Some Results on Sparse Block Factorization Iterative Methods

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ABSTRACT

We consider the general sparse block factorization iterative methods as Beauwens and Ben Bouzid did. We develop some existence and convergence theorems for general nonsingular H-matrices.

1. INTRODUCTION

Consider the linear system

$$Ax = b, \tag{1.1}$$

where A is a nonsingular large sparse complex matrix. A conjugate gradient type method to solve (1.1) has been proposed and studied in [2] and [3]. As is well known, we should generally apply the method to the preconditioned system

$$M^{-1}Ax = M^{-1}b, \tag{1.2}$$

where M is a preconditioning matrix, in some sense an approximation of A. M is constructed so that a system with M as coefficient matrix is easy to solve and

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the condition number of $M^{-1}A$ is much smaller than that of A. The sparse block factorization method is a very efficient one for constructing such an M. Meanwhile it produces automatically an iterative method for solving (1.1), the sparse block factorization iterative method.

2. SPARSE BLOCK FACTORIZATION ITERATIVE METHODS

Let $A \in C^{n,n}$ be partitioned into block matrix form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}. \tag{2.1}$$

Here $A_{ij} \in C^{n_i, n_j}$, $1 \leqslant n_i, n_j \leqslant n$. Matrices $\alpha, \beta \in C^{n,n}$ are partitioned in the same manner.

Compute recursively

$$P_{11} = L_{11} = U_{11} = \alpha_{11} * A_{11},$$

$$U_{1j} = \alpha_{1j} * A_{1j}, \qquad 1 < j \leqslant m,$$

$$L_{j1} = \alpha_{j1} * A_{j1}, \qquad 1 < j \leqslant m,$$

$$P_{ii} = L_{ii} = U_{ii} = \alpha_{ii} * A_{ii} - \beta_{ii} * \left(\sum_{s=1}^{i-1} L_{is} K_{ss} U_{si} \right), \qquad 1 < i \leqslant m, \tag{2.2}$$

$$U_{ij} = \alpha_{ij} * A_{ij} - \beta_{ij} * \left(\sum_{s=1}^{i-1} L_{is} K_{ss} U_{sj} \right), \qquad 1 < i < j \leqslant m,$$

$$L_{ji} = \alpha_{ji} * A_{ji} - \beta_{ji} * \left(\sum_{s=1}^{i-1} L_{js} K_{ss} U_{si} \right), \qquad 1 < i < j \leqslant m.$$

Here $*$ is Hadamard multiplication, and K_{ss} an approximation to P_{ss}^{-1}.

From (2.2) we get the block diagonal matrix P, lower block triangular matrix L, and upper block triangular matrix U; their sparsity can be controlled through the sparsity of α, β, and the block diagonal matrix K. If P is nonsingular, then $M = LP^{-1}U$ is called a sparse block factorization or block OBV factorization of A. With α, β, and K properly chosen, M can be used as a preconditioning matrix for (1.1). On the other hand, $A = M - R$ represents a splitting of the matrix A, and associated with this splitting is an iterative method

$$Mx_{n+1} = Rx_n + b, \tag{2.3}$$

or

$$x_{n+1} = x_n + M^{-1}(b - Ax_n), \tag{2.4}$$

which, being the standard stationary iterative method for the given splitting, is called the sparse block factorization iterative method. We remark that the convergence of (2.3) indicates, to some extent, the effect of M as a preconditioning matrix. The existence of sparse block factorizations (or the nonsingularity of P) and the convergence of corresponding iterative methods have been studied for nonsingular M-matrices in [1]; see also [4], [5], and [6]. In this paper, we'll develop some existence and convergence theorems for general nonsingular H-matrices.

3. EXISTENCE

Let E denote the matrix all of whose entries are equal to unity; set $\operatorname{offdiag}(A) = A - \operatorname{diag}(A)$.

THEOREM 3.1 [1]. *Let A be a nonsingular M-matrix, partitioned as in* (2.1). *If $\alpha, \beta \in R^{n,n}$, partitioned as in* (2.1), *satisfy the inequalities*

$$\operatorname{diag}(\alpha) \geqslant I, \qquad 0 \leqslant \operatorname{offdiag}(\alpha) \leqslant E, \qquad 0 \leqslant \beta \leqslant E,$$

and if $K_{ii} \in R^{n_i, n_i}$ are chosen arbitrarily under the restriction

$$0 \leqslant K_{ii} \leqslant P_{ii}^{-1}, \qquad i = 1, \dots, m,$$

then for α, β, and K, the matrices P, L, U computed from (2.2) *are all nonsingular M-matrices.*

For $A \in C^{n,n}$, its comparison matrix $\mathscr{M}(A) = [m_{ij}]$ is defined by

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \qquad i \neq j, \quad 1 \leqslant i, j \leqslant n,$$

and A is said to be a nonsingular H-matrix if $\mathscr{M}(A)$ is a nonsingular M-matrix.

LEMMA 3.1 [7, Theorem 2.2]. *Let $A \in R^{n,n}$ be a nonsingular M-matrix. If the elements of $B \in R^{n,n}$ satisfy the relations*

$$b_{ii} \geqslant a_{ii}, \quad a_{ij} \leqslant b_{ij} \leqslant 0, \qquad i \neq j, \quad 1 \leqslant i, j \leqslant n,$$

then B is also a nonsingular M-matrix.

THEOREM 3.2. *Let $A \in C^{n,n}$ be a nonsingular H-matrix, and A^0 its comparison matrix, both partitioned as* (2.1). *By Theorem* 4.3(3) *of* [1], *there exist lower and upper block triangular nonsingular M-matrices L^0 and U^0, respectively, such that $A^0 = L^0(P^0)^{-1}U^0$, where P^0 is block diagonal and $P_{ii}^0 = L_{ii}^0 = U_{ii}^0$, $i = 1, \dots, m$. If $\alpha, \beta \in C^{n,n}$, partitioned as in* (2.1), *satisfy the inequalities*

$$|\operatorname{diag}(\alpha)| \geqslant I, \qquad |\operatorname{offdiag}(\alpha)| \leqslant E, \qquad |\beta| \leqslant E,$$

and if $K_{ii} \in C^{n_i, n_i}$ are chosen arbitrarily under the restriction

$$|K_{ii}| \leqslant \left(P_{ii}^0\right)^{-1}, \qquad i = 1, \dots, m, \tag{3.1}$$

then for α, β, and K, the matrices P, L, U computed from (2.2) *are all nonsingular H-matrices. In particular, P is nonsingular.*

Proof. It's easily seen that the block entries $L_{ij}^0, P_{ii}^0, U_{ij}^0$ of L^0, P^0, U^0 can be computed from the block entries A_{ij}^0 of A^0 by (2.2) with $\alpha = \beta = E$, $K_{ii} = (P_{ii}^0)^{-1}$. Therefore from (2.2) it's immediate that the inequalities

$$|\operatorname{diag}(P_{tt})| \geqslant \left|\operatorname{diag}\left(P_{tt}^0\right)\right|, \quad |\operatorname{offdiag}(P_{tt})| \leqslant \left|\operatorname{offdiag}\left(P_{tt}^0\right)\right|,$$

$$|U_{tj}| \leqslant \left|U_{tj}^0\right|, \quad |L_{jt}| \leqslant \left|L_{jt}^0\right|, \qquad t < j \leqslant m, \tag{3.2}$$

hold for $t = 1$. Now assume (3.2) is true for $t = 1, \dots, i-1$ $(i \leqslant m)$; then for $t = i$, using (2.2), we have

$$\begin{aligned}
|\mathrm{diag}(P_{ii})| &\geqslant |\mathrm{diag}(\alpha_{ii} * A_{ii})| - \left|\mathrm{diag}\left(\beta_{ii} * \left(\sum_{s=1}^{i-1} L_{is} K_{ss} U_{si}\right)\right)\right| \\
&\geqslant |\mathrm{diag}(A_{ii})| - \mathrm{diag}\left(\sum_{s=1}^{i-1} |L_{is}|\ |K_{ss}|\ |U_{si}|\right) \\
&\geqslant |\mathrm{diag}(A_{ii}^0)| - \mathrm{diag}\left(\sum_{s=1}^{i-1} L_{is}^0 (P_{ss}^0)^{-1} U_{si}^0\right) \\
&= \mathrm{diag}(P_{ii}^0) = |\mathrm{diag}(P_{ii}^0)|,
\end{aligned}$$

$$\begin{aligned}
|\mathrm{offdiag}(P_{ii})| &\leqslant |\mathrm{offdiag}(\alpha_{ii} * A_{ii})| + \left|\mathrm{offdiag}\left(\beta_{ii} * \left(\sum_{s=1}^{i-1} L_{is} K_{ss} U_{si}\right)\right)\right| \\
&\leqslant |\mathrm{offdiag}(A_{ii})| + \mathrm{offdiag}\left(\sum_{s=1}^{i-1} |L_{is}|\,|K_{ss}|\,|U_{si}|\right) \\
&\leqslant -\mathrm{offdiag}(A_{ii}^0) + \mathrm{offdiag}\left(\sum_{s=1}^{i-1} L_{is}^0 (P_{ss}^0)^{-1} U_{si}^0\right) \\
&= -\mathrm{offdiag}(P_{ii}^0) = |\mathrm{offdiag}(P_{ii}^0)|,
\end{aligned}$$

$$\begin{aligned}
|U_{ij}| &\leqslant |\alpha_{ij} * A_{ij}| + \left|\beta_{ij} * \left(\sum_{s=1}^{i-1} L_{is} K_{ss} U_{sj}\right)\right| \\
&\leqslant -A_{ij}^0 + \sum_{s=1}^{i-1} L_{is}^0 (P_{ss}^0)^{-1} U_{sj}^0 \\
&= -U_{ij}^0 = |U_{ij}^0|, \qquad i < j \leqslant m.
\end{aligned}$$

Similarly,

$$|L_{ji}| \leqslant |L_{ji}^0|, \qquad i < j \leqslant m.$$

Therefore we have proven by induction that (3.2) is true for $t = 1, \dots, m$.

Since L^0, U^0 are nonsingular M-matrices, by (3.2) and Lemma 3.1 we can easily deduce that P, L, U are nonsingular H-matrices. In particular, P is nonsingular. ■

Theorem 3.2 provides the existence of sparse block factorizations of nonsingular H-matrices for a wide range of α, β, and K. However, P_{ii}^0 are not computed in practice, so we cannot choose K_{ii} according to (3.1). Instead we wish that K_{ii} could be conveniently determined from P_{ii} in one way or another as in the case of nonsingular M-matrices. In what follows we'll see that the commonly used techniques for determining K_{ii} in the case of nonsingular M-matrices are still valid in the case of nonsingular H-matrices.

The following result, due to A. M. Ostrowski, can be found in [8].

LEMMA 3.2. *Let A be a nonsingular M-matrix of order n, and B a complex matrix of the same order. If* $|\text{diag}(B)| \geqslant \text{diag}(A)$ *and* $|\text{offdiag}(B)| \leqslant |\text{offdiag}(A)|$, *then B is nonsingular; moreover,* $|B^{-1}| \leqslant A^{-1}$.

THEOREM 3.3. *Let A, α, β be as in Theorem 3.2. If $K_{ii} \in C^{n_i, n_i}$ are chosen such that*

$$|K_{ii}| \leqslant |P_{ii}^{-1}|, \qquad i = 1, \dots, m, \tag{3.3}$$

then the conclusion of Theorem 3.2 is still valid.

Proof. From the proof of Theorem 3.2 we know that (3.2) holds for $t = 1$. So by Lemma 3.2 we have $|P_{11}^{-1}| \leqslant (P_{11}^0)^{-1}$; therefore $|K_{11}| \leqslant |P_{11}^{-1}| \leqslant (P_{11}^0)^{-1}$. Now, again from the proof of Theorem 3.2, we know that (3.2) holds for $t = 2$, and similarly we have $|K_{22}| \leqslant (P_{22}^0)^{-1}$. Continuing in this way, we can prove $|K_{ii}| \leqslant (P_{ii}^0)^{-1}$, $i = 1, \dots, m$. ■

We note that recently Polman [9] essentially proved Theorem 3.3, using a different approach, for the special case that A is block tridiagonal, all the $A_{i,i+1}$ are diagonal, and

$$\alpha_{ij} = B_{ij} = \begin{cases} E, & |i-j| \leqslant 1, \\ 0 & \text{otherwise}. \end{cases}$$

The most frequently used technique of determining K_{ii} is to take $K_{ii} = \gamma * P_{ii}^{-1}$, where γ is a matrix all of whose entries are unity or zero. For K_{ii} so chosen, (3.3) is obviously valid.

We now shift our attention to another technique for determining K_{ii}: Neumann approximate inverses. Suppose that a matrix P has a sparse (point) factorization $P = (I - L)\Sigma(I - U)$, where L, U are strictly lower and upper triangular matrices, respectively, and Σ is diagonal. The technique is to take $K = (I + U + \cdots + U^s)\Sigma^{-1}(I + L + \cdots + L^t)$ as approximations to P^{-1}, where s, t are small nonnegative integers. We'll prove that if in Theorem 3.2 K_{ii} are Neumann approximate inverses of P_{ii}, then (3.1) is valid.

For this purpose we need the following result.

LEMMA 3.3. *Under the conditions of Lemma* 3.2, *the factorizations*

$$A = (I - L)\Sigma(I - U), \qquad \Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n), \tag{3.4}$$

$$B = (I - \tilde{L})\tilde{\Sigma}(I - \tilde{U}), \qquad \tilde{\Sigma} = \operatorname{diag}(\tilde{\sigma}_1, \dots, \tilde{\sigma}_n), \tag{3.5}$$

exist. Moreover,

$$|\tilde{L}| \leqslant L, \qquad |\tilde{U}| \leqslant U, \qquad |\tilde{\Sigma}| \geqslant \Sigma > 0. \tag{3.6}$$

Proof. From Corollary 3.2 of [10], we already know that the factorization (3.4) exists; moreover, $\Sigma > 0$, $L \geqslant 0$, $U \geqslant 0$. It remains to show that $|\tilde{\Sigma}| \geqslant \Sigma$, $|\tilde{L}| \leqslant L$, $|\tilde{U}| \leqslant U$. In fact, writing (3.4), (3.5) as recursive relations of elements of the matrices and using induction, we can easily prove that

$$|\tilde{\sigma}_t| \geqslant \sigma_t, \quad |\tilde{l}_{jt}| \leqslant l_{jt}, \quad |\tilde{\sigma}_t \tilde{u}_{tj}| \leqslant \sigma_t u_{tj}, \qquad j = t+1, \dots, n, \quad t = 1, \dots, n. \tag{3.7}$$

Putting (3.7) in matrix form, we have $|\tilde{\Sigma}| \geqslant \Sigma$, $|\tilde{L}| \leqslant L$, and $|\tilde{\Sigma}\tilde{U}| \leqslant \Sigma U$, from which (3.6) follows. ■

We remark that Lemma 3.2 is an easy consequence of Lemma 3.3 and that the existence of (3.5) has been established elsewhere (see [11] and [9, Theorem 4]).

THEOREM 3.4. *Let* A, α, β *be as in Theorem* 3.2. *If* $K_{ii} \in C^{n_i, n_i}$ *are Neumann approximate inverses of* P_{ii}, *then the conclusion of Theorem* 3.2 *is still valid.*

Proof. From the proof of Theorem 3.2 we know that (3.2) holds for $t = 1$. So by Lemma 3.3, the factorizations

$$P_{11} = (I - L_{P_{11}})\Sigma_{P_{11}}(I - U_{P_{11}}), \qquad P_{11}^0 = (I - L_{P_{11}^0})\Sigma_{P_{11}^0}(I - U_{P_{11}^0})$$

exist; moreover,

$$|L_{P_{11}}| \leqslant L_{P_{11}^0}, \qquad |U_{P_{11}}| \leqslant U_{P_{11}^0}, \qquad |\Sigma_{P_{11}}| \geqslant \Sigma_{P_{11}^0} > 0;$$

therefore

$$\begin{aligned} |K_{11}| &= \left|\left(I + U_{P_{11}} + \cdots + U_{P_{11}}^{S_1}\right)\Sigma_{P_{11}}^{-1}\left(I + L_{P_{11}} + \cdots + L_{P_{11}}^{t_1}\right)\right| \\ &\leqslant \left(I + U_{P_{11}}^0 + \cdots + U_{P_{11}^0}^{s_1}\right)\Sigma_{P_{11}^0}^{-1}\left(I + L_{P_{11}^0} + \cdots + L_{P_{11}^0}^{t_1}\right) \\ &\leqslant \left(I - U_{P_{11}^0}\right)^{-1}\Sigma_{P_{11}^0}^{-1}\left(I - L_{P_{11}^0}\right)^{-1} = \left(P_{11}^0\right)^{-1}. \end{aligned}$$

The remainder follows as in the proof of Theorem 3.3. ■

We now consider one more technique for determining K_{ii}: polynomial approximations. Suppose that a matrix P admits a convergent splitting $P = B - C$. The technique is to take $K = f(B^{-1}C)B^{-1}$ as an approximation to P^{-1}, where $f(x)$ is some polynomial approximation of $(1 - x)^{-1}$ ($|x| < 1$). Here we'll only consider the case that the splitting $P = B - C$ is obtained by the incomplete factorizations of P and $f(x) = 1 + x + \cdots + x^s$. We'll prove that if in Theorem 3.2 K_{ii} are the polynomial approximations to P_{ii}^{-1}, then (3.1) is valid.

Let G be an index set such that

$$\left\{(i,i)\middle|1 \leqslant i \leqslant n\right\} \subset G \subset \left\{(i,j)\middle|1 \leqslant i,j \leqslant n\right\}.$$

Then the (point) incomplete factorization of A is just the sparse block factorization of A with all n_i's equal to unity in (2.1), $K_{ii} = 1/P_{ii}$, and

$$\alpha_{ij} = \beta_{ij} = \begin{cases} 1, & (i,j) \in G, \\ 0, & (i,j) \notin G. \end{cases}$$

The set G is called the nonzero set for the incomplete factorization.

LEMMA 3.4 (Cf. [10, Corollary 3.3]). *Under the conditions of Lemma 3.2, incomplete factorization with any nonzero set G will yield*

$$A = L\Sigma U - R, \qquad \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n),$$

$$B = \hat{L}\hat{\Sigma}\hat{U} - \hat{R}, \qquad \hat{\Sigma} = \operatorname{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_n),$$

where L ($\hat{L}$) and U ($\hat{U}$) are unit lower and unit upper triangular. Let

$$L = I - T, \qquad U = I - V, \qquad \hat{L} = I - \hat{T}, \qquad \hat{U} = I - \hat{V}.$$

Then

$$|\hat{T}| \leqslant T, \qquad |\hat{V}| \leqslant V, \qquad |\hat{\Sigma}| \geqslant \Sigma > 0, \qquad |\hat{R}| \leqslant R.$$

Proof. This is a generalization of Lemma 3.3 and Corollary 3.3 of [10]. The proof of all the conclusions except $|\hat{R}| \leqslant R$ is similar to that of Lemma 3.3. That is, we can similarly prove

$$|\hat{\sigma}_i| \geqslant \sigma_i, \quad |\hat{t}_{ji}| \leqslant t_{ji}, \quad |\hat{\sigma}_i\hat{v}_{ij}| \leqslant \sigma_i v_{ij}, \qquad j = i+1,\dots,n, \quad i = 1,\dots,n.$$

Now we come to prove $|\hat{r}_{ij}| \leqslant r_{ij}$, $i,j = 1,\dots,n$. If $(i,j)\in G$, then $|\hat{r}_{ij}| = r_{ij} = 0$. If $(i,j)\notin G$, then

$$\begin{aligned}
\hat{r}_{ij} &= \sum_{k=1}^{\min(i,j)-1} \hat{l}_{ik}\hat{\sigma}_k\hat{u}_{kj} - b_{ij} \\
&= \sum_{k=1}^{\min(i,j)-1} \hat{t}_{ik}\hat{\sigma}_k\hat{v}_{kj} - b_{ij}, \\
|\hat{r}_{ij}| &\leqslant \sum_{k=1}^{\min(i,j)-1} |\hat{t}_{ik}|\,|\hat{\sigma}_k\hat{v}_{kj}| + |b_{ij}| \\
&\leqslant \sum_{k=1}^{\min(i,j)-1} t_{ik}\sigma_k v_{kj} - a_{ij} \\
&= \sum_{k=1}^{\min(i,j)-1} l_{ik}\sigma_k u_{kj} - a_{ij} = r_{ij}.
\end{aligned}$$

■

THEOREM 3.5. *Let A, α, β be as in Theorem 3.2. If K_{ii} are polynomial approximations of P_{ii}^{-1}, then the conclusion of Theorem 3.2 is still valid.*

Proof. The proof of this theorem is similar to that of Theorem 3.4. ■

In Theorem 3.3 we have reached the following conclusion: If A is an H-matrix, then for any partitioning (2.1), any α, β such that $|\text{diag}(\alpha)| \geqslant I$, $|\text{offdiag}(\alpha)| \leqslant E$, $|\beta| \leqslant E$, and any K_{ii} satisfying $|K_{ii}| \leqslant P_{ii}^{-1}|$, all the P_{ii}

computed from (2.2) are nonsingular. It's natural to ask whether this conclusion can be extended to a wider class of matrices.

THEOREM 3.6 [11, Theorem 1]. *Let $A \in C^{n,n}$ be given. If for any $B \in C^{n,n}$ such that $|b_{ii}| = |a_{ii}|$, $|b_{ij}| \leqslant |a_{ij}|$, $1 \leqslant i, j \leqslant n$, the matrix B admits an incomplete factorization with respect to any nonzero set G, then A is a nonsingular H-matrix.*

Now we can give a negative answer to the above question.

COROLLARY 3.1. *Let $A \in C^{n,n}$ have the form* (2.1) *with all n_i equal to unity. If for any α, β such that $|\mathrm{diag}(\alpha)| = I$, $|\mathrm{offdiag}(\alpha)| \leqslant E$, $0 \leqslant \beta \leqslant E$, and $K_{ii} = 1/P_{ii}$, the P_{ii} computed from* (2.2) *don't all vanish, then A is necessarily a nonsingular H-matrix.*

4. CONVERGENCE

In this section, we further study the convergence of the sparse block factorization iterative method (2.3).

THEOREM 4.1 [1]. *Under the same assumptions as in Theorem* 3.1, *the splitting*

$$A = LP^{-1}U - R$$

is regular and therefore convergent.

THEOREM 4.2. *Let A, A^0 be as in Theorem* 3.2, *and $\alpha = \alpha^0$, $\beta = \beta^0$ be such that*

$$\mathrm{diag}(\alpha) \geqslant I, \qquad 0 \leqslant \mathrm{offdiag}(\alpha) \leqslant E, \qquad 0 \leqslant \beta \leqslant E.$$

Take $K_{ii} = S_i * P_{ii}^{-1}$, $K_{ii}^0 = S_i * (P_{ii}^0)^{-1}$, where $0 \leqslant S_i \leqslant E$. From (2.2) *and by Theorem* 3.3 *we have splittings*

$$A = LP^{-1}U - R \equiv M - R, \qquad A^0 = L^0\left(P^0\right)^{-1}U^0 - R^0 \equiv M^0 - R^0.$$

Then

$$\rho\left(M^{-1}R\right) \leqslant \rho\left(\left(M^0\right)^{-1}R^0\right) < 1.$$

Proof. By Theorem 4.1, $A^0 = M^0 - R^0$ is a regular splitting; hence $\rho((M^0)^{-1}R^0) < 1$. It remains to prove $\rho(M^{-1}R) \leqslant \rho((M^0)^{-1}R^0)$. By Theorem 2.8 of [12, p. 47], it's enough to show $|M^{-1}R| \leqslant (M^0)^{-1}R^0$.

From Theorem 3.1 we know that P^0, L^0, U^0 are M-matrices. A proof similar to that of (3.2) and Lemma 3.2 yield

$$|P^{-1}| \leqslant (P^0)^{-1}, \qquad |L_{ji}| \leqslant -L_{ji}^0, \ |U_{ij}| \leqslant -U_{ij}^0,$$
$$i < j \leqslant m, \ i = 1, \ldots, m-1. \quad (4.1)$$

Now

$$M = LP^{-1}U = LP^{-1}PP^{-1}U, \qquad M^0 = L^0(P^0)^{-1}P^0(P^0)^{-1}U^0.$$

Let

$$LP^{-1} = I - \tilde{L}, \qquad P^{-1}U = I - \tilde{U},$$
$$L^0(P^0)^{-1} = I - \tilde{L}^0, \qquad (P^0)^{-1}U^0 = I - \tilde{U}^0.$$

From (4.1) it's easily seen that

$$|\tilde{L}| \leqslant \tilde{L}^0, \qquad |\tilde{U}| \leqslant \tilde{U}^0;$$

therefore

$$\left|(LP^{-1})^{-1}\right| = \left|(I - \tilde{L})^{-1}\right| = |I + \tilde{L} + \cdots + \tilde{L}^{m-1}|$$
$$\leqslant I + \tilde{L}^0 + \cdots + (\tilde{L}^0)^{m-1} = \left[L^0(P^0)^{-1}\right]^{-1}.$$

Similarly

$$\left|(P^{-1}U)^{-1}\right| \leqslant \left[(P^0)^{-1}U^0\right]^{-1};$$

hence $|M^{-1}| \leqslant (M^0)^{-1}$.

Now we come to prove $|R| \leqslant R^0$. In fact (2.2) can be put into the following matrix form:

$$L + U - P = \alpha * A - \beta * [(L-P)K(U-P)];$$

therefore

$$
\begin{aligned}
R &= LP^{-1}U - A = L + U - P - A + (L-P)P^{-1}(U-P)\\
&= \alpha * A - A + (L-P)(P^{-1} - K)(U-P)\\
&\quad + (E-\beta)*[(L-P)K(U-P)],\\
|R| &\leqslant |\alpha * A - A| + |L-P|\,|P^{-1} - K|\,|U-P|\\
&\quad + (E-\beta)*(|L-P|\,|K|\,|U-P|)\\
&\leqslant \alpha^0 * A^0 - A^0 + (L^0 - P^0)[(E-S)*|P^{-1}|](U^0 - P^0)\\
&\quad + (E-\beta^0)*[(L^0-P^0)(S*|P^{-1}|)(U^0-P^0)]\\
&\leqslant \alpha^0 * A^0 - A^0 + (L^0 - P^0)[(P^0)^{-1} - K^0](U^0 - P^0)\\
&\quad + (E-\beta^0)*[(L^0-P^0)K^0(U^0-P^0)]\\
&= R^0,
\end{aligned}
$$

where S is the block diagonal matrix with S_i as diagonal blocks. ■

Similarly we can prove the following result.

THEOREM 4.3. *If in Theorem 4.2 K_{ii} are the Neumann approximate inverses of P_{ii} or polynomial approximations of P_{ii}^{-1} mentioned in the last section, and K_{ii}^0 are determined in the same manner with P_{ii} replaced by P_{ii}^0, then the conclusion of Theorem 4.2 is still valid.*

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