Convergence of the solution of a nonsymmetric matrix Riccati differential equation to its stable equilibrium solution

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Abstract

We consider the initial value problem for a nonsymmetric matrix Riccati differential equation, where the four coefficient matrices form an M-matrix. We show that for a wide range of initial values the Riccati differential equation has a global solution $X(t)$ on $[0, \infty)$ and $X(t)$ converges to the stable equilibrium solution as t goes to infinity.

Key words: Nonsymmetric Riccati differential equation; Global solution; Stable equilibrium solution; Nonsymmetric algebraic Riccati equation; Nonnegative solution

1 Introduction

We consider the nonsymmetric matrix Riccati differential equation (RDE)

$$X'(t) = X(t)CX(t) - X(t)D - AX(t) + B, \quad X(0) = X_0, \tag{1}$$

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where A, B, C, D are real matrices of sizes $m \times m, m \times n, n \times m, n \times n$, respectively, such that

$$K = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \tag{2}$$

is a nonsingular M-matrix, or an irreducible singular M-matrix. Some relevant definitions are given below.

For any matrices $A, B \in \mathbb{R}^{m \times n}$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ for all i, j. We can then define positive matrices, nonnegative matrices, etc. The spectrum of a square matrix A will be denoted by $\sigma(A)$. The open left half-plane, the open right half-plane, the closed left half-plane and the closed right half-plane will be denoted by $\mathbb{C}_<$, $\mathbb{C}_>$, $\mathbb{C}_\leq$ and $\mathbb{C}_\geq$, respectively. A real square matrix A is called a Z-matrix if all its off-diagonal elements are nonpositive. It is clear that any Z-matrix A can be written as $sI - B$ with $B \geq 0$. A Z-matrix A is called an M-matrix if $s \geq \rho(B)$, where $\rho(\cdot)$ is the spectral radius. It is called a singular M-matrix if $s = \rho(B)$; it is called a nonsingular M-matrix if $s > \rho(B)$. It follows immediately that $\sigma(A) \subset \mathbb{C}_>$ for any nonsingular M-matrix A and $\sigma(A) \subset \mathbb{C}_\geq$ for any singular M-matrix A. Note also that for any nonsingular M-matrix A there is a positive vector v such that $Av > 0$ (see [1], for example).

We will use several results from the theory of nonnegative matrices (see [1,7,15]). They are summarized below.

Theorem 1 *Let $A, B, C \in \mathbb{R}^{n \times n}$ with $A \geq 0$. Then*

(i) *If $A \leq B$, then $\rho(A) \leq \rho(B)$.*
(ii) *If $A \leq B \leq C$, $A \neq B \neq C$, and B is irreducible, then $\rho(A) < \rho(B) < \rho(C)$.*
(iii) *If $Av < kv$ for a positive vector v, then $\rho(A) < k$.*
(iv) *If $Av = kv$ for a positive vector v, then $\rho(A) = k$.*
(v) *If A is irreducible, then $\rho(A)$ is a simple eigenvalue of A and there is a positive vector u such that $Au = \rho(A)u$.*

It follows easily form Theorem 1 (i) and (ii) that the matrices A and D in (1) are both nonsingular M-matrices when K in (2) is a nonsingular M-matrix or an irreducible singular M-matrix.

In this paper we will show that the initial value problem (1) has a solution $X(t)$ on $[0, \infty)$ for a suitable initial value X_0 and $X(t)$ converges to the stable equilibrium solution of (1).

For symmetric RDEs, problems like these have been studied in [2,13,14]. For a particular nonsymmetric RDE, these problems have been studied in [8]. As shown in Proposition 3.4 of [4], that nonsymmetric matrix RDE is a special case of (1). Moreover, the condition imposed on X_0 in this paper is much weaker than that in [8].

2 Results on nonsymmetric algebraic Riccati equations

The equilibrium solutions of (1) are the solutions of the algebraic Riccati equation (ARE)

$$XCX - XD - AX + B = 0. \tag{3}$$

Nonsymmetric AREs of this type appear in transport theory (see [9]) and Wiener–Hopf factorization of Markov chains (see [12]). The solution of practical interest is the minimal nonnegative solution.

In this section, we summarize some main results about (3). See [4–6] for more details.

Theorem 2 *If K in* (2) *is a nonsingular M-matrix, then* (3) *has a minimal nonnegative solution S_1 and $D - CS_1$ is a nonsingular M-matrix. Moreover, $S_1 v_1 < v_2$, where v_1 and v_2 are positive vectors such that $K(v_1^T \ v_2^T)^T > 0$. If K is an irreducible M-matrix, then* (3) *has a minimal nonnegative solution S_1 and $S_1 > 0$. Moreover, $D - CS_1$ and $A - S_1 C$ are irreducible M-matrices.*

We will also need the dual equation of (3)

$$XBX - XA - DX + C = 0. \tag{4}$$

Since

$$\widetilde{K} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

is a nonsingular M-matrix (a singular M-matrix, an irreducible matrix) if and only if K is so, the results in Theorem 2 can be applied to (4) directly. Therefore, when K is a nonsingular M-matrix, (4) has a minimal nonnegative solution $\widetilde{S}_1$, $A - B\widetilde{S}_1$ is a nonsingular M-matrix and $\widetilde{S}_1 v_2 < v_1$. When K is an irreducible M-matrix, (4) has a minimal nonnegative solution $\widetilde{S}_1$, $\widetilde{S}_1 > 0$ and $A - B\widetilde{S}_1$ is an irreducible M-matrix.

If K is an irreducible singular M-matrix (then so is K^T), we let $u_1, u_2, v_1, v_2 > 0$ be such that $K(v_1^T \ \ v_2^T)^T = 0$ and $(u_1^T \ \ u_2^T)K = 0$. We know from Theorem 1 (v) that the vectors $(v_1^T \ \ v_2^T)$ and $(u_1^T \ \ u_2^T)$ are each unique up to a scalar multiple. We will also need the matrix

$$H = \begin{pmatrix} D & -C \\ B & -A \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} K. \tag{5}$$

Theorem 3 *If K in* (2) *is a nonsingular M-matrix or an irreducible singular M-matrix, then we have the so-called Wiener–Hopf factorization for K:*

$$H \begin{pmatrix} I & \widetilde{S}_1 \\ S_1 & I \end{pmatrix} = \begin{pmatrix} I & \widetilde{S}_1 \\ S_1 & I \end{pmatrix} \begin{pmatrix} G_1 & 0 \\ 0 & -G_2 \end{pmatrix}, \tag{6}$$

where $G_1 = D - CS_1$ and $G_2 = A - B\widetilde{S}_1$. If (2) *is an irreducible singular M-matrix with $u_1^T v_1 \neq u_2^T v_2$, then one of G_1 and G_2 is nonsingular; if $u_1^T v_1 = u_2^T v_2$, then both G_1 and G_2 are singular. Moreover, the matrix $\begin{pmatrix} I & \widetilde{S}_1 \\ S_1 & I \end{pmatrix}$ is nonsingular if* (2) *is a nonsingular M-matrix or an irreducible singular M-matrix with $u_1^T v_1 \neq u_2^T v_2$.*

Let all eigenvalues of H be arranged in an descending order by their real parts, and be denoted by $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \cdots, \lambda_{n+m}$. When (2) is an irreducible nonsingular M-matrix or an irreducible singular M-matrix with $u_1^T v_1 \neq u_2^T v_2$, we see from Theorem 3 that $\lambda_1, \ldots, \lambda_n$ are eigenvalues of G_1 and $\lambda_{n+1}, \cdots, \lambda_{n+m}$ are the negative of eigenvalues of G_2. In particular, $\lambda_n > \lambda_{n+1}$ are real simple eigenvalues of H.

The next result can be found in [11].

Lemma 4 *If S is any solution of* (3), *then*

$$\begin{pmatrix} I & 0 \\ S & I \end{pmatrix}^{-1} \begin{pmatrix} D & -C \\ B & -A \end{pmatrix} \begin{pmatrix} I & 0 \\ S & I \end{pmatrix} = \begin{pmatrix} D - CS & -C \\ 0 & -(A - SC) \end{pmatrix}.$$

Thus, the eigenvalues of $D - CS$ are eigenvalues of (5) *and the eigenvalues of $A - SC$ are the negative of the remaining eigenvalues of* (5).

Note that $\mathcal{R}(X) = XCX - XD - AX + B$ defines a mapping from $\mathbb{R}^{m\times n}$ into itself. The first Fréchet derivative of $\mathcal{R}$ at a matrix X is a linear operator

$\mathcal{R}'_X : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ given by

$$\mathcal{R}'_X(Z) = -\left((A - XC)Z + Z(D - CX)\right).$$

Since the eigenvalues of the operator $\mathcal{R}'_X$ are the eigenvalues of the matrix $-(I \otimes (A - XC) + (D - CX)^T \otimes I)$, where $\otimes$ denotes the Kronecker product, an equilibrium solution X of (1) is (asymptotically) stable if all eigenvalues of $I \otimes (A - XC) + (D - CX)^T \otimes I$ are in $\mathbb{C}_>$. Note that any eigenvalue of $I \otimes (A - XC) + (D - CX)^T \otimes I$ is the sum of an eigenvalue of $A - XC$ and an eigenvalue of $D - CX$ (see [10], for example).

It follows from Theorem 3 that S_1 is a stable equilibrium solution of (1) when (2) is a nonsingular M-matrix or an irreducible singular M-matrix with $u_1^T v_1 \neq u_2^T v_2$. We also know (from Theorem 4.1 of [6], for example) that no other solution of (3) can be a stable equilibrium solution of (1). When (2) is an irreducible singular M-matrix with $u_1^T v_1 = u_2^T v_2$, S_1 is not a stable solution since $I \otimes (A - S_1 C) + (D - CS_1)^T \otimes I$ has a zero eigenvalue by Theorem 3.

3 Global existence of solutions of the matrix Riccati differential equation

Since the matrices A and D in (1) are nonsingular M-matrices, they can be decomposed (in many different ways) as $A = A_1 - A_2$ and $D = D_1 - D_2$, where $A_2, D_2 \geq 0$ and A_1, D_1 are nonsingular M-matrices. Then (1) becomes

$$X' + XD_1 + A_1X = XCX + XD_2 + A_2X + B, \quad X(0) = X_0. \tag{7}$$

The initial value problem (7) can be written in its equivalent integral form; namely, premultiplying and postmultiplying the differential equation in (7) by the integrating factors $e^{-(t-s)A_1}$ and $e^{-(t-s)D_1}$, respectively, and integrating the resulting equation with respect to s from 0 to t, we obtain as in [8]

$$\begin{aligned} X(t) = e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}(X(s)CX(s) \\ + X(s)D_2 + A_2X(s) + B)e^{-(t-s)D_1}ds. \end{aligned} \tag{8}$$

We will establish the global existence of solutions to (1) for suitable initial values X_0 by using the Picard iteration:

$$X^{(0)}(t) = 0,$$

$$X^{(m)}(t) = e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}(X^{(m-1)}(s)CX^{(m-1)}(s) + X^{(m-1)}(s)D_2 + A_2X^{(m-1)}(s) + B)e^{-(t-s)D_1}ds.$$

We start with a simple result on matrix exponential.

Lemma 5 *If A is a Z-matrix, then $e^{-tA} \geq 0$ for $t \geq 0$.*

PROOF. Since A is a Z-matrix, we can write $A = sI - B$ with $B \geq 0$ and $s \in \mathbb{R}$. Then $e^{-tA} = e^{-stI}e^{tB}$ (by the fact that if two matrices M, N commute then $e^{M+N} = e^M e^N$). Thus, $e^{-tA} = e^{-st}\sum_{n=0}^{\infty}\frac{1}{n!}(tB)^n \geq 0$. □

Theorem 6 *Assume K in* (2) *is a nonsingular M-matrix or an irreducible singular M-matrix. If $0 \leq X_0 \leq S$, where S is any nonnegative solution of* (3)*. Then:*

(i) $0 \leq X^{(m-1)}(t) \leq X^{(m)}(t) \leq S$ *for all $t \geq 0$ and all $m \in \mathbb{N}$.*
(ii) $X^{(m)}(t)$ *converges pointwise to a continuous function $X(t)$ on $[0, \infty)$, which is a global solution to* (1)*.*

PROOF. (i) The first two inequalities can easily be shown by induction. To see the last inequality, we proceed by induction as well. Assuming $X^{(m-1)}(t) \leq S$, we get by Lemma 5

$$\begin{aligned} X^{(m)}(t) &\leq e^{-tA_1}X_0e^{-tD_1} \\ &\quad + \int_0^t e^{-(t-s)A_1}(SCS + SD_2 + A_2S + B)e^{-(t-s)D_1}ds \\ &= e^{-tA_1}X_0e^{-tD_1} + \int_0^t e^{-(t-s)A_1}(A_1S + SD_1)e^{-(t-s)D_1}ds \\ &= S - e^{-tA_1}(S - X_0)e^{-tD_1} \leq S. \end{aligned}$$

This completes the proof of (i).

(ii) It follows from (i) that $X^{(m)}(t)$ converges pointwise to a function $X(t)$ on $[0, \infty)$, and $X(t) \leq S$. Letting $m \to \infty$ in the Picard iteration and applying the monotone convergence theorem for Lebesgue integrals, we conclude that the limit function $X(t)$ satisfies (8). It then follows from the boundedness of $X(t)$ that $X(t)$ is also continuous, which in turn implies that $X(t)$ is differential and is a solution to (1). □

4 Convergence to the stable equilibrium solution

It is well known that the initial value problem (1) is related to the initial value problem for the corresponding linear system:

$$\begin{pmatrix} Y'(t) \\ Z'(t) \end{pmatrix} = \begin{pmatrix} D & -C \\ B & -A \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix}, \quad \begin{pmatrix} Y \\ Z \end{pmatrix}(0) = \begin{pmatrix} I \\ X_0 \end{pmatrix}. \tag{9}$$

The following result is a special case of the so-called Radon's lemma (see [3], for example).

Lemma 7 *The initial value problem* (1) *has a solution* $X(t)$ *on* $[0, \infty)$ *if and only if* $Y(t)$ *is nonsingular on* $[0, \infty)$ *for the solution* $\begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix}$ *to* (9)*. In this case,* $X(t) = Z(t)Y^{-1}(t)$.

This result will be fundamental in proving that the solution $X(t)$ to (1) converges to S_1 for suitable initial values X_0, where S_1 is the minimal nonnegative solution of (3) and also the asymptotically stable equilibrium solution of (1). We will also need the following result.

Lemma 8 (i) *If* A *is a nonsingular* M*-matrix, then* $\lim_{t\to\infty} e^{-At} = 0$.
(ii) *If* A *is an irreducible singular* M*-matrix, then* e^{-At} *is bounded on* $[0, \infty)$.

PROOF. (i) Let the Jordan canonical form of A be $A = MJM^{-1}$ and let $f(x) = e^{-tx}$. Then for the $r \times r$ Jordan block

$$J_k = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

we have (see [10], for example)

$$f(J_k) = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \cdots & \frac{f^{(r-1)}(\lambda)}{(r-1)!} \\ 0 & f(\lambda) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{f'(\lambda)}{1!} \\ 0 & \cdots & 0 & f(\lambda) \end{pmatrix}.$$

Since $\sigma(A) \subset \mathbb{C}_>$, it follows that $\lim_{t\to\infty} f(J_k) = 0$ for each Jordan block J_k. Then

$$\lim_{t\to\infty} e^{-At} = \lim_{t\to\infty} f(A) = M \lim_{t\to\infty} f(J) M^{-1} = 0.$$

(ii) Since A is an irreducible singular M-matrix, it can be written as $A = sI - B$, where $s = \rho(B)$ and $B \geq 0$ is irreducible. By Theorem 1 (v), 0 is a simple eigenvalue of A and the remaining eigenvalues of A are in $\mathbb{C}_>$. Thus, e^{-At} converges to a nonzero matrix in this case. $\square$

Theorem 9 *Assume K in* (2) *is a nonsingular M-matrix. If* $0 \leq X_0 \leq S_1$ *and $X(t)$ is the solution of* (1). *Then $X(t) \to S_1$ as $t \to \infty$.*

PROOF. The existence of $X(t)$ on $[0, \infty)$ is guaranteed by Theorem 6, and we have by Lemma 7

$$\begin{pmatrix} I \\ X(t) \end{pmatrix} = \begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} Y^{-1}(t) = e^{Ht} \begin{pmatrix} I \\ X_0 \end{pmatrix} Y^{-1}(t), \tag{10}$$

where H is the matrix in (5). By Theorem 3 we have

$$H = U \begin{pmatrix} G_1 & 0 \\ 0 & -G_2 \end{pmatrix} U^{-1},$$

where G_1 and G_2 are nonsingular M-matrices and

$$U = \begin{pmatrix} I & \widetilde{S}_1 \\ S_1 & I \end{pmatrix}.$$

Then (10) becomes

$$\begin{pmatrix} I \\ X(t) \end{pmatrix} = U \begin{pmatrix} e^{G_1 t} & 0 \\ 0 & e^{-G_2 t} \end{pmatrix} U^{-1} \begin{pmatrix} I \\ X_0 \end{pmatrix} Y^{-1}(t). \tag{11}$$

Let

$$U^{-1} \begin{pmatrix} I \\ X_0 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{12}$$

Then $(I - \widetilde{S}_1 S_1)V_1 = I - \widetilde{S}_1 X_0$. Since $S_1 v_1 < v_2$ and $\widetilde{S}_1 v_2 < v_1$ by Theorem 2, we have $\widetilde{S}_1 S_1 v_1 \le \widetilde{S}_1 v_2 < v_1$. It follows from Theorem 1 (iii) that $\rho(\widetilde{S}_1 S_1) < 1$. Since $0 \le X_0 \le S_1$, we also have $\rho(\widetilde{S}_1 X_0) \le \rho(\widetilde{S}_1 S_1) < 1$ by Theorem 1 (i). Therefore, the matrix V_1 is nonsingular. Now, we have by (11) and (12)

$$\begin{pmatrix} I \\ X(t) \end{pmatrix} = U \begin{pmatrix} I \\ W_1(t) \end{pmatrix} W_2(t), \tag{13}$$

where

$$W_1(t) = e^{-G_2 t} V_2 V_1^{-1} e^{-G_1 t}, \quad W_2(t) = e^{G_1 t} V_1 Y^{-1}(t).$$

By Lemma 8 we have $\lim_{t\to\infty} W_1(t) = 0$. From (13) we have

$$W_2(t) = (I + \widetilde{S}_1 W_1(t))^{-1}, \quad X(t) = (S_1 + W_1(t))W_2(t).$$

Thus, $\lim_{t\to\infty} W_2(t) = I$ and $\lim_{t\to\infty} X(t) = S_1$, as required. $\square$

In the remaining part of this section, we will show that the convergence of $X(t)$ to S_1 can be guaranteed for a wider range of initial values X_0 when K in (2) is an irreducible nonsingular M-matrix or an irreducible singular M-matrix with $u_1^T v_1 \neq u_2^T v_2$. Recall that in both cases, $\lambda_n > \lambda_{n+1}$ are real simple eigenvalues of H in (5).

When K is an irreducible M-matrix, the matrices $D - CS_1$ and $A - S_1 C$ are also irreducible M-matrices by Theorem 2. Note that $(D - CS_1)^T$ is also an irreducible M-matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. By Lemma 4, the eigenvalues of $A - S_1 C$ are $-\lambda_{n+m}, \ldots, -\lambda_{n+1}$. Since $A - S_1 C$ and $(D - CS_1)^T$ can be written in the form $sI - N$, where $N \geq 0$ is irreducible, it follows form Theorem 1 (v) that there exist unique positive vectors a and b with unit 1-norm (i.e., $a^T e = b^T e = 1$, where e is the column vector of 1's.) such that

$$(A - S_1 C)a = -\lambda_{n+1} a, \quad b^T(D - CS_1) = \lambda_n b^T. \tag{14}$$

Since K is irreducible, we have $C \neq 0$ and thus $b^T C a > 0$.

Theorem 10 *Assume that K in* (2) *is an irreducible nonsingular M-matrix or an irreducible singular M-matrix with $u_1^T v_1 \neq u_2^T v_2$. Then there exists a second positive solution S_2 of* (3) *given by*

$$S_2 = S_1 + kab^T, \tag{15}$$

where the vectors a, b are specified in (14) *and $k = (\lambda_n - \lambda_{n+1})/b^T C a$.*

PROOF. Notice that

$$\begin{aligned}\mathcal{R}(S_2) &= \mathcal{R}(S_1) + k^2 ab^T C ab^T - kab^T(D - CS_1) - k(A - S_1 C)ab^T \\ &= k^2(b^T C a)ab^T - k\lambda_n ab^T + k\lambda_{n+1} ab^T \\ &= (k(b^T C a) - \lambda_n + \lambda_{n+1})kab^T.\end{aligned}$$

This completes the proof. □

Lemma 11 *Under the assumption of Theorem* 10*, we have*

$$\sigma(D - CS_2) = \{\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_{n+1}\}.$$

PROOF. For the vector b in (14) we have by (15)

$$\begin{aligned}b^T(D - CS_2) &= b^T(D - CS_1) - k(b^T C a)b^T \\ &= \lambda_n b^T - (\lambda_n - \lambda_{n+1})b^T = \lambda_{n+1} b^T.\end{aligned} \tag{16}$$

Let $(D - CS_2)^T = sI - N$, where $N \geq 0$ is irreducible. Then $(sI - N)b = \lambda_{n+1} b$ and $Nb = (s - \lambda_{n+1})b$. It follows from Theorem 1 that $s - \lambda_{n+1} = \rho(N)$ is a simple eigenvalue of N. Thus, λ_{n+1} is an eigenvalue of $D - CS_2$ and all other eigenvalues of $D - CS_2$ have strictly larger real parts. By Lemma 4 the eigenvalues of $D - CS_2$ are part of the eigenvalues of H. Therefore, we only need to show that λ_n is not an eigenvalue of $D - CS_2$. Suppose, for contradiction, that $(D - CS_2)z = \lambda_n z$ for $z \neq 0$. From this and (16) we get $\lambda_n b^T z = b^T(D - CS_2)z = \lambda_{n+1} b^T z$. So $b^T z = 0$. Then $(D - CS_1)z = (D - CS_2)z = \lambda_n z$. Since λ_n is a simple eigenvalue of $D - CS_1$, z is a scalar multiple of a positive eigenvector of $D - CS_1$ corresponding to λ_n, which is contradictory to $b^T z = 0$. $\square$

By Lemma 11 the solution S_2 can be found directly by using the Schur method in much the same way as we found S_1 using the Schur method in [4]. Lemma 11 is also needed in the next result.

Lemma 12 *Let S_2 be as in Theorem 10 and $\widetilde{S}_1$ be the minimum positive solution of (4). Then $\rho(\widetilde{S}_1 S_2) = 1$.*

PROOF. Since S_2 is also a solution of (3), we have

$$\begin{pmatrix} D & -C \\ B & -A \end{pmatrix} \begin{pmatrix} I \\ S_2 \end{pmatrix} = \begin{pmatrix} I \\ S_2 \end{pmatrix} (D - CS_2).$$

So we can replace the matrix S_1 in (6) by S_2 and get

$$\begin{pmatrix} D & -C \\ B & -A \end{pmatrix} \begin{pmatrix} I & \widetilde{S}_1 \\ S_2 & I \end{pmatrix} = \begin{pmatrix} I & \widetilde{S}_1 \\ S_2 & I \end{pmatrix} \begin{pmatrix} D - CS_2 & 0 \\ 0 & -(A - B\widetilde{S}_1) \end{pmatrix}. \quad (17)$$

By Theorem 2, $D - CS_1$ and $A - B\widetilde{S}_1$ are irreducible M-matrices. The former implies that $D - CS_2$ is an irreducible Z-matrix. Then, as before, there exist positive vectors u and v such that $(A - B\widetilde{S}_1)u = -\lambda_{n+1} u$ and $(D - CS_2)v = \lambda_{n+1} v$ (Lemma 11 is used here).

Postmultiplying (17) by

$$\begin{pmatrix} 0 \\ u \end{pmatrix} \text{ and } \begin{pmatrix} v \\ 0 \end{pmatrix},$$

respectively, we see that the vectors

$$\begin{pmatrix} \widetilde{S}_1 u \\ u \end{pmatrix} \text{ and } \begin{pmatrix} v \\ S_2 v \end{pmatrix} \quad (18)$$

are eigenvectors of H corresponding to the eigenvalue λ_{n+1}. Since λ_{n+1} is a simple eigenvalue of H, the first vector in (18) is a scalar multiple, $k > 0$, of the second. So $\widetilde{S}_1 u = kv$ and $u = kS_2 v$. Thus, $\widetilde{S}_1 S_2 v = v$ for a positive vector v, which implies $\rho(\widetilde{S}_1 S_2) = 1$. □

Theorem 13 *Assume that K in* (2) *is an irreducible nonsingular M-matrix or an irreducible singular M-matrix with $u_1^T v_1 \neq u_2^T v_2$. If $0 \leq X_0 \leq S_2$, $X_0 \neq S_2$ and $X(t)$ is the solution of* (1)*. Then $X(t) \to S_1$ as $t \to \infty$.*

PROOF. We proceed as in the proof of Theorem 9. Only two changes are needed. The first change is about the invertibility of the matrix V_1 in that proof. Recall that $(I - \widetilde{S}_1 S_1)V_1 = I - \widetilde{S}_1 X_0$. Since $\widetilde{S}_1 S_2 > 0$, $0 \leq \widetilde{S}_1 X_0 \leq \widetilde{S}_1 S_2$ and $\widetilde{S}_1 X_0 \neq \widetilde{S}_1 S_2$, it follows from Theorem 1 (ii) that $\rho(\widetilde{S}_1 X_0) < \rho(\widetilde{S}_1 S_2)$. Thus, $\rho(\widetilde{S}_1 X_0) < 1$ by Lemma 12. So $I - \widetilde{S}_1 X_0$ is nonsingular and thus V_1 is nonsingular as well. The second change is about the matrices G_1 and G_2. We now know that G_1 and G_2 are both M-matrices and at least one of them is nonsingular. Therefore, by Lemma 8 we still have $\lim_{t\to\infty} W_1(t) = 0$ for the matrix $W_1(t)$ in the proof of Theorem 9. □

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