (18). If \check{K} were a regular M -matrix, we could apply the result $\Phi v_1 \leq v_2$ to the equation (22) to get $\Phi^T u_2 \leq u_1$, or $u_2^T \Phi \leq u_1^T$. However, \check{K} is not always regular when K is, as seen from $K = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$. To prove $\Phi^T u_2 \leq u_1$, we first assume that D has no zero diagonal entries and proceed as in the proof of Lemma 1. Let A_1, A_2, D_1, D_2 be as given there. Let $Z_0 = 0$ and consider the fixed-point iteration

$$D_1 Z_{i+1} + Z_{i+1} A_1 = Z_i C^T Z_i + Z_i A_2^T + D_2^T Z_i + B^T.$$

It is easily shown by induction that $Z_i \leq Z_{i+1} \leq \Phi^T$ for all $i \geq 0$. It follows that $\lim_{i \rightarrow \infty} Z_i = Z_*$ exists and is a nonnegative solution of (22). Since $Z_* \leq \Phi^T$ and Φ^T is the minimal nonnegative solution, we have $Z_* = \Phi^T$. As in the proof of Lemma 1, we have $Z_i u_2 \leq u_1$ for all $i \geq 0$. Thus $\Phi^T u_2 \leq u_1$. When D has zero diagonal entries, we can still show that $\Phi^T u_2 \leq u_1$ using the procedure in the proof of Proposition 4.

By applying the results $\Phi v_1 \leq v_2$ and $u_2^T \Phi \leq u_1^T$ to the dual equation (13), we have $\Psi v_2 \leq v_1$ and $u_1^T \Psi \leq u_2^T$. \square

Theorem 10. *Let K be a regular singular M -matrix with Assumption 1 and let u_1, u_2, v_1, v_2 be as in (18). Then*

- (i) *If $u_1^T v_1 > u_2^T v_2$, then $D - C\Phi$ is singular and $A - \Phi C$ is nonsingular, $\Phi v_1 = v_2$, $\Psi v_2 \neq v_1$, $u_1^T \Psi = u_2^T$, $u_2^T \Phi \neq u_1^T$.*
- (ii) *If $u_1^T v_1 < u_2^T v_2$, then $D - C\Phi$ is nonsingular and $A - \Phi C$ is singular, $\Phi v_1 \neq v_2$, $\Psi v_2 = v_1$, $u_1^T \Psi \neq u_2^T$, $u_2^T \Phi = u_1^T$.*
- (iii) *If $u_1^T v_1 = u_2^T v_2$, then $D - C\Phi$ and $A - \Phi C$ are both singular, $\Phi v_1 = v_2$, $\Psi v_2 = v_1$, $u_1^T \Psi = u_2^T$, $u_2^T \Phi = u_1^T$.*

PROOF. Rewriting (17) as

$$H = \begin{bmatrix} I_n & 0 \\ \Phi & I_m \end{bmatrix} \begin{bmatrix} D - C\Phi & -C \\ 0 & -(A - \Phi C) \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -\Phi & I_m \end{bmatrix}$$

and using (20), we get

$$(u_1^T - u_2^T \Phi)(D - C\Phi) = 0, \quad (A - \Phi C)(\Phi v_1 - v_2) = 0. \quad (23)$$

For the dual equation (13), we have the corresponding result

$$(u_2^T - u_1^T \Psi)(A - B\Psi) = 0, \quad (D - \Psi B)(\Psi v_2 - v_1) = 0. \quad (24)$$

If $D - C\Phi$ is nonsingular, then $u_1^T = u_2^T \Phi$ by (23) and $u_1^T v_1 = u_2^T \Phi v_1 \leq u_2^T v_2$ by Lemma 9. If $A - B\Psi$ is nonsingular, then $u_2^T = u_1^T \Psi$ by (24) and $u_2^T v_2 = u_1^T \Psi v_2 \leq u_1^T v_1$ by Lemma 9.

We now prove (i). When $u_1^T v_1 > u_2^T v_2$, $D - C\Phi$ is singular, and then $A - \Phi C$ and $A - B\Psi$ are nonsingular by Lemma 8, Theorem 6 and Corollary 7. It follows from (23) and (24) that $\Phi v_1 = v_2$ and $u_1^T \Psi = u_2^T$. We now show that $\Psi v_2 \neq v_1$. This is trivial when $v_2 = 0$. Suppose $v_2 \neq 0$ and $\Psi v_2 = v_1$. Then $(A - B\Psi)v_2 = Av_2 - Bv_1 = 0$, contradictory to the nonsingularity of $A - B\Psi$. Finally, $u_2^T \Phi \neq u_1^T$ is trivial when $u_2 = 0$. Suppose $u_2 \neq 0$ and $u_2^T \Phi = u_1^T$. Then $u_2^T (A - \Phi C) = -u_1^T C + u_2^T A = 0$, contradictory to the nonsingularity of $A - \Phi C$.

The results in (ii) are obtained by applying the results in (i) to the dual equation (13) and using Corollary 7.

To prove (iii), we consider (as in [14])

$$K(\alpha) = \begin{bmatrix} D & -C \\ -\alpha B & \alpha A \end{bmatrix}$$

for $\alpha > 1$. It is clear that $K(\alpha)$ is still a regular singular M -matrix, and the null spaces of $K(\alpha)$ and $K(\alpha)^T$ are still one-dimensional. We now have

$$u_1(\alpha) = u_1, \quad u_2(\alpha) = \alpha^{-1}u_2, \quad v_1(\alpha) = v_1, \quad v_2(\alpha) = v_2.$$

So we have $u_1(\alpha)^T v_1(\alpha) > u_2(\alpha)^T v_2(\alpha)$. Let $\Phi(\alpha)$ and $\Psi(\alpha)$ be the minimal nonnegative solutions of equations (1) and (13), respectively, corresponding to $K(\alpha)$. We have by (i) that $D - C\Phi(\alpha)$ is singular, $\Phi(\alpha)v_1 = v_2$, $u_1^T \Psi(\alpha) = \alpha^{-1}u_2^T$. Using the notation in Lemma 5, H has m_1 eigenvalues in $\mathbb{C}_<$, n_1 eigenvalues in $\mathbb{C}_>$, and r zero eigenvalues. By Assumption 1, H has one $r \times r$ Jordan block associated with the zero eigenvalue. Let

$$H(\alpha) = \begin{bmatrix} D & -C \\ \alpha B & -\alpha A \end{bmatrix}.$$

For any $\alpha > 1$ sufficiently close to 1, $m - m_1$ zero eigenvalues of H are perturbed to eigenvalues of $H(\alpha)$ in $\mathbb{C}_<$ and the real parts of these $m - m_1$ eigenvalues of $H(\alpha)$ are strictly larger than those of the remaining m_1 eigenvalues of $H(\alpha)$ in $\mathbb{C}_<$; $n - n_1$ zero eigenvalues of H are perturbed to eigenvalues of $H(\alpha)$ in $\mathbb{C}_>$, and the real parts of these $n - n_1$ eigenvalues of $H(\alpha)$ are strictly smaller than those of the remaining

n_1 eigenvalues of $H(\alpha)$ in $\mathbb{C}_>$. Let \mathcal{V} and $\mathcal{V}(\alpha)$ be the column spaces of $\begin{bmatrix} I_n \\ \Phi \end{bmatrix}$ and $\begin{bmatrix} I_n \\ \Phi(\alpha) \end{bmatrix}$, respectively. By Assumption 1, H has one $r \times r$ Jordan block associated with the zero eigenvalue of multiplicity r . So every invariant subspace of H corresponding to the zero eigenvalue is uniquely determined by its dimension ℓ ($1 \leq \ell \leq r$), namely $\text{Ker}(H^\ell)$. Let \mathcal{V}_1 be the invariant subspace of H corresponding to its n_1 eigenvalues in $\mathbb{C}_>$ and \mathcal{V}_2 be the invariant subspace of H spanned by a Jordan chain of length $n - n_1$ corresponding to the zero eigenvalue. Let $\mathcal{V}_1(\alpha)$ be the invariant subspace of $H(\alpha)$ corresponding to its n_1 eigenvalues with largest real parts and $\mathcal{V}_2(\alpha)$ be the invariant subspace of $H(\alpha)$ corresponding to its remaining $n - n_1$ eigenvalues in \mathbb{C}_\geq . By Theorem 6, we have $\mathcal{V} = \mathcal{V}_1 \dot{+} \mathcal{V}_2$ and $\mathcal{V}(\alpha) = \mathcal{V}_1(\alpha) \dot{+} \mathcal{V}_2(\alpha)$. As in the proof of [14, Theorem 3.3], we have $\theta(\mathcal{V}_1(\alpha), \mathcal{V}_1) \leq c_1 \|K(\alpha) - K\|$ for some constant $c_1 > 0$, where θ denotes the gap between two subspaces [8]. By using the perturbation theory in section 16.5 of [8] and section 5 of [9] (see [8, Theorem 16.4.1] or [9, Theorem 5.1] in particular), we have $\theta(\mathcal{V}_2(\alpha), \mathcal{V}_2) \leq c_2 \|K(\alpha) - K\|^{1/r}$ for some constant $c_2 > 0$. It follows as in [14] that $\theta(\mathcal{V}(\alpha), \mathcal{V}) \leq c_3 \|K(\alpha) - K\|^{1/r}$ and that

$$\|\Phi(\alpha) - \Phi\| \leq c_4 \|K(\alpha) - K\|^{1/r}$$

for some constants $c_3, c_4 > 0$. In particular, $\lim_{\alpha \rightarrow 1^+} \Phi(\alpha) = \Phi$. Similarly, we can show that $\lim_{\alpha \rightarrow 1^+} \Psi(\alpha) = \Psi$. Letting $\alpha \rightarrow 1^+$ in our conclusions about $\Phi(\alpha)$ and $\Psi(\alpha)$, we obtain that $D - C\Phi$ is singular, $\Phi v_1 = v_2$, $u_1^T \Psi = u_2^T$. The other results in (iii) follows by duality. \square

Corollary 11. *Let K be a regular singular M -matrix with Assumption 1. If $u_1^T v_1 = u_2^T v_2$, then $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are regular singular M -matrices.*

PROOF. By Proposition 4, $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are regular M -matrices. By Theorem 10, $\Phi\Psi v_2 = \Phi v_1 = v_2$ and $\Psi\Phi v_1 = \Psi v_2 = v_1$. So v_1 and v_2 are both nonzero vectors, and 1 is an eigenvalue of $\Phi\Psi$ and $\Psi\Phi$. Thus $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are both singular. \square

We also have the following conjecture.

Conjecture 12. *Let K be a regular singular M -matrix with Assumption 1. If $u_1^T v_1 \neq u_2^T v_2$, then $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are nonsingular M -matrices.*

The conjecture is known to be true when K is irreducible [15]. Attempts to find counterexamples for the reducible case have failed and we are led to believe that the conjecture is true. But at present we can prove it only under a positivity assumption.

Proposition 13. *Let K be a regular singular M -matrix with Assumption 1. Then $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are nonsingular M -matrices if $u_1^T v_1 \neq u_2^T v_2$ and at least one of Φ and Ψ is positive.*

PROOF. Assume $u_1^T v_1 > u_2^T v_2$. If $\Phi > 0$, we know from Lemma 9 and Theorem 10 that $\Phi\Psi v_2 < \Phi v_1 = v_2$. If $\Psi > 0$, we know from Lemma 9 and Theorem 10 that $u_2^T \Phi\Psi < u_1^T \Psi = u_2^T$. In either case we have $\rho(\Phi\Psi) = \rho(\Psi\Phi) < 1$ (see [2]). So $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are nonsingular M -matrices. The result for the case $u_1^T v_1 < u_2^T v_2$ follows by duality. \square

As we mentioned earlier, Φ and Ψ are always positive when K is irreducible. When K is reducible, they may still be positive. We also have the following related result.

Proposition 14. *Let K be a regular M -matrix. Then the equation (1) can have at most two positive solutions if the eigenvalues of H are all simple.*

PROOF. Let $w > 0$ be such that $Kw \geq 0$, and let $K_\epsilon = K + \epsilon w^T w I - \epsilon w w^T$. Then $K_\epsilon w = Kw \geq 0$. So K_ϵ is an irreducible M -matrix for any $\epsilon > 0$. Suppose that the equation (1) (corresponding to K) has more than two positive solutions. Then each of these solutions is determined by n eigenvectors of H corresponding to n of its simple eigenvalues. So each solution changes continuous as K is perturbed to K_ϵ . It then follows that the equation (1) corresponding to K_ϵ would have more than two positive solutions for $\epsilon > 0$ sufficiently small, which is contradictory to [4, Theorem 2.13]. \square

4. Numerical methods for the minimal nonnegative solution

When the matrix K in (2) is a nonsingular M -matrix or an irreducible singular M -matrix, various numerical methods are available to find the minimal nonnegative solution of the equation (1). See [1, 3, 4, 5, 7, 10, 12, 13, 14, 15, 16, 17, 18, 22, 23]. In this section we assume that K is a regular M -matrix that is singular and reducible. We have already seen from Theorem 2 that the case where A has zero diagonal entries can be reduced to the case where A has no zero diagonal entries, and we know from Lemma 1 that the minimal nonnegative solution of the reduced equation can be found by a simple fixed-point iteration. However, some other fixed-point iterations described in [10] may not be well-defined when the matrix $I \otimes A + D^T \otimes I$ is a singular M -matrix, where \otimes is the Kronecker product, even if all diagonal entries of A and D are positive. We now assume that the matrix H in (10) has a simple zero eigenvalue. Recall that zero is a simple eigenvalue of H if and only if $u_1^T v_1 \neq u_2^T v_2$ for the vectors u_1, u_2, v_1, v_2 in (18). We then know from Theorem 10 that the matrix $I \otimes (A - \Phi C) + (D - C\Phi)^T \otimes I$ is a nonsingular M -matrix. It follows that the matrix $I \otimes A + D^T \otimes I$ is also a nonsingular M -matrix.

The class of fixed-point iterations discussed in [10] can then be applied to find the minimal nonnegative solution Φ . The theoretical results in [10] (Theorems 2.3, 2.4, 2.5, 2.7) are all applicable. The convergence of these fixed-point iterations is linear.

Since zero is a simple eigenvalue of H , we have $\lambda_n > \lambda_{n+1}$ in Theorem 6 and the minimal nonnegative solution Φ can be computed by finding the invariant subspace of H corresponding to the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ using the Schur method, as in [10]. When the matrix K is such that $Ke = 0$, as in the study of Markov chains, the eigenvector corresponding to the zero eigenvalue is known exactly and the modified Schur method described in [12] can be used to find Φ with much better accuracy when $\lambda_n \approx \lambda_{n+1}$.

The minimal nonnegative solution Φ can also be found by Newton's method. Indeed, Theorem 2.3 of [14] is still valid. The proof there for the case where K is a nonsingular M -matrix is valid for the current situation since $I \otimes A + D^T \otimes I$ and $I \otimes (A - \Phi C) + (D - C\Phi)^T \otimes I$ are nonsingular M -matrices by our assumption that zero is a simple eigenvalue of H . The convergence of Newton's method is quadratic.

A structure-preserving doubling algorithm (SDA) has been presented in [17] to find the minimal nonnegative solutions Φ and Ψ simultaneously when K is a nonsingular M -matrix. It is easy to see that the main results in [17] (Theorems 3.1 and 4.1) still

hold when K is a regular singular M -matrix. One would only need to modify some statements in their proofs. Specifically, we now have that the matrices $R = D - C\Phi$ and $S = A - B\Psi$ are regular M -matrices (so all eigenvalues of R and S are in \mathbb{C}_{\geq}), the matrices D and A are regular M -matrices, and $\rho(R_\gamma) \leq 1, \rho(S_\gamma) \leq 1$ for all $\gamma > 0$, where $R_\gamma = (R + \gamma I_n)^{-1}(R - \gamma I_n)$ and $S_\gamma = (S + \gamma I_m)^{-1}(S - \gamma I_m)$. The SDA in [17] generates four sequences $\{E_k\}, \{F_k\}, \{G_k\}, \{H_k\}$ and [17, Theorem 4.1] implies that

$$\limsup_{k \rightarrow \infty} \sqrt[2^k]{\|H_k - \Phi\|} \leq \rho(R_\gamma)\rho(S_\gamma), \quad \limsup_{k \rightarrow \infty} \sqrt[2^k]{\|G_k - \Psi\|} \leq \rho(R_\gamma)\rho(S_\gamma) \quad (25)$$

for $\gamma > \max\{\max a_{ii}, \max d_{ii}\}$.

The SDA of [17] is further studied in [15]. When K is an irreducible singular M -matrix, it is shown in [15] that the SDA is well defined and (25) holds for $\gamma \geq \max\{\max a_{ii}, \max d_{ii}\}$. Moreover, among these γ values, the SDA will have fastest convergence for $\gamma = \max\{\max a_{ii}, \max d_{ii}\}$. The proof in [15] is based on the fact that $0 \leq G_k < \Psi$, $0 \leq H_k < \Phi$, and $\rho(G_k H_k) < \rho(\Psi\Phi) \leq 1$. When K is a nonsingular M -matrix, we can reach the same conclusion about the SDA since we now have $0 \leq G_k \leq \Psi$, $0 \leq H_k \leq \Phi$, and $\rho(G_k H_k) \leq \rho(\Psi\Phi) < 1$.

When K is a reducible singular M -matrix and H has a simple zero eigenvalue, we know (from Lemma 8, Theorem 10 and Corollary 7) that one of $\rho(R_\gamma)$ and $\rho(S_\gamma)$ is equal to 1 and the other is strictly smaller than 1. It follows that $H_k \rightarrow \Phi$ and $G_k \rightarrow \Psi$, both quadratically. We can allow $\gamma = \max\{\max a_{ii}, \max d_{ii}\}$ if we have $\rho(\Phi\Psi) < 1$. Note that this special γ value is positive since K can have at most one zero diagonal entry when H has a simple zero eigenvalue. Note also that we do have $\rho(\Phi\Psi) < 1$ if Conjecture 12 is true.

A negative answer to Conjecture 12 is also interesting. Suppose there is an example with singular $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ when K is a reducible singular M -matrix and H has a simple zero eigenvalue. Then for this example, the matrices to be inverted in each step of the SDA, $I_m - H_k G_k$ and $I_n - G_k H_k$, will be nearly singular near convergence. On the other hand, the matrix $I \otimes (A - \Phi C) + (D - C\Phi)^T \otimes I$ associated with Newton's method is known to be a nonsingular M -matrix. To compute Φ with high accuracy, one would have to use Newton's method, whose computational work is roughly 3 times that for the SDA in each iteration.

In the SDA of [17], the parameter γ is used in a Cayley transform applied to the matrix H . In [22], a generalized Cayley transform involving two parameters α and β is applied to H , and the resulting doubling algorithm is called ADDA. The iteration formulas in the ADDA are exactly the same as in [17], but the initial matrices E_0, F_0, G_0, H_0 for ADDA (G_0 and H_0 are denoted by Y_0 and X_0 in [22]) are determined by

$$\begin{aligned} \begin{bmatrix} E_0 & -G_0 \\ -H_0 & F_0 \end{bmatrix} &= \begin{bmatrix} D + \alpha I_n & -C \\ B & -A - \beta I_m \end{bmatrix}^{-1} \begin{bmatrix} D - \beta I_n & -C \\ B & -A + \alpha I_m \end{bmatrix} \\ &= \begin{bmatrix} D + \alpha I_n & -C \\ -B & A + \beta I_m \end{bmatrix}^{-1} \begin{bmatrix} D - \beta I_n & -C \\ -B & A - \alpha I_m \end{bmatrix}. \end{aligned} \quad (26)$$

The above compact form for determining E_0, F_0, G_0, H_0 is based on a result of Poloni [19] (see also [4, Theorem 5.5]). One can verify that these initial matrices are the same as those determined in [22]. When $\alpha = \beta = \gamma$, the ADDA is reduced to the SDA and the initial matrices E_0, F_0, G_0, H_0 from (26) are exactly the same as those in [17].

When K is a nonsingular M -matrix or an irreducible singular M -matrix, the convergence theory of the ADDA has been given in [22]. The following result allows K to be singular and reducible.

Theorem 15. *Let K be a regular singular M -matrix and assume that $\alpha > \max a_{ii}$ and $\beta > \max d_{ii}$. Then the ADDA is well defined with $I - G_k H_k$ and $I - H_k G_k$ being nonsingular M -matrices for each $k \geq 0$. Moreover, $E_0 \leq 0, F_0 \leq 0, E_k \geq 0, F_k \geq 0, 0 \leq H_{k-1} \leq H_k \leq \Phi, 0 \leq G_{k-1} \leq G_k \leq \Psi$ for all $k \geq 1$, and*

$$\limsup_{k \rightarrow \infty} \sqrt[2^k]{\|H_k - \Phi\|} \leq r(\alpha, \beta), \quad \limsup_{k \rightarrow \infty} \sqrt[2^k]{\|G_k - \Psi\|} \leq r(\alpha, \beta), \quad (27)$$

where $r(\alpha, \beta) = \rho((R + \alpha I)^{-1}(R - \beta I)) \cdot \rho((S + \beta I)^{-1}(S - \alpha I))$ with $R = D - C\Phi, S = A - B\Psi$.

PROOF. The proof is largely the same as in [22]. The main difference is that the approach in [15] cannot be used to prove that $I - G_k H_k$ and $I - H_k G_k$ are nonsingular M -matrices for each $k \geq 0$. Instead, this property is proved using the approach in [17], which does not allow $\alpha = \max a_{ii}$ or $\beta = \max d_{ii}$. \square

When H has a simple zero eigenvalue, we know from Lemma 8, Theorem 10, and Corollary 7 that one of the matrices R and S is singular and the other is nonsingular. It follows from [22, Theorem 2.3] that $r(\alpha, \beta) < 1$ in Theorem 15. Thus, H_k converges to Φ quadratically, and G_k converges to Ψ quadratically. Since K has at most one zero diagonal entry when H has a simple zero eigenvalue, we have $\max a_{ii} > 0$ and $\max d_{ii} > 0$ when $m, n \geq 2$ (which we shall assume). By [22, Theorem 2.3] $r(\alpha, \beta)$ is strictly increasing in α for $\alpha \geq \max a_{ii}$, and is strictly increasing in β for $\beta \geq \max d_{ii}$. If Conjecture 12 is true, then $\rho(\Phi\Psi) < 1$ when H has a simple zero eigenvalue. In this case we can use the approach in [15] to prove that the ADDA is well defined with $I - G_k H_k$ and $I - H_k G_k$ being nonsingular M -matrices for each $k \geq 0$ even when $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$. Therefore, if Conjecture 12 is true, we should normally use the optimal values $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$ for the ADDA. Otherwise, we may have to take α and β to be slightly larger. When $\max a_{ii}$ is much larger (or smaller) than $\max d_{ii}$, the convergence of ADDA may be significantly faster than that of the SDA in [17], it is also faster than the SDA-ss in [5]. When $\max a_{ii} \approx \max d_{ii}$ but $\max a_{ii} \gg \min a_{ii}$ and/or $\max d_{ii} \gg \min d_{ii}$, all three doubling algorithms take roughly the same number of iterations for convergence, which may be significantly larger than that required for the Newton iteration, as noted in [13].

We also remark that the modified Schur method described earlier can achieve very good normwise accuracy, but this method is unable to compute any tiny positive entries in the minimal nonnegative solution with good relative accuracy, while the fixed-point iterations, the Newton iteration, the doubling algorithms can be implemented in such a way that these tiny positive entries can be computed with high relative accuracy [23].

The discussion of the numerical methods in this section typically requires the assumption that zero is a simple eigenvalue of H , which is stronger than Assumption 1 we made for the theoretical results in the previous section. The more involved critical case in Theorem 10 (iii) is barely treated here. Although Theorem 15 is valid for the critical case, the conclusions in (27) do not imply the convergence of H_k and G_k to Φ and Ψ respectively, since $r(\alpha, \beta) = 1$ in the critical case.

5. Some illustrating examples

When the matrix K in (2) is a regular M -matrix that is singular and reducible and the matrix H in (10) has a simple zero eigenvalue, the numerical algorithms reviewed in the previous section can be applied without difficulty (with the help of our new theoretical results) and the numerical behaviour of these algorithms is very similar to that for the equation (1) corresponding to a nearby nonsingular M -matrix or irreducible singular M -matrix. So in this section, we only present a few examples of small sizes to illustrate some of the theoretical results established in this paper.

Take

$$K = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Then K is a regular M -matrix that is singular and reducible, with $Ke = 0$. The 2×2 matrices A, B, C, D are then determined by (2). We find that the eigenvalues of H are $2, 1, 0, -3$. By Theorem 6 the minimal nonnegative solution Φ of (1) is determined by the eigenvectors corresponding to the eigenvalues 2 and 1, and is found to be

$$\begin{bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}.$$

As shown in Lemma 9 and Theorem 10 (ii), Φ is substochastic.

We now take

$$K = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 2 & -2 \\ -1 & -1 & 0 & 2 \end{bmatrix}.$$

Then K is a regular M -matrix that is singular and reducible, with $Ke = 0$. The 2×2 matrices A, B, C, D are again determined by (2). We find that the eigenvalues of H are $3, 0, -1, -2$. The minimal nonnegative solution Φ of (1) is determined by the eigenvectors corresponding to the eigenvalues 3 and 0, and is found to be

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

As shown in Theorem 10 (i), Φ is stochastic. By Proposition 14 the equation (1) has at most two positive solutions. For this example, (1) has exactly two positive solutions. As suggested by [6, Lemma 11], the other positive solution of (1) is determined by the eigenvectors corresponding to the eigenvalues 3 and -1 , and is found to be

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

For this example, Proposition 13 also applies, and we know that $I - \Phi\Psi$ and $I - \Psi\Phi$ are nonsingular M -matrices.

Finally we take

$$K = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Then K is a regular M -matrix that is singular and reducible, with $Ke = 0$. The 2×2 matrices A, B, C, D are again determined by (2). We find that the eigenvalues of H are $1, 0, 0, -1$ and that there is only one linearly independent eigenvector corresponding to the double eigenvalue 0. The minimal nonnegative solution of (1) is determined by the eigenvectors corresponding to the eigenvalues 1 and 0, and is found to be

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify that this is the only nonnegative solution of (1). For this example, we have case (iii) in Theorem 10 and Φ is stochastic.

6. Conclusions

In this paper we have studied algebraic Riccati equations associated with regular M -matrices. Our results extend previous results for algebraic Riccati equations associated with nonsingular M -matrices or irreducible singular M -matrices. In the future, we can use the term M -matrix algebraic Riccati equation to refer to equation (1) for which the matrix K in (2) is a regular M -matrix. While we have been able to prove a number of theoretical results for this larger class of Riccati equations (sometimes with proofs simpler than previous proofs for special cases; see proof of Theorem 6 for example), we are unable to prove the statement given in Conjecture 12. We have pointed out that an answer to the conjecture (whether it is positive or negative) has interesting implications. Some further research is needed to settle this conjecture.

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