On absolute value equations associated with M-matrices and H-matrices

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Abstract

We consider the absolute value equation (AVE) $Ax-|x|=b$, where the diagonal entries of $A\in\mathbb{R}^{n\times n}$ are all greater than 1 and $\langle A\rangle - I$ is an irreducible singular M-matrix ($\langle A\rangle$ is the comparison matrix of A). We investigate the existence and uniqueness of solutions for the AVE. The AVE does not necessarily have a unique solution for every $b\in\mathbb{R}^n$, so most of the existing convergence results for various iterative methods are not generally applicable. Moreover, the generalized Newton method may break down. We show that if the AVE has a solution x^* with at least one negative component, then the sequence generated by the generalized Gauss–Seidel iteration converges to x^* linearly for any initial vector.

Keywords: Absolute value equation, Existence and uniqueness, Generalized Gauss–Seidel iteration, Global convergence, Linear convergence
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1. Introduction

We consider the absolute value equation (AVE)

$$Ax-|x|=b, \tag{1}$$

where $A\in\mathbb{R}^{n\times n}$ and $b\in\mathbb{R}^n$ are given, and $|\cdot|$ denotes absolute value. The AVE can be obtained [9] by reformulating the linear complementarity problem (LCP), which appears in many mathematical programming problems. It is also closely related to the piecewise linear system $x^+ + Tx = c$. In [3] a Newton-type iterative procedure for solving certain piecewise linear systems that arise from numerical modelling of free-surface hydrodynamics is derived and investigated. In view of the discussions in [6], the matrix T in [3] is an irreducible symmetric M-matrix (nonsingular or singular). Using $x^+=\frac{1}{2}(x+|x|)$, the piecewise linear system becomes $-(2T+I)x-|x|=-2c$, and becomes $(2T+I)x-|x|=-2c$ if we replace x by $-x$. So the problem studied in [3] can also be solved by working

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on the AVE (1) for which $A - I$ is an irreducible symmetric M-matrix. The analysis in [6, 10] shows that the symmetry assumption is not needed and that the irreducibility is required only when $A - I$ is a singular M-matrix. The class of AVEs for which $A - I$ is a nonsingular M-matrix or an irreducible singular M-matrix is then formally introduced in [6]. AVEs from LCP do not always have such nice sign pattern for the matrix A. In [4] a generalized Gauss–Seidel iteration is introduced and studied for the AVE (1) for which the diagonal entries of A are all greater than 1 and $\langle A \rangle - I$ is a *nonsingular* M-matrix. In that case, (1) has a unique solution for all b (see [7]). The research in this paper will then be on AVEs where all diagonal entries of A are greater than 1 and $\langle A \rangle - I$ is an irreducible singular M-matrix. The situation is significantly more difficult to handle, both in terms of the existence and uniqueness of solutions and in terms of numerical solution.

2. Preliminaries

The jth component of a vector u is denoted by $(u)_j$ and the jth diagonal entry of a diagonal matrix D is denoted by $(D)_j$. The identity matrix is denoted by I. We use $\|\cdot\|$ to denote the vector 2-norm, and use $\rho(A)$ to denote the spectral radius of a square matrix A. The superscript T denotes the transpose of a vector or matrix, and $|A|$ denotes the matrix $[|a_{ij}|]$ for any $n \times k$ matrix $A = [a_{ij}]$. An $n \times k$ real matrix $A = [a_{ij}]$ is called nonnegative (positive) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for all i and j. For real matrices A and B of the same size, we write $A \geq B$ ($A > B$) if $A - B$ is nonnegative (positive).

A real square matrix A is called a Z-matrix if all its off-diagonal entries are nonpositive. Any Z-matrix A can be written as $sI - B$ with $B \geq 0$; it is called a nonsingular M-matrix if $s > \rho(B)$, and a singular M-matrix if $s = \rho(B)$. A Z-matrix A is a nonsingular M-matrix if and only if $A^{-1} \geq 0$ (see [2]).

A square matrix A is said to be a nonsingular H-matrix if its comparison matrix $\langle A \rangle = [\hat{a}_{ij}]$ is a nonsingular M-matrix, where $\hat{a}_{ij} = |a_{ij}|$ for $i = j$, and $\hat{a}_{ij} = -|a_{ij}|$ for $i \neq j$. When $\langle A \rangle$ is a singular M-matrix, we say A is an H-matrix (we refrain from calling it a singular H-matrix because the matrix may be nonsingular). When $\langle A \rangle$ is a nonsingular M-matrix, A is indeed nonsingular and $|A^{-1}| \leq \langle A \rangle^{-1}$ (see [5]).

Let M_s be the set of all irreducible singular M-matrices, and let $H_s^+ = \{A \in \mathbb{R}^{n \times n} \mid a_{ii} > 0, i = 1, \ldots, n, \langle A \rangle \in M_s\}$. We use $D_{\{1,-1\}}$ to denote the set of all diagonal matrices whose first diagonal entry is 1 and other diagonal entries are 1 or -1.

We collect below some results that will be needed in the next section.

Lemma 1. *[2, Theorem 6.4.16] If $A \in M_s$, then there is a vector $u > 0$ such that $Au = 0$. The vector u is uniquely determined up to a scalar multiple.*

Lemma 2. *[2, Corollary 2.1.5] If $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$. If, in addition, $A \neq B$ and B is irreducible, then $\rho(A) < \rho(B)$.*

The next result follows from Lemma 2 and the definition of M-matrices.

Lemma 3. *Let A be a nonsingular M-matrix or an irreducible singular M-matrix. If $B \geq 0$ is nonzero and $A+B$ is a Z-matrix, then $A+B$ is a nonsingular M-matrix. In particular, the diagonal entries of an irreducible singular M-matrix are all positive.*

Lemma 4. *[5] Let A and B be nonsingular M-matrices with $A \geq B$. Then $0 \leq A^{-1} \leq B^{-1}$.*

Lemma 5. *[11, Theorem 3.31] Let $A = M - N$ be a nonsingular M-matrix, where M is obtained by setting certain off-diagonal entries of A to zero. Then $\rho(M^{-1}N) < 1$.*

Lemma 6. *[12, Theorem 3.2] The AVE* (1) *has a unique solution for every b if and only if the interval matrix $[A - I, A + I]$ is regular, i.e., all matrices X with $A - I \leq X \leq A + I$ are nonsingular.*

3. AVEs associated with M-matrices and H-matrices

When $A - I \in M_s$, we also have $(A - I)^T \in M_s$. So by Lemma 1 there is $v > 0$ such that $v^T(A - I) = 0$ and v is uniquely determined up to a scalar multiple. In this case, the existence and uniqueness of solutions of the AVE (1) has been settled completely.

Theorem 7. *([6, 10]) Suppose $A - I \in M_s$. Then*

1. *If $v^T b < 0$, then* (1) *has a unique solution x^*, which has at least one negative component.*
2. *If $v^T b = 0$, then* (1) *has infinitely many solutions and all of them are nonnegative.*
3. *If $v^T b > 0$, then* (1) *has no solutions.*

Regarding the numerical solution of the AVE (1) with $A - I \in M_s$, the main issue is to find the unique solution x^* when $v^T b < 0$. This solution can be found [6, 10] by using the generalized Newton method (GNM) in [8], which will terminate with the exact solution in at most $n + 1$ iterations [6], starting with any vector with at least one negative component. GNM requires $O(n^3)$ operations each iteration for a dense A, while the idea of Gauss–Seidel iteration leads to iterative methods with $O(n^2)$ operations each iteration.

Let $A = D - E - F$, where D, E and F are diagonal, strictly lower triangular and strictly upper triangular matrices, respectively. The generalized Gauss–Seidel iteration (GGS) for solving the AVE (1) is as follows [4].

$$(D - E)x^{(k)} - |x^{(k)}| = Fx^{(k-1)} + b, \quad k = 1, 2, \ldots, \tag{2}$$

where $x^{(0)}$ is a given initial guess. For given $x^{(k-1)}$, $x^{(k)}$ can be found by forward substitution, where n scalar AVEs are easily solved [4]. The Gauss-Seidel-Newton iteration (GSN) proposed in [1] for piecewise linear systems can be translated to a GSN for AVEs, which turns out to be a variant of GGS, but is more difficult to analyze. The convergence results established in [1] do not cover all cases for which a Newton-type iterative procedure is studied in [3]. On the other hand, the convergence results in [4] and in our new results in this section cover all those cases and much more.

Theorem 8. *If $A - I \in M_s$ and $v^T b < 0$, then the sequence $\{x^{(k)}\}$ generated by GGS converges linearly to the unique solution x^* of (1) for any initial vector $x^{(0)}$.*

Proof. Since x^* is a solution of (1), we get from (2)

$$(D - E)(x^{(k)} - x^*) = |x^{(k)}| - |x^*| + F(x^{(k-1)} - x^*).$$

Let $S^{(k)}$ be the diagonal matrix with diagonal entries

$$(S^{(k)})_i = \begin{cases} \frac{|(x^{(k)})_i| - |(x^*)_i|}{(x^{(k)})_i - (x^*)_i}, & \text{if } (x^{(k)})_i \neq (x^*)_i, \\ -1, & \text{if } (x^{(k)})_i = (x^*)_i. \end{cases}$$

Then $-I \leq S^{(k)} \leq I$ and

$$(D - E)(x^{(k)} - x^*) = S^{(k)}(x^{(k)} - x^*) + F(x^{(k-1)} - x^*).$$

Note that $D - E - S^{(k)}$ is a lower triangular matrix with positive diagonal entries. So

$$x^{(k)} - x^* = (D - E - S^{(k)})^{-1} F(x^{(k-1)} - x^*). \tag{3}$$

We also know by Lemma 3 that $D - E - S^{(k)} = (A - I) + (I - S^{(k)} + F)$ is a nonsingular M-matrix since $F \neq 0$ by the irreducibility of $A - I$. Let $u > 0$ be such that $(D - E - F - I)u = 0$. So $(D - E - F - S^{(k)})u \geq 0$ and then $(D - E - S^{(k)})^{-1} F u \leq u$. For given $x^{(0)}$ we can find a constant c such that $|x^{(0)} - x^*| \leq cu$. Suppose $|x^{(k-1)} - x^*| \leq cu$ for $k \geq 1$. Then by (3)

$$|x^{(k)} - x^*| \leq (D - E - S^{(k)})^{-1} F |x^{(k-1)} - x^*| \leq (D - E - S^{(k)})^{-1} F cu \leq cu.$$

Thus $|x^{(k)} - x^*| \leq cu$ for all $k \geq 0$ and then for some $\beta > 0$ we have $|(x^{(k)})_i| \leq \beta$ for all k and i. We know from Theorem 7 that $(x^*)_\ell < 0$ for an index ℓ. When $(x^{(k)})_\ell \leq 0$, we have $(S^{(k)})_\ell = -1$. When $(x^{(k)})_\ell > 0$,

$$(S^{(k)})_\ell = \frac{|(x^{(k)})_\ell| - |(x^*)_\ell|}{(x^{(k)})_\ell - (x^*)_\ell} = \frac{(x^{(k)})_\ell + (x^*)_\ell}{(x^{(k)})_\ell - (x^*)_\ell} \leq \frac{\beta + (x^*)_\ell}{\beta - (x^*)_\ell} < 1.$$

Let $\gamma = \frac{\beta + (x^*)_\ell}{\beta - (x^*)_\ell}$ and S be the matrix obtained from I by replacing its ℓth diagonal entry with γ. Then we have by (3) and Lemma 4

$$|x^{(k)} - x^*| \leq (D - E - S)^{-1} F |x^{(k-1)} - x^*| \leq \left((D - E - S)^{-1} F\right)^k |x^{(0)} - x^*|.$$

Since $D - E - F - S$ is a nonsingular M-matrix by Lemma 3, we know from Lemma 5 that $\rho\left((D - E - S)^{-1} F\right) < 1$ and thus $\{x^{(k)}\}$ converges to x^* linearly. □

Theorem 9. *Suppose $A - I \in H_s^+$. If (1) has a solution x^* with at least one negative component, then x^* is the unique solution of (1) and the sequence $\{x^{(k)}\}$ generated by GGS converges linearly to x^* for any $x^{(0)}$.*

Proof. Let x^* be a fixed solution of (1) with at least one negative component and $x^{(0)}$ be any initial vector. We proceed as in the proof of Theorem 8 and get

$$x^{(k)} - x^* = (D - E - S^{(k)})^{-1} F(x^{(k-1)} - x^*). \tag{4}$$

Now $D - |E| - S^{(k)} = (\langle A \rangle - I) + (I - S^{(k)} + |F|)$ is a nonsingular M-matrix and $|(D - E - S^{(k)})^{-1}| \leq (D - |E| - S^{(k)})^{-1}$. So by (4)

$$|x^{(k)} - x^*| \leq (D - |E| - S^{(k)})^{-1} |F| |x^{(k-1)} - x^*|.$$

By Assumption, $(x^*)_\ell < 0$ for an index ℓ. Let $\hat{u} > 0$ be such that $(D - |E| - |F| - I)\hat{u} = 0$. We can proceed as in the proof of Theorem 8 to show that

$$|x^{(k)} - x^*| \leq \left((D - |E| - S)^{-1} |F|\right)^k |x^{(0)} - x^*|, \tag{5}$$

where S be the matrix obtained from I by replacing its ℓth diagonal entry with some $\gamma \in (0, 1)$. Since $D - |E| - |F| - S$ is a nonsingular M-matrix, $\{x^{(k)}\}$ converges to x^* linearly. Let $\hat{x}$ be any solution of (1) and take $x^{(0)} = \hat{x}$. Then $x^{(k)} = \hat{x}$ for all $k \geq 0$. The convergence of $x^{(k)}$ to x^* implies that $\hat{x} = x^*$. So x^* is the unique solution. □

We note that the inequality in (5) is valid for all $k \geq 0$. This proves the global convergence. Once global convergence is proved, we can have a better estimate of the asymptotic rate of convergence.

Theorem 10. *The asymptotic rate of convergence of the sequence $\{x^{(k)}\}$ to x^* in Theorem 9 is such that*

$$\limsup \sqrt[k]{\|x^{(k)} - x^*\|} \leq \rho\left((D - |E| - S)^{-1} |F|\right) < 1,$$

where S is the diagonal matrix given by

$$(S)_i = \begin{cases} 1, & \text{if } (x^*)_i \geq 0, \\ -1, & \text{if } (x^*)_i < 0. \end{cases}$$

Proof. When $(x^*)_i < 0$, we have $(x^{(k)})_i < 0$ and $(S^{(k)})_i = -1$ for all k large enough. □

Note that with the new definition of S, $\rho\left((D - |E| - S)^{-1} |F|\right)$ gets smaller (see Lemmas 2 and 4).

We may also try to apply GNM to the AVE (1) with $A - I \in H_s^+$ to find a solution x^* with at least one negative component (it it exists). If the sign pattern of the components of $x^{(0)}$ match that of x^*, except the zero component of x^*, then $x^{(1)} = x^*$ (see [6]). But GNM may break down if $x^{(0)}$ does not have the correct sign pattern.

Example 1. *Consider the AVE* (1) *with*

$$A = \begin{bmatrix} 2 & 0.5 \\ 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1.6 \\ 1.6 \end{bmatrix}.$$

It is easily verified that $(2, -0.8)^T$ *is a solution to* (1)*. By Theorem 9, this is the unique solution and can be found by GGS with any initial vector. We now apply GNM with* $x^{(0)} = (\alpha, \beta)^T$*. If* $x^{(0)} > 0$*, then GNM breaks down in the first iteration. If* $\alpha < 0$*, then* $x^{(1)} > 0$ *for any* β *and GNM breaks down in the second iteration.*

In Theorem 9, the existence of a solution x^* is assumed. So we now examine the existence and uniqueness of solutions of (1) when $A - I \in H_s^+$.

Lemma 11. *Suppose* $A - I \in H_s^+$*. Then* $A - I$ *is singular if and only if* $A = \Delta\langle A\rangle\Delta$ *for some* $\Delta \in D_{\{1,-1\}}$*.*

Proof. We have $\langle A\rangle - I = rI - B$ with $B \geq 0$ and $r = \rho(B)$. Then $A - I = rI - C$ with $|C| = B$. So $A - I$ is singular if and only if r is an eigenvalue of C. Then by [2, Theorem 2.2.14 (and its proof)], r is an eigenvalue of C if and only if $C = \Delta B \Delta$ for some $\Delta \in D_{\{1,-1\}}$ (because C is a real matrix in our case). Therefore $A - I$ is singular if and only if $A = \Delta\langle A\rangle\Delta$. □

Theorem 12. *Suppose* $A - I \in H_s^+$ *and that there are no matrices* $\Delta \in D_{\{1,-1\}}$ *such that* $A = \Delta\langle A\rangle\Delta$*. Then* $A - I$ *is nonsingular and the AVE* (1) *has a unique solution for every* b*. If* $(A - I)^{-1}b \geq 0$*, then* $(A - I)^{-1}b$ *is the unique solution. Otherwise, the unique solution* x^* *has at least one negative component and the sequence* $\{x^{(k)}\}$ *generated by GGS converges linearly to* x^* *for any* $x^{(0)}$*.*

Proof. With these assumptions, $A - I$ is nonsingular by Lemma 11. All other matrices G in the interval $[A - I, A + I]$ are such that $\langle G\rangle = \langle A\rangle - I + D$, where $D \geq 0$ is a nonzero diagonal matrix. So $\langle A\rangle - I + D$ is a nonsingular M-matrix by Lemma 3 and thus G is also nonsingular. Now the interval matrix $[A - I, A + I]$ is regular and therefore (1) has a unique solution for every b. □

We now suppose $A - I \in H_s^+$ and that $A = \Delta\langle A\rangle\Delta$ for some $\Delta \in D_{\{1,-1\}}$. When $\Delta = I$, the existence and uniqueness of solutions of (1) have been fully characterized by Theorem 7. In particular, (1) may have no solutions for some vectors b. However, we can prove that (1) always has a solution when $\Delta \neq I$.

Lemma 13. *Suppose* $A - I \in H_s^+$ *and* $A = \Delta\langle A\rangle\Delta$ *for some* $\Delta \in D_{\{1,-1\}}$*. Then zero is the unique solution of* $Ax - |x| = 0$ *if and only if* $\Delta \neq I$*.*

Proof. Since zero is a solution of $Ax - |x| = 0$, all solutions are nonnegative by Theorem 9. To find all nonnegative solutions, we consider the linear system $(A - I)x = 0$, which is equivalent to $(\langle A\rangle - I)\Delta x = 0$. Thus $\Delta x = tu$ for a vector $u > 0$ and a scalar t such that $x = t\Delta u \geq 0$. When $\Delta \neq I$, t must be 0, so zero is the unique solution. When $\Delta = I$, all solutions are given by $x = tu$ with $t \geq 0$. □

Theorem 14. *Suppose $A - I \in H_s^+$ and $A = \Delta\langle A\rangle\Delta$ with $\Delta \in D_{\{1,-1\}} \setminus \{I\}$. Then (1) has a solution for every b. All nonnegative solutions (if any) can be found by solving the linear system $(A - I)x = b$. If there are no nonnegative solutions, then (1) has a unique solution x^* (with at least one negative component) and the sequence $\{x^{(k)}\}$ generated by GGS converges linearly to x^* for any $x^{(0)}$.*

Proof. For any integer $m \geq 1$ and any $b \neq 0$, the AVE $(A + \frac{1}{m}I)x - |x| = b$ has a unique solution $x^{(m)} \neq 0$ since $[A + \frac{1}{m}I - I, A + \frac{1}{m}I + I]$ is regular. Now the sequence $\{\frac{x^{(m)}}{\|x^{(m)}\|}\}$ is bounded. Suppose the sequence $\{x^{(m)}\}$ itself is unbounded. Then there is a subsequence $\{x^{(m_k)}\}$ such that $\lim_{k\to\infty} \|x^{(m_k)}\| = \infty$ and $\lim_{k\to\infty} \frac{x^{(m_k)}}{\|x^{(m_k)}\|} = d \neq 0$. Letting $k \to \infty$ in

$$(A + \frac{1}{m_k}I)\frac{x^{(m_k)}}{\|x^{(m_k)}\|} - \left|\frac{x^{(m_k)}}{\|x^{(m_k)}\|}\right| = \frac{b}{\|x^{(m_k)}\|},$$

we get $Ad - |d| = 0$, which is contradictory to Lemma 13. Therefore, the sequence $\{x^{(m)}\}$ is bounded and $\lim_{k\to\infty} x^{(m_k)} = w$ exists for a subsequence $\{x^{(m_k)}\}$. Letting $k \to \infty$ in

$$(A + \frac{1}{m_k}I)x^{(m_k)} - \left|x^{(m_k)}\right| = b,$$

we get $Aw - |w| = b$. We have thus shown that (1) has a solution for every b. The remaining statements in the theorem are straightforward, in view of Theorem 9. □

If (1) has a nonnegative solution $\hat{x}$ in Theorem 14 (then all solutions are nonnegative by Theorem 9), we can also determine whether the solution is unique. Using the notation in the proof of Lemma 13, all nonnegative solutions x are given by $x = \hat{x} + t\Delta u \geq 0$ for all possible numbers t. If $\hat{x} > 0$, then the solution set consists of infinitely many nonnegative solutions. If $\hat{x} \geq 0$ and $Q = \{i \mid \hat{x}_i = 0\}$, then the solution set consists of infinitely many nonnegative solutions if $(\Delta)_i$ have the same sign for all $i \in Q$, and consists of the unique solution $\hat{x}$ otherwise.

We end the paper by commenting on the numerical performance of GGS for the AVE (1) when $A - I \in H_s^+$. We take $n = 1000$ and use random matrices to generate matrices $B \in M_s$ and then let $A = I + B$. We update A in two ways: (a) change the sign of the off-diagonal entries of A randomly, and have the case covered by Theorem 12; (b) change A to $\Delta A\Delta$ with a random $\Delta \in D_{\{1,-1\}}$, and have the situation in Theorem 14. With the updated A, we pick x^* randomly (which has at least one negative component) and let $b = Ax^* - |x^*|$. We then take a random $x^{(0)}$ and run GGS with stopping criterion $\|Ax^{(k)} - |x^{(k)}| - b\|/\|b\| < 10^{-6}$. We find that GGS converges in just 4 iterations for all 1000 trials with the updating in (a). When the updating in (b) is used, GGS often requires more than 1000 iterations. As suggested by Theorem 10, GGS usually requires more iterations when x^* has just one negative component. In this case, we find that GGS often requires more than 10000 iterations. It would be interesting to develop numerical methods that have guaranteed convergence and better efficiency for this difficult situation.

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