Comments on: "Two efficient iteration methods for solving the absolute value equations"

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Abstract

Two iterative methods for solving the absolute value equations are recently proposed and analyzed in the paper by Yu and Wu (Appl. Numer. Math. 208 (2025) 148–159). We point out that the convergence analysis for both methods is incorrect and that the second method with "optimal" parameters is always slightly *less* efficient than the well-known generalized Newton method.

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Two iterative methods are given in [5] for solving the absolute value equation (AVE)

$$Ax - |x| = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular, $b \in \mathbb{R}^n$, and $|\cdot|$ denotes absolute value.

The first method is the AGSOR-like iteration method, with iteration formula

$$\begin{pmatrix} A & 0 \\ -\beta D(x_{k+1}) & I \end{pmatrix}\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} (1-\alpha)A & \alpha I \\ 0 & (1-\beta)I \end{pmatrix}\begin{pmatrix} x_k \\ y_k \end{pmatrix}+\begin{pmatrix} \alpha b \\ 0 \end{pmatrix},$$

where $D(x) = diag(sign(x))$ and α, β are nonzero parameters. The authors then introduce the iteration matrix

$$M_{\alpha,\beta} = \begin{pmatrix} A & 0 \\ -\beta D(x_{k+1}) & I \end{pmatrix}^{-1}\begin{pmatrix} (1-\alpha)A & \alpha I \\ 0 & (1-\beta)I \end{pmatrix}.$$

From the proof and later use of [5, Theorem 3.1], that result should be restated as follows.

Theorem 1. *Suppose that $\sigma_{\min}(A) > 1$. If λ is any eigenvalue of $M_{\alpha,\beta}$ with $\lambda \neq 1-\alpha$ and $\lambda \neq 1-\beta$, then*

$$\lambda^2 - (\alpha\beta\mu - \alpha - \beta + 2)\lambda + \alpha\beta - \alpha - \beta + 1 = 0$$

for some eigenvalue μ of $A^{-1}D(x_{k+1})$.

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In [5, Theorem 3.2], the assumption that all eigenvalues of $A^{-1}D(x_{k+1})$ are real is very restrictive and not so practical, but is not a main issue here. We point out that the proof of convergence for the AGSOR-like method is not valid. That proof goes as follows: When $0 < \alpha\beta < \alpha + \beta < 2$, $|\lambda| < 1$ for all eigenvalues of $M_{\alpha,\beta}$ (using Theorem 1 and $|\mu| < 1$), and therefore the AGSOR-like method is convergent. The argument is not valid because the matrices $M_{\alpha,\beta}$ are dependent on k. For the special case $b = 0$, the authors have effectively used the assertion that the sequence $\{z_k\}$ defined by $z_{k+1} = M_k z_k$ is convergent for any z_0 if $\rho(M_k) < 1$ for all $k \geq 0$. This assertion is obviously false, as shown by the example

$$z_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad M_{2k} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad M_{2k+1} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad k = 0, 1, \ldots.$$

The proof of [5, Theorem 3.2] is therefore incorrect.

Whether the conclusion of [5, Theorem 3.2] is true is not an interesting question because, in general, verifying the assumption in [5, Theorem 3.2] is more time-consuming than solving the AVE inefficiently (by looking for a solution of any sign pattern and solving the corresponding ordinary linear system). Moreover, when $(\alpha, \beta) = (1, 1)$, AGSOR-like method with $x_0 = y_0 = 0$ is reduced to the Picard iteration $x_{k+1} = A^{-1}(|x_k| + b)$ with $x_0 = 0$, whose convergence is well known under the usual assumption that $\sigma_{\min}(A) > 1$. When $(\alpha, \beta) \neq (1, 1)$, the convergence of AGSOR-like method is not known even with the very restrictive additional assumption on $A^{-1}D(x_{k+1})$. It is even more difficult to choose (α, β) such that AGSOR-like method is faster than Picard iteration. In fact, the authors of [5] used $(\alpha, \beta) = (0.96, 1.04)$ for their Examples 5.1 and 5.2, and used $(\alpha, \beta) = (1.00, 1.01)$ for their Example 5.3. Both choices are outside their claimed convergence region in [5, Theorem 3.2]. Presumably they did not find good parameters inside their claimed convergence region. Moreover, their choice $(\alpha, \beta) = (0.96, 1.04)$, for example, does not appear to be better than the choice $(\alpha, \beta) = (1, 1)$, as we illustrate below.

In the spirit of [5, Example 5.1], we consider a very simple AVE (1) with unique solution x_*, where

$$A = \begin{bmatrix} 2.1 & -1 \\ -1 & 2.1 \end{bmatrix}, \quad x_* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} 2.1 \\ -4.1 \end{bmatrix}.$$

We take $x_0 = 0$ and after 20 iterations of Picard iteration we get approximate solution $(0.999995296, -0.999995296)^T$. We then take $x_0 = y_0 = 0$ and after 20 iterations of AGSOR-like method with $(\alpha, \beta) = (0.96, 1.04)$ we get approximate solution $(0.999994519, -0.999995766)^T$. So Picard iteration gives a better approximation in terms of 1-norm, 2-norm, and ∞-norm.

We also mention that the statements in [5, Theorem 3.3] are not valid since the $\mu_{\max}$ there is k-dependent and (α, β) is supposed to be k-independent.

The second method in [5] is the PGSOR-like iteration method, given by

$$\begin{aligned} x_{k+1} &= (1-\alpha)x_k + (\omega A - D(x_k))^{-1}[\alpha(\omega - 1)y_k + \alpha\omega b], \\ y_{k+1} &= \frac{\alpha}{1+\omega}(Ax_{k+1} + \omega|x_{k+1}|) + (1-\alpha)y_k - \frac{\alpha}{1+\omega}b. \end{aligned}$$

[5, Theorem 4.3] says that the optimal parameters for the PGSOR-like method is $(\alpha, \omega) = (1, 1)$ under certain conditions and the authors use the optimal parameters for PGSOR in numerical experiments and compare PGSOR with the generalized Newton method (GN). But when $(\alpha, \omega) = (1, 1)$, the PGSOR-like method becomes

$$x_{k+1} = (A - D(x_k))^{-1} b, \tag{2}$$

$$y_{k+1} = \frac{1}{2}(Ax_{k+1} + |x_{k+1}|) - \frac{1}{2}b. \tag{3}$$

Note that equation (2) is precisely GN and equation (3) is now useless. Therefore, the PGSOR-like method with the optimal parameters is always slightly *less* efficient than GN.

The proof of [5, Theorem 4.1] is incorrect exactly like the proof of [5, Theorem 3.2] is incorrect. Also, "$\omega > 1$" in [5, Theorem 4.1] is believed to be a typo of "$\omega \geq 1$". In fact, the (main) statement and (incorrect) proof of [5, Theorem 4.1] remain the same if $\omega > 1$ is replaced by $\omega \geq 1$, and actually $(\alpha, \omega) = (1, 1)$ is claimed to be optimal in [5, Theorem 4.3]. With $(\alpha, \omega) = (1, 1)$, equation (15) of [5] holds trivially. So if [5, Theorem 4.1] were true with $(\alpha, \omega) = (1, 1)$, one would have the following: GN converges whenever $\|A^{-1}\| < 1$, where $\|\cdot\|$ is the matrix 2-norm. This assertion about GN can be shown to be false by the following example, suggested by [2, Proposition 7.7]. Therefore, [5, Theorem 4.1] is false with $(\alpha, \omega) = (1, 1)$.

Example 1. *Consider the AVE* (1) *with* $n = 3$ *and*

$$A = \begin{pmatrix} 0 & 1.6 & 0 \\ 0 & 0 & 1.6 \\ 1.6 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1.6 \\ 1.6 \\ 1.6 \end{pmatrix}.$$

Then $\|A^{-1}\| = 0.625$. *Take* $x_0 = (c_1, c_2, c_3)^T$ *with*

$$c_1 = \frac{1.6^3 + 1.6^2 + 1.6}{1.6^3 + 1}, \quad c_2 = \frac{1.6^3 - 1.6^2 - 1.6}{1.6^3 + 1}, \quad c_3 = \frac{1.6^3 + 1.6^2 - 1.6}{1.6^3 + 1}.$$

Then we find

$$x_1 = (c_3, c_1, c_2)^T, \quad x_2 = (c_2, c_3, c_1)^T, \quad x_3 = (c_1, c_2, c_3)^T = x_0.$$

So the sequence $\{x_k\}$ *is not converging.*

In practice, however, GN often works well in terms of the number of iterations. GN has the finite termination property, as discussed in [4]. The finite termination property has been stated in [3] more generally: GN has finite termination property whenever it has convergence, even when the AVE has infinitely many solutions. In practice, GN often terminates after just 2 or 3 iterations, as also shown in the numerical results reported in [5] for GN and PGSOR-like (with $(\alpha, \omega) = (1, 1)$). Why GN would terminate with so few iterations (even

with a very large n) remains a mystery. It is certainly not due to the smallness of the spectral radii of some matrices.

The paper [5] seems to be following closely the approach used in [1], where the authors claim that "the convergence and optimal parameters of NSOR method for solving the AVE are studied in detail". In view of my comments here on the AGSOR-like method in [5], we can see that there is no convergence analysis of NSOR in [1] either.

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